Applied Analysis and Optimization
Volume 7, Number 2, 2023, 115–133

Yokohama Publishers
ISSN 2189-1664 Online Journal

QUALITATIVE PROPERTIES OF THE SOLUTIONS OF A MECHANICAL SYSTEM OF VIBRATION REDUCTION THROUGH A GENERALIZED VARIANT OF THE KRASNOSELSKII FIXED POINT THEOREM

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Dedicated to Professor Adrian Petruşel on the occasion of his sixtieth anniversary

Abstract. In this paper we investigate nonlinear systems of second order ODEs describing the dynamics of a mechanical system of vibration reduction. We deduce, under certain assumptions, some stability results for the null solution. We also show that for any solution (x, y) of the system we have $\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} \dot{x}(t) = \lim_{t\to+\infty} y(t) = \lim_{t\to+\infty} \dot{y}(t) = 0$, for small initial data. The proofs are mainly based on a generalized variant of the Krasnoselskii fixed point theorem. Our theoretical results are confirmed by numerical simulations.

1. Introduction

Two central results of the fixed point theory are the Schauder theorem and the Banach contraction mapping principle. Krasnoselskii combined them into the following fascinating result (see, e.g., [19, Theorem 2], [27, Theorem A]).

Theorem 1.1 (Krasnoselskii)**.** *Let M be a closed, convex, and bounded subset of a Banach space X. Suppose that A and B map M into X such that:*
(1) *A is a contraction;*
(2) *B is a compact operator;*
(3) *$Ax + By \in M$, for all x, $y \in M$.*

Then there exists $y \in M$, such that

$$y = Ay + By. \tag{1.1}$$

We recall that B is a compact operator iff it is continuous and maps bounded sets into relatively compact sets. Theorem 1.1 has powerful applications to differential equations and integral equations. The proof idea is based on the fact that (1) ensures the existence and the continuity of the operator $(I - A)^{-1}$ (where I denotes the identity operator). Then the solutions of (1.1) are exactly the fixed points of the operator $(I - A)^{-1} B$; but this operator $(I - A)^{-1} B : M \to M$ is compact and so the existence of a fixed point is ensured by the Schauder theorem.

This result has known several extensions and improvements and we refer the reader, e.g., to [1], [4], [6–8], [27], [29], [31], [32], [46]. In the present paper we will apply a generalized variant of Theorem 1.1 (see Theorem 3.3 below) for studying the large-time behavior of solutions of a system of vibration reduction. Specifically,

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2020 *Mathematics Subject Classification.* 34D20.
Key words and phrases. Coupled oscillators, uniform stability, asymptotic stability, uniform asymptotic stability, vibration reduction.

we consider a mechanical system as sketched in Figure 1.1. The block of mass m_1 is anchored to a fixed horizontal wall and to the block of mass m_2 by springs and dampers, and the block of mass m_2 is also attached to the wall by a pair of springs and dampers. Suppose that the stiffnesses and the dampings are represented by the functions $k_i : \mathbb{R}_+ \to \mathbb{R}_+$ and $d_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i \in \{1, 2, 3\}$, and $\widehat{g}_i : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i \in \{1, 2\}$, denote external forces acting on the blocks, where $\mathbb{R}_+ := [0, +\infty)$. We assume that, when the two blocks are in their equilibrium positions, the springs and the dampers are also in their equilibrium positions. Let $x(t)$ and $y(t)$ be the vertical displacements of the blocks from their equilibrium positions.

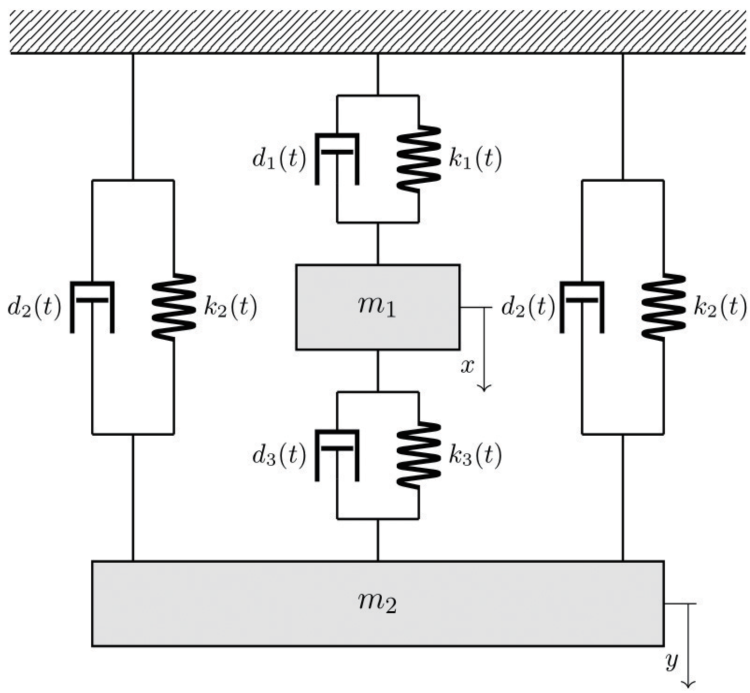


Figure 1.1

Then the system of ODEs describing the motion is (see, e.g., [50])

$$\begin{cases} m_1\ddot{x} + k_1(t)x + d_1(t)\dot{x} - k_3(t)(y-x) - d_3(t)(\dot{y}-\dot{x}) = \widehat{g}_1(t,x,y), \\ m_2\ddot{y} + 2k_2(t)y + 2d_2(t)\dot{y} + k_3(t)(y-x) + d_3(t)(\dot{y}-\dot{x}) = \widehat{g}_2(t,x,y), \end{cases}$$

or

$$\begin{cases} \ddot{x} + 2f_1(t)\dot{x} - f_3(t)\dot{y} + \beta(t)x - \gamma_1(t)y + g_1(t,x,y) = 0, \\ \ddot{y} + 2f_2(t)\dot{y} - f_4(t)\dot{x} - \gamma_2(t)x + \delta(t)y + g_2(t,x,y) = 0, \end{cases} \tag{1.2}$$

where

$$\begin{aligned} f_1(t) &:= \frac{1}{2m_1}(d_1(t) + d_3(t)), \; f_2(t) := \frac{1}{2m_2}(2d_2(t) + d_3(t)), \\ f_3(t) &:= \frac{1}{m_1}d_3(t), \; f_4(t) := \frac{1}{m_2}d_3(t), \end{aligned}$$

$$\begin{aligned}
\beta(t) &:= \frac{1}{m_1}\left(k_1(t)+k_3(t)\right),\ \delta(t) := \frac{1}{m_2}\left(2k_2(t)+k_3(t)\right),\\
\gamma_1(t) &:= \frac{1}{m_1}k_3(t),\ \gamma_2(t) := \frac{1}{m_2}k_3(t),\\
g_1(t,x,y) &:= -\frac{1}{m_1}\widehat{g}_1(t,x,y),\ g_2(t,x,y) := -\frac{1}{m_2}\widehat{g}_2(t,x,y).
\end{aligned}$$

In [26] we studied the system (1.2) by two approaches, based on differential inequalities and on the Lyapunov method. In Theorem 2.2 below we provide some results on the stability of the equilibrium of (1.2) using a generalized variant of the Krasnoselskii theorem, on the metrizable locally convex space of the continuous functions defined on a half-line, endowed with two countable families of seminorms as chosen as to determine the same topology, of the uniform convergence on the compact subsets of this interval. We will also show that for any solution (x, y) to system (1.2) we have $\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} \dot{x}(t) = \lim_{t\to+\infty} y(t) = \lim_{t\to+\infty} \dot{y}(t) = 0$, for small initial data, in the case when the nonlinearities are not necessarily locally Lipschitz functions (hence the uniqueness is not guaranteed).

In [25] we researched the large-time behavior of the solutions of a system of two coupled damped nonlinear oscillators using a generalized form of the Schauder-Tychonoff fixed point theorem. For other results regarding the stability of the equilibria of coupled damped nonlinear oscillators, we refer the reader to [15], [23], [24], [34–37], [44], and the references therein. Investigations on the stability of the equilibrium of a single damped nonlinear oscillator can be found, e.g., in [2], [5], [13], [14], [16], [17], [21], [22], [43], [45], and the references therein. For fundamental concepts and results in stability theory, see, e.g., [3], [9], [11], [12], [20], [33], and for comprehensive studies on the fixed point theory we refer the reader to the monographs [30], [38–42], [49].

The model in Figure 1.1 could be used, e.g., to describe the dynamics in vertical direction of vibration reduction systems for horizontal cranes with loadings suspended in two sides ([18], [47]). For other models of coupled oscillators or for models from electric circuit theory, structural dynamics, described by systems of type (1.2), we refer the reader to the monographs [10], [28], [48].

2. General framework and main result

The following hypotheses will be admitted:

(H1) $f_i \in C^1(\mathbb{R}_+;\mathbb{R}_+)$, $f_j \in C(\mathbb{R}_+;\mathbb{R}_+)$, and $\int_0^{+\infty} f_j(t)\,\mathrm{d}t < +\infty$, $\forall i \in \{1,2\}$, $\forall j \in \{3,4\}$;

(H2) there exist constants h, K_1, $K_2 \geq 0$ such that

$$\left|\dot{f}_i(t) + f_i^2(t)\right| \leq K_i \widetilde{f}(t),\ \forall t \in [h,+\infty),\ \forall i \in \{1,2\},$$

where $\widetilde{f}(t) := \min\{f_1(t), f_2(t)\}$, $\forall t \in \mathbb{R}_+$;

(H3) $\int_0^{+\infty} \widetilde{f}(t)\,\mathrm{d}t = +\infty$.

(H4) $\beta,\ \delta \in C^1(\mathbb{R}_+;\mathbb{R}_+)$, $\beta,\ \delta$ are decreasing, and

$$\beta_0 := \lim_{t\to+\infty} \beta(t) > 0,\ \delta_0 := \lim_{t\to+\infty} \delta(t) > 0,$$

are such that

$$\frac{K_1}{\sqrt{\beta_0}} + \frac{K_2}{\sqrt{\delta_0}} < 1; \tag{2.1}$$

(H5) $\gamma_i \in C(\mathbb{R}_+;\mathbb{R}_+)$ and $\int_0^{+\infty} \gamma_i(t)\,\mathrm{d}t < +\infty$, $\forall i \in \{1,2\}$;

(H6) $g_i = g_i(t,x,y) \in C(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R};\mathbb{R})$ and fulfill the relations

$$|g_1(t,x,y)| \le r_1(t)\,\mathrm{O}(|x|), \quad \forall t \in \mathbb{R}_+,\ \forall y \in \mathbb{R}, \tag{2.2}$$

$$|g_2(t,x,y)| \le r_2(t)\,\mathrm{O}(|y|), \quad \forall t \in \mathbb{R}_+,\ \forall x \in \mathbb{R}, \tag{2.3}$$

where $r_i \in C(\mathbb{R}_+;\mathbb{R}_+)$, $\int_0^{+\infty} r_i(t)\,\mathrm{d}t < +\infty$, $\forall i \in \{1,2\}$, and $\mathrm{O}(|x|)$ denotes the big-O Landau symbol as $x \to 0$ (similarly for $\mathrm{O}(|y|)$);

(H7) $g_i = g_i(t,x,y)$ is locally Lipschitzian in x, y, $\forall i \in \{1,2\}$.

(H8) There is a $p > 0$, such that $f_i(t) \ge p$, $\forall t \ge 0$, $\forall i \in \{1,2\}$.

Remark 2.1. As in, e.g., [26, Remark 2.1], we deduce that if (H1) and (H2) hold, then f_i, $\dot{f}_i$ are bounded, $i \in \{1,2\}$.

The main result of this paper in the following.

Theorem 2.2.

i) *Suppose that the hypotheses (H1)-(H6) are fulfilled. Then for every solution* (x,y) *of the system* (1.2), *we have*

$$\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} \dot{x}(t) = \lim_{t\to+\infty} y(t) = \lim_{t\to+\infty} \dot{y}(t) = 0,$$

for small initial data.

ii) *If the hypotheses (H1)-(H7) are fulfilled, then the null solution of* (1.2) *is asymptotically stable.*

iii) *If the hypotheses (H1), (H2), (H4)-(H7) are fulfilled, then the null solution of* (1.2) *is uniformly stable.*

iv) *If the hypotheses (H1), (H2), (H4)-(H8) are fulfilled, then the null solution of (1.1) is uniformly asymptotically stable.*

Remark 2.3. *Let us note that in order to prove i), the hypothesis (H7), which ensures the uniqueness of the solution of any initial value problem associated to the system* (1.2)*, is not needed. Hence, while ii)-iv) are comparable to the stability results reported in* [26, Theorem 2.1]*, the statement i) is new and is obtained by using a generalized variant of the Krasnoselskii fixed point theorem (see Section* 3 *below). So we emphasize the efficiency of the fixed point method in studying the behavior at infinity of the solutions of* (1.2).

3. Proof of Theorem 2.2

Using the transformation (similar to that introduced by Burton and Furumochi in [5]),

$$\begin{cases} \dot{x} = u - f_1(t)\, x \\ \dot{u} = \left[\dot{f}_1(t) + f_1^2(t) - \beta(t)\right] x - f_1(t)\, u + \left[\gamma_1(t) - f_2(t) f_3(t)\right] y \\ \qquad + f_3(t)\, v - g_1(t,x,y) \\ \dot{y} = v - f_2(t)\, y \\ \dot{v} = \left[\gamma_2(t) - f_1(t) f_4(t)\right] x + f_4(t)\, u + \left[\dot{f}_2(t) + f_2^2(t) - \delta(t)\right] y \\ \qquad - f_2(t)\, v - g_2(t,x,y) \end{cases} \tag{3.1}$$

the system (1.2) becomes, as in [26],

$$\dot{z} = U(t)\, z + V(t)\, z + F(t,z), \tag{3.2}$$

where

$$z = \begin{pmatrix} x \\ u \\ y \\ v \end{pmatrix}, \; U(t) = \begin{pmatrix} -f_1(t) & 1 & 0 & 0 \\ -\beta(t) & -f_1(t) & \gamma_1(t) & 0 \\ 0 & 0 & -f_2(t) & 1 \\ \gamma_2(t) & 0 & -\delta(t) & -f_2(t) \end{pmatrix},$$

$$V(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \dot{f}_1(t) + f_1^2(t) & 0 & -f_2(t) f_3(t) & f_3(t) \\ 0 & 0 & 0 & 0 \\ -f_1(t) f_4(t) & f_4(t) & \dot{f}_2(t) + f_2^2(t) & 0 \end{pmatrix},$$

$$F(t,z) = \begin{pmatrix} 0 \\ -g_1(t,x,y) \\ 0 \\ -g_2(t,x,y) \end{pmatrix}.$$

and our large time behavior question reduces to the large time behavior of the solutions of (3.2).

Take $t_0 \geq 0$ and let

$$Z(t,t_0) = \left(a_{ij}(t,t_0)\right)_{i,j \in \overline{1,4}}, \; t \geq t_0,$$

be the fundamental matrix of the system

$$\dot{z} = U(t)\, z,$$

which equals the identity matrix for $t = t_0$. Let $\|\cdot\|_0$ be the norm in $\mathbb{R}^4$ defined by

$$\|z\|_0 = \left(\beta_0 x^2 + u^2 + \delta_0 y^2 + v^2\right)^{1/2}, \text{ for } z = (x,u,y,v)^\top, \tag{3.3}$$

which is obviously equivalent to the Euclidean norm.

For $z_0 = (x_0, u_0, y_0, v_0)^\top \in \mathbb{R}^4$, we deduce as in [25], [26], for all $t \geq t_0$

$$\|Z(t,t_0)\, z_0\|_0 \leq \lambda \|z_0\|_0\, \alpha(t_0) \mathrm{e}^{\int_{t_0}^t \left[-\widetilde{f}(u) + \frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right] \mathrm{d}u} \tag{3.4}$$

and for all $t \geq s \geq t_0 \geq 0$

$$\left\| Z(t,t_0)\, Z(s,t_0)^{-1} e_2 \right\|_0 \le \mathrm{e}^{\int_s^t \left[-\widetilde{f}(u) + \frac{\gamma(u)}{2\sqrt{\zeta(u)}} \right] \mathrm{d}u}, \tag{3.5}$$
$$\left\| Z(t,t_0)\, Z(s,t_0)^{-1} e_4 \right\|_0 \le \mathrm{e}^{\int_s^t \left[-\widetilde{f}(u) + \frac{\gamma(u)}{2\sqrt{\zeta(u)}} \right] \mathrm{d}u},$$

$$\left\| Z(t,t_0)\, Z(s,t_0)^{-1} z_0 \right\|_0 \le \Lambda \left\| z_0 \right\|_0 \mathrm{e}^{\int_s^t \left[-\widetilde{f}(u) + \frac{\gamma(u)}{2\sqrt{\zeta(u)}} \right] \mathrm{d}u}, \tag{3.6}$$

where $\lambda := \max\left\{1, 1/\sqrt{\beta_0}, 1/\sqrt{\delta_0}\right\}$, $\alpha(t_0) := \sqrt{\beta(t_0) + \delta(t_0) + 2}$,
$\Lambda := \max\left\{ \sqrt{\beta(0)/\beta_0}, \sqrt{\delta(0)/\delta_0} \right\}$, $\gamma(t) := \max\{\gamma_1(t), \gamma_2(t)\}$,
$\zeta(t) := \min\{\beta(t), \delta(t)\}$, $\forall t \in \mathbb{R}_+$, $e_2 = (0,1,0,0)^\top$, and $e_4 = (0,0,0,1)^\top$.

For $t_0 \ge 0$ we consider the functional space

$$C_c(t_0) := \left\{ z : [t_0, +\infty) \to \mathbb{R}^4,\ z \text{ continuous} \right\},$$

which becomes a complete metrizable locally convex space (i.e., a Fréchet space) with respect to each of the countable families of seminorms

$$\|z\|_n := \sup_{t \in [t_0, n]} \left\{ \|z(t)\|_0 \right\},\ n \in \mathbb{N},\ n > t_0, \tag{3.7}$$

and

$$\|z\|_{\lambda_n} := \sup_{t \in [t_0, n]} \left\{ \|z(t)\|_0\, \mathrm{e}^{-\lambda_n (t - t_0)} \right\},\ n \in \mathbb{N},\ n > t_0, \tag{3.8}$$

where $\lambda_n > 0$ ($n \in \mathbb{N}$, $n > t_0$) are positive numbers to be specified later.

Remark 3.1. It is readily seen that a sequence $(w_m)_m \subset C_c(t_0)$ is convergent to $w \in C_c(t_0)$, with respect to the family of seminorms $(3.8) \iff (w_m)_m \subset C_c(t_0)$ is convergent to $w \in C_c(t_0)$, with respect to the family of seminorms (3.7).

Indeed, let us consider $(w_m)_m \subset C_c(t_0)$ convergent to $w \in C_c(t_0)$, with respect to the family of seminorms (3.8). Let $n \in \mathbb{N}$, $n > t_0$ and $\varepsilon > 0$ be given. Then there is $M \in \mathbb{N}$, such that $\forall m \ge M$

$$\|w_m(t) - w(t)\|_0\, \mathrm{e}^{-\lambda_n (t - t_0)} < \varepsilon \mathrm{e}^{-\lambda_n (n - t_0)},\ \forall t \in [t_0, n].$$

Hence

$$\|w_m(t) - w(t)\|_0 < \varepsilon \mathrm{e}^{-\lambda_n (n - t)} \le \varepsilon,\ \forall t \in [t_0, n]$$

and so $w_m \to w$, with respect to the family of seminorms (3.7). The converse can be easily deduced.

Therefore, the families of seminorms (3.7) and (3.8) define on $C_c(t_0)$ the same topology, of the uniform convergence on the compact subsets of $[t_0, +\infty)$, for every sequence λ_n. We also mention that $\mathcal{A} \subset C_c(t_0)$ is relatively compact if and only if it is equicontinuous and uniformly bounded on the compact subsets of $[t_0, +\infty)$ (the Arzelà-Ascoli Theorem).

Let $t_0 \ge 0$ and $z_0 \in \mathbb{R}^4$ be arbitrary. We define on $C_c(t_0)$ the operators

$$(Aw)(t) := Z(t,t_0)\, z_0 + \int_{t_0}^t Z(t,t_0)\, Z^{-1}(s,t_0)\, V(s)\, w(s)\, \mathrm{d}s, \tag{3.9}$$

$$(Bw)(t) := \int_{t_0}^{t} Z(t,t_0) Z^{-1}(s,t_0) F(s,w(s)) \mathrm{d}s, \tag{3.10}$$

for all $w \in C_c(t_0)$ and for all $t \geq t_0$ and, obviously, the set of the solutions of (3.2) fulfilling the initial condition $z(t_0) = z_0$ is equal to the set of the fixed points of $A+B$.

From (2.2), (2.3) we infer that there exist M_i, $l_i > 0$, $i \in \{1,2\}$, such that

$$\begin{array}{l} |g_1(t,x,y)| \leq M_1 r_1(t) |x|, \text{ if } |x| < l_1, \\ |g_2(t,x,y)| \leq M_2 r_2(t) |y|, \text{ if } |y| < l_2. \end{array} \tag{3.11}$$

Let $q(t)$ be the unique solution of the initial value problem

$$\begin{array}{l} \dot{q}(t) = \left[-\widetilde{f}(t) + \frac{\left|\dot{f}_1(t)+f_1^2(t)\right|}{\sqrt{\beta_0}} + \frac{\left|\dot{f}_2(t)+f_2^2(t)\right|}{\sqrt{\delta_0}} + \nu(t) \right] q(t), \ t \geq t_0, \\ q(t_0) = \lambda \|z_0\|_0 \alpha(t_0), \end{array} \tag{3.12}$$

where

$$\begin{aligned} \nu(t) \ &:= \ \frac{\gamma(t)}{2\sqrt{\zeta(t)}} + f_3(t) + f_4(t) + \frac{f_1(t) f_4(t)}{\sqrt{\beta_0}} + \frac{f_2(t) f_3(t)}{\sqrt{\delta_0}} \\ &\quad + \frac{M_1 r_1(t)}{\sqrt{\beta_0}} + \frac{M_2 r_2(t)}{\sqrt{\delta_0}}, \ \forall t \geq 0. \end{aligned}$$

Obviously, for $t \geq t_0$, we have

$$q(t) = \lambda \|z_0\|_0 \alpha(t_0) \mathrm{e}^{-\int_{t_0}^t \widetilde{f}(s)\mathrm{d}s} \mathrm{e}^{\int_{t_0}^t \left[\frac{\left|\dot{f}_1(s)+f_1^2(s)\right|}{\sqrt{\beta_0}} + \frac{\left|\dot{f}_2(s)+f_2^2(s)\right|}{\sqrt{\delta_0}} \right]\mathrm{d}s} \mathrm{e}^{\int_{t_0}^t \nu(s)\mathrm{d}s}$$

and, due to the hypotheses (H1), (H2), (H4)-(H6) and Remark 2.1, we obtain $\int_0^{+\infty} \nu(s)\mathrm{d}s < +\infty$.

Now, consider the set

$$S(t_0,\rho) := \{ w \in C_c(t_0) \mid \|w(t)\|_0 \leq \rho \text{ and } \|w(t)\|_0 \leq q(t), \ \forall t \geq t_0 \},$$

for $t_0 \geq 0$ and $\rho > 0$. Since $w_0(t) := \min\{\rho, q(t)\} (0,1,0,0)^\top$, $\forall t \geq t_0$, is contained in $S(t_0,\rho)$, it follows that the set $S(t_0,\rho)$ is nonempty. Obviously $S(t_0,\rho)$ is a complete and convex subset of $C_c(t_0)$, $\forall t_0 \geq 0$, $\forall \rho > 0$.

We first state and prove the following useful result.

Lemma 3.2. *There exists $l > 0$, such that for all $t_0 \geq 0$ and for all $\rho \in (0,l)$, there exists $a > 0$, such that for all z_0 with $\|z_0\|_0 \in (0,a)$ and for all w_1, $w_2 \in S(t_0,\rho)$, we have $Aw_1 + Bw_2 \in S(t_0,\rho)$.*

Proof. Let $l := \min\left\{\sqrt{\beta_0} l_1, \sqrt{\delta_0} l_2\right\}$, $\rho \in (0,l)$, $t_0 \geq 0$, and $z_0 \in \mathbb{R}^4 \setminus \{0\}$ with $\|z_0\|_0$ small enough. Consider some arbitrary w_1, $w_2 \in S(t_0,\rho)$. Obviously, $Aw_1 + Bw_2 \in C_c(t_0)$.

From (3.4) – (3.6), (3.9), (3.10) we deduce (as in [26])

$$\begin{aligned} \|(Aw_1)(t)\|_0 \ &\leq \ \lambda \|z_0\|_0 \alpha(t_0) \mathrm{e}^{\int_{t_0}^t \left[-\widetilde{f}(u) + \frac{\gamma(u)}{2\sqrt{\zeta(u)}} \right]\mathrm{d}u} \\ &\quad + \int_{t_0}^t \mathrm{e}^{\int_s^t \left[-\widetilde{f}(u) + \frac{\gamma(u)}{2\sqrt{\zeta(u)}} \right]\mathrm{d}u} \left[\frac{\left|\dot{f}_1(s) + f_1^2(s)\right|}{\sqrt{\beta_0}} \right. \end{aligned} \tag{3.13}$$

$$+\frac{\left|\dot{f}_2(s)+f_2^2(s)\right|}{\sqrt{\delta_0}}+f_3(s)+f_4(s)$$
$$\left.+\frac{f_1(s)\,f_4(s)}{\sqrt{\beta_0}}+\frac{f_2(s)\,f_3(s)}{\sqrt{\delta_0}}\right]\|w_1(s)\|_0\,\mathrm{d}s$$

and

$$\|(Bw_2)(t)\|_0\le\int_{t_0}^{t}\mathrm{e}^{\int_s^t\left[-\widetilde{f}(u)+\frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u}\left[\frac{M_1r_1(s)}{\sqrt{\beta_0}}+\frac{M_2r_2(s)}{\sqrt{\delta_0}}\right]\|w_2(s)\|_0\,\mathrm{d}s, \tag{3.14}$$

for all $t\ge t_0$.

Since $\|w_i(s)\|_0\le q(s)$, $\forall s\ge t_0$, $\forall i\in\{1,2\}$, we obtain from (3.13) and (3.14)

$$\begin{aligned}\|(Aw_1+Bw_2)(t)\|_0 &\le \lambda\|z_0\|_0\,\alpha(t_0)\mathrm{e}^{\int_{t_0}^t\left[-\widetilde{f}(u)+\frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u}\\
&\quad+\int_{t_0}^{t}\mathrm{e}^{\int_s^t\left[-\widetilde{f}(u)+\frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u}\left[\frac{\left|\dot{f}_1(s)+f_1^2(s)\right|}{\sqrt{\beta_0}}\right.\\
&\quad+\frac{\left|\dot{f}_2(s)+f_2^2(s)\right|}{\sqrt{\delta_0}}+f_3(s)+f_4(s)+\frac{f_1(s)\,f_4(s)}{\sqrt{\beta_0}}\\
&\quad\left.+\frac{f_2(s)\,f_3(s)}{\sqrt{\delta_0}}+\frac{M_1r_1(s)}{\sqrt{\beta_0}}+\frac{M_2r_2(s)}{\sqrt{\delta_0}}\right]q(s)\,\mathrm{d}s\\
&=:\sigma(t),\ \forall t\ge t_0.\end{aligned}$$

Easy computations show that $\sigma(t)$ fulfills the initial value problem (3.12) and so $\sigma(t)=q(t)$, $\forall t\ge t_0$. Hence $\|(Aw_1+Bw_2)(t)\|_0\le q(t)$, $\forall t\ge t_0$.

Thus, for all $t\ge t_0$

$$\begin{aligned}\|(Aw_1+Bw_2)(t)\|_0 &\le \lambda\|z_0\|_0\,\alpha(t_0)\mathrm{e}^{-\int_{t_0}^t\widetilde{f}(s)\mathrm{d}s}\mathrm{e}^{\int_{t_0}^t\nu(s)\mathrm{d}s}\\
&\quad\times\mathrm{e}^{\int_{t_0}^t\left[\frac{\left|\dot{f}_1(s)+f_1^2(s)\right|}{\sqrt{\beta_0}}+\frac{\left|\dot{f}_2(s)+f_2^2(s)\right|}{\sqrt{\delta_0}}\right]\mathrm{d}s}.\end{aligned}\tag{3.15}$$

Heaving in mind the hypothesis (H2), we distinguish two cases.

Case 1: $t_0\in[0,h)$.

Since $f_i\in C^1([0,h];\mathbb{R}_+)$, $f_j, r_i, \beta, \delta, \gamma\in C([0,h];\mathbb{R}_+)$, $\forall i\in\{1,2\}$, $\forall j\in\{3,4\}$, from (3.15) we derive that there exists a constant $D>0$, such that

$$\|(Aw_1+Bw_2)(t)\|_0\le\lambda\|z_0\|_0\,\alpha(t_0)\mathrm{e}^{Dh},\ \forall t\in[t_0,h].$$

By (3.15) and hypothesis (H2) we get for all $t\ge h$

$$\begin{aligned}\|(Aw_1+Bw_2)(t)\|_0 &\le \lambda\|z_0\|_0\,\alpha(t_0)\mathrm{e}^{Dh}\mathrm{e}^{-K\int_h^t\widetilde{f}(s)\mathrm{d}s}\mathrm{e}^{\int_h^t\nu(s)\mathrm{d}s}\\
&=:\Pi_1(t),\end{aligned}$$

where $K:=1-\frac{K_1}{\sqrt{\beta_0}}-\frac{K_2}{\sqrt{\delta_0}}\in(0,1]$.

Let

$$a := \rho \mathrm{e}^{-Dh} \mathrm{e}^{-\int_h^{+\infty} \nu(s)\mathrm{d}s} / \left(\lambda \sqrt{\beta(t_0) + \delta(t_0) + 2}\right).$$

From (2.1) we deduce that if $\|z_0\|_0 < a$, then

$$\|(Aw_1 + Bw_2)(t)\|_0 < \rho, \ \forall t \geq t_0.$$

Case 2: $t_0 \geq h$.

We obtain similarly for all $t \geq t_0$

$$\begin{aligned} \|(Aw_1 + Bw_2)(t)\|_0 &\leq \lambda \|z_0\|_0 \alpha(t_0) \mathrm{e}^{-K\int_{t_0}^t \widetilde{f}(s)\mathrm{d}s} \mathrm{e}^{\int_{t_0}^t \nu(s)\mathrm{d}s} \\ &= : \Pi_2(t) \end{aligned}$$

and, with the same a as in Case 1, $\|z_0\|_0 < a$ implies

$$\|(Aw_1 + Bw_2)(t)\|_0 < \rho, \ \forall t \geq t_0.$$

□

Taking into account Lemma 3.2, for proving the part i) of Theorem 2.2, it suffices to show that the system (3.2) admits solutions defined on $\mathbb{R}_+$ for initial data small enough. We will do this using the following generalized variant of the Krasnoselskii fixed point theorem, which can be found, e.g., in [8, Theorem 3.1].

Theorem 3.3. *Let X be a Hausdorff locally convex topological vector space and $\mathcal{P}$ a family of seminorms which generates the topology of X. Let D be a convex and complete subset of X and let A, B be operators on D into X such that $Ax + By \in D$ for every pair x, $y \in D$. Suppose A is a $\mu-$contraction for every $\mu \in \mathcal{P}$, B is continuous, and $B(D)$ is contained in a compact set. Then there is a point $\overline{x}$ in D such that $A\overline{x} + B\overline{x} = \overline{x}$.*

We recall that A is *$\mu-$contraction* iff there is $L_\mu \in [0, 1)$ such that for all x, $y \in D$,

$$\mu(Ax - Ay) \leq L_\mu \mu(x - y).$$

Let $t_0 \geq 0$ and $\rho \in (0, l)$, where l is given by Lemma 3.2.

We set $X = C_c(t_0)$. Let A, B be given by (3.9), (3.10), and $D = S(t_0, \rho)$.

Step 1. Let $n \in \mathbb{N}$, $n > t_0$ be arbitrary. We prove that A is $n-$contraction. Let w_1, $w_2 \in S(t_0, \rho)$ be given and $t \in [t_0, n]$. Then

$$\begin{aligned} \|(Aw_1 - Aw_2)(t)\|_0 &= \left\| \int_{t_0}^t Z(t, t_0) Z^{-1}(s, t_0) V(s) [w_1(s) - w_2(s)] \mathrm{d}s \right\|_0 \\ &\leq \int_{t_0}^t \mathrm{e}^{\int_s^t \left[-\widetilde{f}(u) + \frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u} \left[\frac{\left|\dot{f}_1(s) + f_1^2(s)\right|}{\sqrt{\beta_0}} \right. \\ &\quad + \frac{\left|\dot{f}_2(s) + f_2^2(s)\right|}{\sqrt{\delta_0}} + \frac{f_2(s) f_3(s)}{\sqrt{\delta_0}} + \frac{f_1(s) f_4(s)}{\sqrt{\beta_0}} \\ &\quad \left. + f_3(s) + f_4(s) \right] \|w_1(s) - w_2(s)\|_0 \mathrm{d}s. \end{aligned}$$

Since f_i, γ, ζ, $\left|\dot{f}_i + f_i^2\right|$, f_j are bounded on $[t_0, n]$, $\forall i \in \{1,2\}$, $\forall j \in \{3,4\}$, there is a constant $c_n > 0$, such that

$$\begin{aligned}\|(Aw_1 - Aw_2)(t)\|_0 &\le c_n \int_{t_0}^{t} \|w_1(s) - w_2(s)\|_0 \, \mathrm{e}^{-\lambda_n(s-t_0)} \mathrm{e}^{\lambda_n(s-t_0)} \mathrm{d}s \\ &\le c_n \|w_1 - w_2\|_{\lambda_n} \int_{t_0}^{t} \mathrm{e}^{\lambda_n(s-t_0)} \mathrm{d}s \\ &< c_n \|w_1 - w_2\|_{\lambda_n} \frac{\mathrm{e}^{\lambda_n(t-t_0)}}{\lambda_n}.\end{aligned}$$

Hence

$$\|(Aw_1 - Aw_2)(t)\|_0 \, \mathrm{e}^{-\lambda_n(t-t_0)} < \frac{c_n}{\lambda_n} \|w_1 - w_2\|_{\lambda_n}, \ \forall t \in [t_0, n],$$

and so

$$\|Aw_1 - Aw_2\|_{\lambda_n} \le \frac{c_n}{\lambda_n} \|w_1 - w_2\|_{\lambda_n}.$$

By taking $\lambda_n > c_n$, it follows that A is an $n-$contraction.

Step 2. We are going to show that B is continuous. Let w_m, $w \in S(t_0, \rho)$ be such that $w_m \to w$ in $C_c(t_0)$.

Consider $n \in \mathbb{N}$, $n > t_0$. Using (3.6) we have for every $t \in [t_0, n]$

$$\begin{aligned}\|(Bw_m - Bw)(t)\|_0 \le \ & \Lambda \int_{t_0}^{t} \mathrm{e}^{\int_s^t \left[-\widetilde{f}(u) + \frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right] \mathrm{d}u} \|F(s, w_m(s)) \\ & -F(s, w(s))\|_0 \, \mathrm{d}s.\end{aligned}$$

Therefore there exists a constant $d_n > 0$, such that

$$\|(Bw_m - Bw)(t)\|_0 \le d_n \int_{t_0}^{n} \|F(s, w_m(s)) - F(s, w(s))\|_0 \, \mathrm{d}s.$$

Since $F(t,z)$ is uniformly continuous for $t \in [t_0, n]$ and $\|z\|_0 \le \rho$, it follows that the sequence $F(t, w_m(t))$ converges uniformly on $[t_0, n]$ to $F(t, w(t))$. Hence $Bw_m \to Bw$ in $C_c(t_0)$, which proves the continuity of H.

Step 3. Finally, we prove that $B(S(t_0, \rho))$ is relatively compact. For this aim, we need to show that for each $n \in \mathbb{N}$, $n > t_0$, the set $\{(Bw)(t)\,|_{t \in [t_0, n]}, \, w \in S(t_0, \rho)\}$ is uniformly bounded and equicontinuous.

Let $n \in \mathbb{N}$, $n > t_0$ be fixed. As at Step 2, we have for every $w \in S(t_0, \rho)$ and $t \in [t_0, n]$,

$$\|(Bw)(t)\|_0 \le d_n \int_{t_0}^{n} \|F(s, w(s))\|_0 \, \mathrm{d}s. \tag{3.16}$$

Since $F(t, w)$ is bounded for $t \in [t_0, n]$, $\|w\|_0 \le \rho$, from (3.16) it follows that there is a positive constant p_n, such that $\|(Bw)(t)\|_0 \le p_n$, $\forall w \in S(t_0, \rho)$, $\forall t \in [t_0, n]$. Hence the set $\left\{(Bw)(t)\,|_{t \in [t_0, n]}, \ w \in S(t_0, \rho)\right\}$ is uniformly bounded in $C_c(t_0)$.

Let $w \in S(t_0, \rho)$ be arbitrary and let $z = Bw$. By differentiating (3.10) with respect to $t \in [t_0, n]$, we obtain

$$\dot{z}(t) = U(t)\, z(t) + F(t, w(t)), \ \forall t \in [t_0, n].$$

Since the functions f_i, γ_i, $\forall i \in \{1,2\}$, β, δ are bounded on $[t_0, n]$ and $F(t,w)$ is bounded for $t \in [t_0, n]$, $\|w\|_0 \le \rho$, it follows that there are positive constants φ_n, ψ_n, such that

$$\|\dot{z}(t)\|_0 \le \varphi_n p_n + \psi_n, \ \forall t \in [t_0, n].$$

So, the family of the derivatives of the functions from $B(S(t_0,\rho))$ is uniformly bounded and we infer that $B(S(t_0,\rho))$ is equicontinuous on the compact subsets of $[t_0, +\infty)$.

By applying Theorem 3.3, it follows that $A+B$ admits fixed points in $S(t_0,\rho)$. Hence a solution $z(t)$ with initial data small enough exists on the whole $\mathbb{R}_+$.

As in the proof of Lemma 3.2, if $t_0 \in [0,h)$ we have

$$q(t) \le \Pi_1(t), \ \forall t \ge h,$$

and if $t_0 \ge h$,

$$q(t) \le \Pi_2(t), \ \forall t \ge t_0.$$

Using the hypotheses (H1), (H3)-(H6), in both cases we deduce

$$\lim_{t\to+\infty} q(t) = 0.$$

Applying Lemma 3.2, it follows $\lim_{t\to+\infty} \|z(t)\|_0 = 0$ and so

$$\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} \dot{x}(t) = \lim_{t\to+\infty} y(t) = \lim_{t\to+\infty} \dot{y}(t) = 0.$$

ii) If g_1, g_2 are locally Lipschitzian with respect to x, y, then the solution exists on the whole $\mathbb{R}_+$ for small initial data and is unique. So we can proceed with the stability question for the null solution of the system (1.2), which, due to the boundedness of the functions f_i, β, δ, γ_i, $\left|\dot{f}_i + f_i^2\right|$, f_j, g_i, $i \in \{1,2\}$, $j \in \{3,4\}$, reduces to the stability of the null solution $z(t) = 0$ of (3.2).

By virtue of i), for proving the asymptotic stability, we need to prove that the null solution of (3.2) is stable.

Let $\varepsilon > 0$ and $\rho \in (0,l)$ be fixed. Consider $t_0 \ge 0$ and $z_0 \in \mathbb{R}^4 \setminus \{0\}$, with $\|z_0\|_0 < a$, where l and a are given by Lemma 3.1. If $z(t,t_0,z_0)$ is the solution of (3.2) which equals z_0 for $t = t_0$, then we have for all $t \ge t_0$

$$\begin{aligned} z(t,t_0,z_0) &= Z(t,t_0) z_0 + \int_{t_0}^{t} Z(t,t_0) Z^{-1}(s,t_0) [V(s) z(s,t_0,z_0) \\ &\quad + F(s, z(s,t_0.z_0)) \, \mathrm{d}s] \end{aligned}$$

and, from i), $\|z(t,t_0,z_0)\|_0 \le \rho$ and $\|z(t,t_0,z_0)\|_0 \le q(t)$, $\forall t \ge t_0$.

We distinguish again two cases.

Case 1: $t_0 \in [0,h)$.

We deduce, as in the proof of Lemma 3.2,

$$\|z(t,t_0,z_0)\|_0 \le \lambda \|z_0\|_0 \alpha(t_0) \mathrm{e}^{Dh}, \ \forall t \in [t_0,h],$$

and

$$\|z(t,t_0,z_0)\|_0 \le \Pi_1(t), \ \forall t \ge h. \tag{3.17}$$

Let

$$\eta = \eta(t_0,\varepsilon) := \varepsilon \mathrm{e}^{-Dh} \mathrm{e}^{-\int_{t_0}^{+\infty} \nu(s) \mathrm{d}s} / (\lambda \alpha(t_0)).$$

Then we can show that if $\|z_0\|_0 < \min\{\eta, a\}$, then $\|z(t, t_0, z_0)\|_0 < \varepsilon$, $\forall t \geq t_0$. From the boundedness of the functions f_i, β, δ, γ_i, $\left|\dot{f}_i + f_i^2\right|$, f_j, g_i, $i \in \{1,2\}$, $j \in \{3,4\}$, we get that $\|\dot{z}(t, t_0, z_0)\|_0$ is also small.

Case 2: $t_0 \geq h$.

We have

$$\|z(t, t_0, z_0)\|_0 \leq \Pi_2(t), \ \forall t \geq t_0. \tag{3.18}$$

With the same η as before, $\|z_0\|_0 < \min\{\eta, a\}$ implies $\|z(t, t_0, z_0)\|_0 < \varepsilon$, $\forall t \geq t_0$. Since $\|\dot{z}(t, t_0, z_0)\|_0$ is also small, it follows that the null solution of (3.2) is stable.

iii) The uniform stability of the null solution of (3.2) can be deduced in the same manner as for the stability, if we consider

$$a := \rho \mathrm{e}^{-Dh} \mathrm{e}^{-\int_h^{+\infty} \nu(s)\mathrm{d}s} / (\lambda\alpha(0)),$$
$$\eta = \eta(\varepsilon) := \varepsilon \mathrm{e}^{-Dh} \mathrm{e}^{-\int_0^{+\infty} \nu(s)\mathrm{d}s} / (\lambda\alpha(0)).$$

iv) We know from iii) that the null solution of (1.2) is uniformly stable. It remains to prove that there exists $\xi > 0$, such that for every $\varepsilon > 0$ there exists $T = T(\varepsilon) > 0$, such that $\|z_0\|_0 < \xi$ implies $\|z(t, t_0, z_0)\|_0 < \varepsilon$, for all $t_0 \geq 0$ and $t \geq t_0 + T$.

Indeed, if (H7) also holds, then $\int_{t_0}^t \widetilde{f}(s)\,\mathrm{d}s \geq p(t - t_0)$, $\forall t \geq t_0 \geq 0$. Let

$$\xi := \frac{1}{\lambda\alpha(0)\mathrm{e}^{Dh} N},$$

where $N := \mathrm{e}^{\int_0^{+\infty} \nu(s)\mathrm{d}s}$, $\varepsilon > 0$ be given, and

$$T = T(\varepsilon) := \begin{cases} h + \frac{1}{Kp} \ln \frac{1}{\varepsilon}, & \text{if } \varepsilon < 1, \\ C, & \text{if } \varepsilon \geq 1, \end{cases}$$

with $C > h$ arbitrary. Let $z_0 \in \mathbb{R}^4$, $z_0 \neq 0$, with $\|z_0\|_0 < \xi$ and $t_0 \geq 0$.

Corresponding to the position of t_0 about h from the hypothesis (H2), we have again two cases.

Case 1: $t_0 \in [0, h)$.

Let $t \geq t_0 + T$ be given. Then $t > t_0 + h \geq h$ and by (3.17) we deduce

$$\begin{aligned} \|z(t, t_0, z_0)\|_0 &\leq \lambda \|z_0\|_0 \alpha(t_0) \mathrm{e}^{Dh} \mathrm{e}^{-K\int_h^t \widetilde{f}(s)\mathrm{d}s} \mathrm{e}^{\int_h^t \nu(s)\mathrm{d}s} \\ &< \mathrm{e}^{-K\int_h^t \widetilde{f}(s)\mathrm{d}s} \leq \mathrm{e}^{-K\int_{t_0+h}^t \widetilde{f}(s)\mathrm{d}s} \\ &\leq \mathrm{e}^{-pK(t-t_0-h)}. \end{aligned} \tag{3.19}$$

From the definition of T we easily derive $\mathrm{e}^{-pK(t-t_0-h)} \leq \varepsilon$ and, by (3.19) we obtain

$$\|z(t, t_0, z_0)\|_0 < \varepsilon,$$

for all $t \geq t_0 + T$.

Case 2: $t_0 \geq h$.

Let $t \geq t_0 + T$ be arbitrary. So $t \geq t_0$ and from (3.18) we get

$$\|z(t, t_0, z_0)\|_0 \leq \lambda \|z_0\|_0 \alpha(t_0) \mathrm{e}^{-K\int_{t_0}^t \widetilde{f}(s)\mathrm{d}s} \mathrm{e}^{\int_{t_0}^t \nu(s)\mathrm{d}s} \tag{3.20}$$

$$< \quad \mathrm{e}^{-K\int_{t_0}^{t}\widetilde{f}(s)\mathrm{d}s} \leq \mathrm{e}^{-Kp(t-t_0)}.$$

Using the definition of T, we similarly infer $\mathrm{e}^{-Kp(t-t_0)} \leq \varepsilon$ and, from (3.20) it follows that

$$\|z\left(t,t_0,z_0\right)\|_0 < \varepsilon,$$

for all $t \geq t_0 + T$. Therefore the null solution of (1.2) is uniformly asymptotically stable.

The proof of Theorem 2.2 is now complete. $\square$

Example 3.4. Here are some examples of functions $f_i,\ f_j, \beta,\ \delta,\ \gamma_i,\ g_i,\ i \in \{1,2\}$, $j \in \{3,4\}$,

$$f_1(t) = \frac{1}{2t+\sqrt{t^2+2}},\ f_2(t) = \frac{1}{t+\sqrt{t^2+1}},\ \forall t \geq 0,$$

$$f_3\left(t\right) = \frac{1}{\left(t+1\right)^4},\ f_4\left(t\right) = \frac{2}{\left(t+1\right)^3},\ \forall t \geq 0,$$

$$\beta(t) = \frac{2t+3}{t+1},\ \delta(t) = \frac{2t^3+5}{t^3+2},\ \gamma_1\left(t\right) = \frac{1}{t\sqrt{t^2+1}+1},\ \gamma_2\left(t\right) = \mathrm{e}^{-\frac{t}{2}},\ \forall t \geq 0,$$

$$g_1(t,x,y) = \mathrm{e}^{-\frac{t^2}{2}}x^3,\ g_2\left(t,x,y\right) = \frac{3}{t^2\sqrt{t+1}}y^4,\ \forall t \geq 0,\ \forall x,\ y \in \mathbb{R}.$$

These functions satisfy the hypotheses (H1)-(H7), with $\beta_0 = 2$, $\delta_0 = 2$, $K_1 = 1/\sqrt{2}$, $K_2 = \left(2+\sqrt{3}\right) \times \left(3-2\sqrt{2}\right)$, $h = 1$, $r_1\left(t\right) = \mathrm{e}^{-\frac{t^2}{2}}$, $r_2\left(t\right) = \frac{3}{t^2\sqrt{t+1}}$, $\forall t \geq 0$. In Figure 3.1 the solution of (1.2) and its derivative are plotted in the case of two time intervals, for the initial data $z_0 = [0.01, 0.01, 0.01, 0.01]$. The plottings of the solution in the planes $(x,\dot{x})$, $(y,\dot{y})$ are given in Figure 3.2.

Example 3.5. If in Example 3.4 we replace only $f_1,\ f_2$ by $f_1(t) = \frac{1}{10} + \frac{1}{t+1}$, respectively $f_2(t) = \frac{1}{5} + \frac{2}{t+1}$, $\forall t \geq 0$, then the hypotheses (H1), (H2), (H4)-(H8) are verified with $K_1 = 1/5$, $K_2 = 4/5$, $h = 7$, $p = \frac{1}{10}$, and the same β_0, δ_0, $r_1\left(t\right), r_2\left(t\right)$), and we obtain the solution of (1.2) and its derivative plotted in Figure 3.3 with the same time intervals and for the same initial data. The plottings of the solution in the planes $(x,\dot{x})$, $(y,\dot{y})$ are given in Figure 3.4.

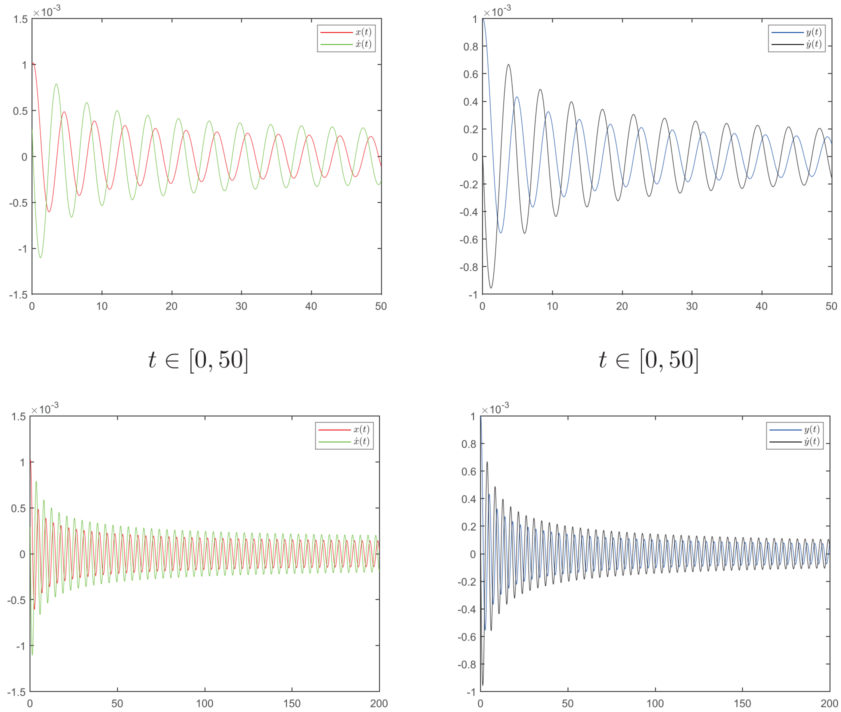
Remark 3.6. Note that the null solution of the system (1.2) can be uniformly stable, but not asymptotically stable. Indeed, this can be seen by considering the following functions

$$f_1(t) = \frac{1}{\left(t+1\right)^2},\ f_2(t) = 2f_1\left(t\right),\ f_3\left(t\right) = \frac{\left|\sin t^2\right|}{t+2},\ f_4\left(t\right) = \frac{\mathrm{e}^{-t^2}}{t+1},\ \forall t \geq 0,$$

$$\beta(t) = 0.04 + \frac{1}{t^2+1},\ \delta(t) = 0.2 + \frac{1}{\sqrt{t^2+2}},\ \forall t \geq 0,$$

$$\gamma_1\left(t\right) = \frac{t}{t+2}\mathrm{e}^{-t^2},\ \gamma_2\left(t\right) = \frac{3\left|\cos t\right|}{\left(t+1\right)^2},\ \forall t \geq 0,$$

$$g_1(t,x,y) = \frac{3x^3}{\left(t^2+2\right)^2},\ g_2\left(t,x,y\right) = \frac{2y^2}{\left(t+1\right)^3},\ \forall t \geq 0,\ \forall x,y \in \mathbb{R}.$$



$t \in [0, 50]$ $t \in [0, 50]$

$t \in [0, 200]$ $t \in [0, 200]$

FIGURE 3.1. The solution of (1.2) and its derivative.

These functions satisfy the hypotheses (H1), (H2), (H4)-(H7), with $K_1 = 1/10$, $K_2 = 1/5$, $h = 20$, $\beta_0 = 0.04$, $\delta_0 = 0.2$, $r_1(t) = \frac{3}{(t^2+2)^2}$, $r_2(t) = \frac{2}{(t+1)^3}$, $\forall t \geq 0$. For the initial data $z_0 = [0.0001, 0.0001, 0.0001, 0.0001]$, the solution of (1.2) and its derivative are plotted in Figure 3.5 on some time intervals. The plottings of the solution in the planes $(x, \dot{x})$, $(y, \dot{y})$ are given in Figure 3.6.

All the graphs in the figures above were obtained using the `Matlab` programming platform.

Remark 3.7. If the block of mass m_1 is subject to the action of a time dependent external force $\widehat{f} : \mathbb{R}_+ \to \mathbb{R}$, then the inhomogeneous system of ODEs describing the dynamics of the mechanical system is

$$\left\{ \begin{array}{l} \ddot{x} + 2f_1(t)\dot{x} - f_3(t)\dot{y} + \beta(t)x - \gamma_1(t)y - f(t) + g_1(t, x, y) = 0, \\ \ddot{y} + 2f_2(t)\dot{y} - f_4(t)\dot{x} - \gamma_2(t)x + \delta(t)y + g_2(t, x, y) = 0, \end{array} \right. \tag{3.21}$$

with the same functions as before and $f(t) := \frac{1}{m_1}\widehat{f}(t)$, $\forall t \in \mathbb{R}_+$. Using the same proof techimques, based on Theorem 3.3, we can derive in this case qualitative properties of the solutions of (3.21) with initial data small enough, similar to those from [26, Theorem 3.1].

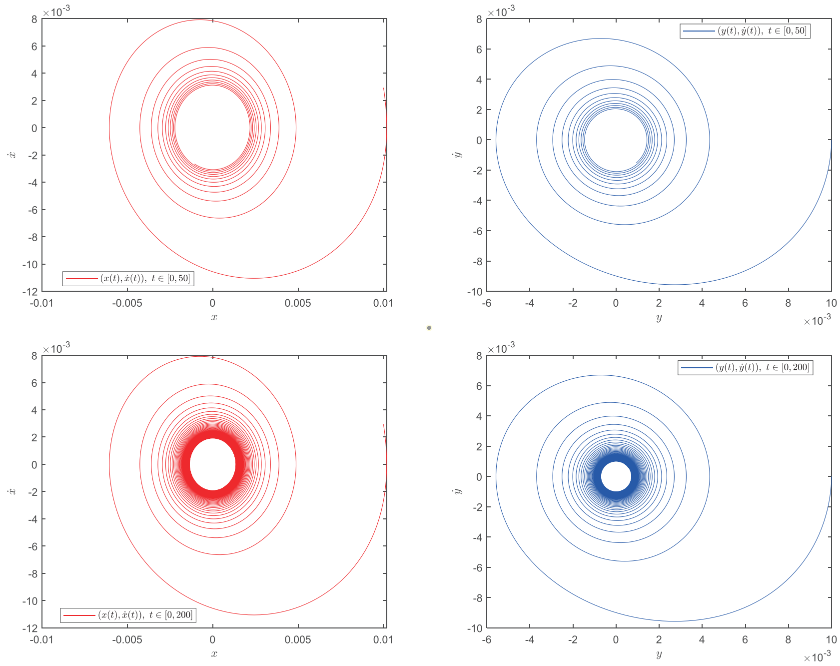
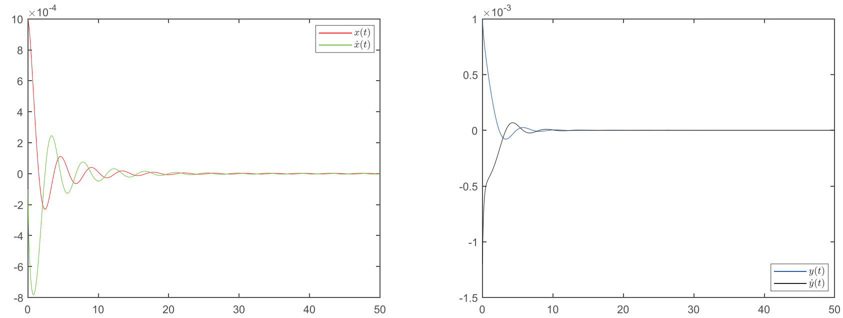
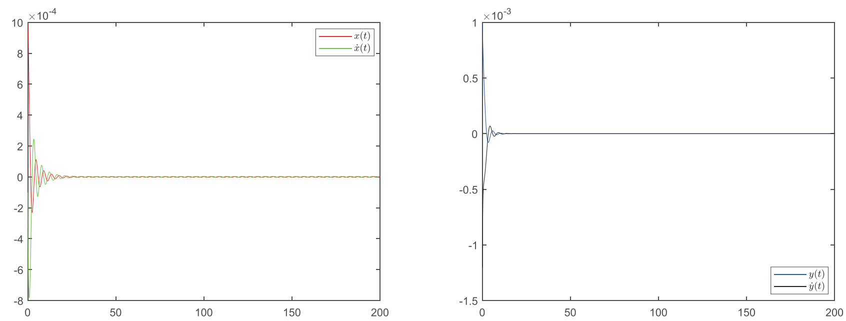


FIGURE 3.2. The solution of (1.2) in the planes $(x, \dot{x})$, $(y, \dot{y})$.



$t \in [0, 50]$ $t \in [0, 50]$



$t \in [0, 200]$ $t \in [0, 200]$

FIGURE 3.3. The solution of (1.2) and its derivative.

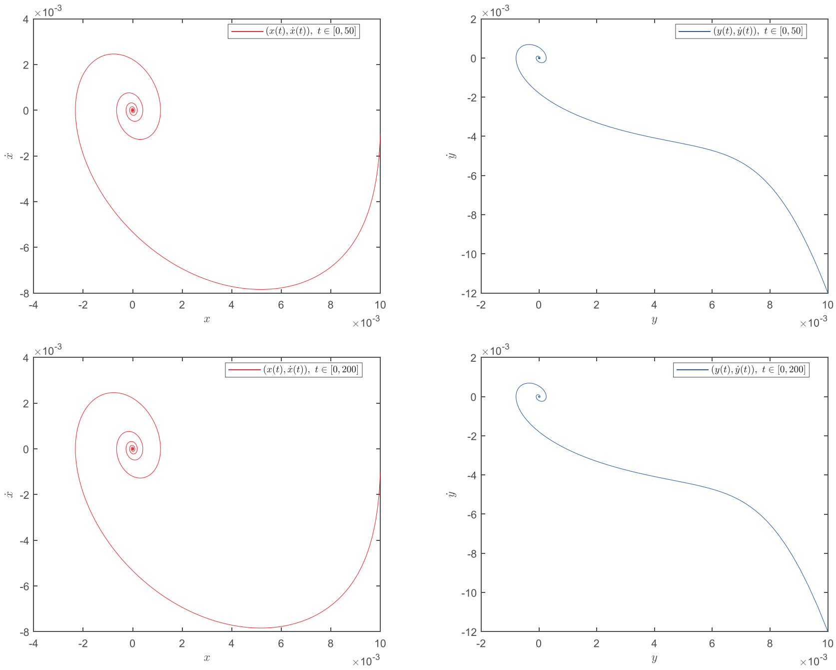
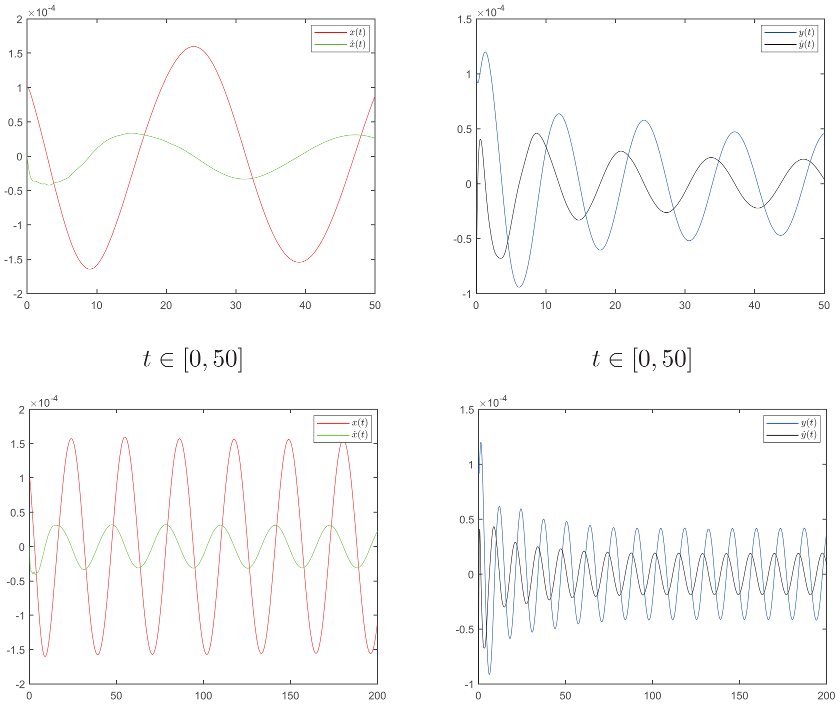


FIGURE 3.4. The solution of (1.2) in the planes $(x, \dot{x})$, $(y, \dot{y})$.



$t \in [0, 50]$ $t \in [0, 50]$

$t \in [0, 200]$ $t \in [0, 200]$

FIGURE 3.5. The solution of (1.2) and its derivative.

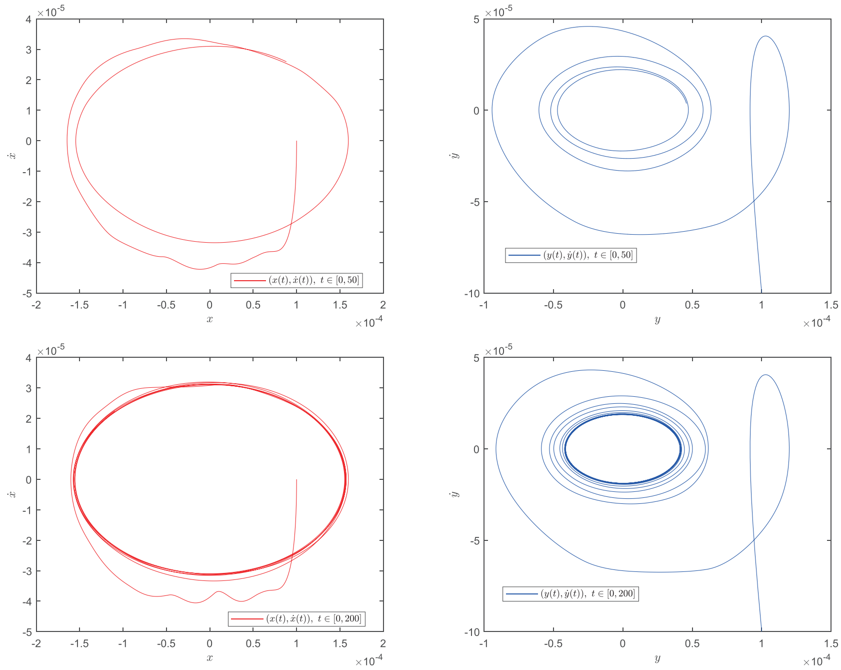


FIGURE 3.6. The solution of (1.2) in the planes $(x,\dot{x})$, $(y,\dot{y})$.

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Manuscript received September 5 2022
revised December 31 2022

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