Applied Analysis and Optimization
Volume 7, Number 1, 2023, 79–97

Yokohama Publishers
ISSN 2189-1664 Online Journal

A FRACTIONAL STUDY OF MHD CASSON FLUID MOTION WITH THERMAL RADIATIVE FLUX AND HEAT INJECTION/SUCTION MECHANISM UNDER RAMPED WALL CONDITION: APPLICATION OF ROBOTNOV EXPONENTIAL KERNEL

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Abstract. The aim of this present research work is to generalize the idea of fractionalized Casson fluid flow. Exact analysis of MHD natural convective flow of the Casson fluid to derive analytical solutions with the non-integer order derivative Yang-Abdel-Cattani (YAC) is reported here. This fractional operator has more generalized exponential kernel. The fluid flow is elaborated near an infinitely vertical plate with characteristics velocity u_0. The modeling of the considered problem is done in terms of the partial differential equations with Newtonian heating and ramped conditions. Introduced the appropriate set of variables to transform the governing equations into dimensionless form. Laplace transform(LT) is operated on the fractional system of equations and results are presented in series form and also presented the solution in the form of special functions. The pertinent parameter's influence such as α, β, Pr, Q, Gr, M, Nr, K on the fluid flow is brought under consideration to reveal the interesting results. In comparison, we noticed the YAC approach shows better results than the existing operators in the literature, and graphs are drawn to show the results. Also, obtained the results in a limiting sense such as Casson and viscous fluid models for classical form from Yang-Abdel-Cattani fractionalized Casson fluid model.

1. Introduction

Owing to the wide range of applications in the modern world, non-Newtonian fluids have to be a great importance for researchers and scientists for last few decades because it effectively used in various fields. The Navier-Stokes equation can not characterize the mechanical features of the most valuable non-Newtonian liquid owing to the complex nature of the non-Newtonian fluids. So, the single constitutive equation that is not able to describe its rheological behavior. The rheological behavior of non-Newtonian fluid makes them important for many industrial and technological applications for instance in petroleum, biological, plastic manufacturing, chemical, textile, and cosmetic industries. There are several models namely viscoplastic model, second grade fluid model, Williamson fluid, Bingham plastic model, power law model, Jeffery model, Brinkman type fluid, Oldroyd-B model, Maxwell model, Walters-B fluid model, tangent hyperbolic fluid and Casson model (shear thinning liquid) accorded to explain the resourceful nature of the non-Newtonian fluids [9, 12, 15, 25, 28, 29]. Among them, Casson fluid is gotten to

2020 *Mathematics Subject Classification.* 26A33, 35K05, 80A19.
Key words and phrases. YAC derivative, porous medium, Caputo derivative, analytical solution, heat transfer, Laplace transformation, Robotnov exponential functions, physical parameters .

be the simplest and generalization of the Newtonian fluid. There exist another one most common type of such fluid that is known as Casson fluid, and this type of non-Newtonian fluid was initially introduced by Casson in1959, with an ambition to determine the regime of pigment-oil suspentions [6]. From the Casson fluid model, the model for viscous fluid can be determine by oversight the impacts of their generalized parameters. Being mindful of its important properties, features and abilities, the wide application of Casson fluid is observed in biological science such as plasma, handling of biological fluid, blood, etc, and in mechanics due to its viscoelastic behavior. According to the prevailing modern scientific challanges, several mathematicians, researchers, scientists and engineers pay extraordinary attention to biology, engineering, chemistry, petroleum industries and physiology as compared to Newtonian fluids, they specially focus to study the Casson fluid due to its natural behaviour. Khalid et al. [11] studied MHD unsteady free convectional transport ofCasson model with computational aspects in porous media. Bhatta charyya et al. [5] described the magnetohydro dynamic flow of Casson fluid velocity in presence of exponentially stretching surface. Oka [19] examined the Casson fluid movement first time along with the convective conditions at the boundary, through permeable stretching sheet and analyzed the results theoretically. The impacts of heat generation on the MHD Maxwell fluid in a permeable medium carried by Riaz et al. [30]. Mernone et al. [14] examined the two-dimentional peristaltic Cassonfluid flow in a channel. Arthur et al. [3] investigated the generalized peristaltic Casson fluid flow in a permeable channel subject to chemical reaction effects. Mukhopadhyay [16] examined the heat transfer phenomenon of MHD cason fluid heat suction/blowing that passed over the stretching plate. Mustafa et al. [18] explained for unsteady flow of Casson model by considering the homotopy analysis to analyzed the heat transfer over movable flatplate. Similar studies on MHD Casson fluid recorded in literature [23, 27, 32] and references therein.

The versatile and valuable impacts of fractional calculus in the field of electrical engineering, electro chemistry, control theory, electromagnetism, mechanics, image processing, bio-engineering, physics, finance, fluid dynamics, and many others make it a valuable tool for study. Fractional derivatives not only keep the record of the present but also the past, so they are very suitable and accurate when the system has long-term memory. It has several applications in physical science as well as in other areas such as biology, astrophysics, ecology, geology and chemistry. The mechanism of non-Newtonian models is elaborated successfully with the fractional calculus in the past decades due to its simple and elegant description of the complexity of its behavior. One of the important feature and most commonly known name of non-Newtonianfluid is viscoelastic fluid that which exhibit the behavior of elasticity and viscosity. Such types of fluid models have great implications in various fields namely polymerization, industrial as well as mechanical engineering and also in the field of automobile industry due to its significance. Fractional calculus is very helpful in the interpretation of the viscoelastic nature of the materials. Taking into account the enormous mentioned properties, many researchers paid attention to analyse the fractional behaviour of different fluid models directly or indirectly in case of derivatives when it is considered as non-integer order from. Bagley and Torvik [4] noted the fractional calculus application on the viscoelastic fluids. Rehman et al. [22] studied the fractional Maxwell fluid and explored the closed solution of shear stress and velocity. M.B. Riaz [31] analyzed the influence of the MHD on the heat transfer of fractionalized Oldroyd-B fluid. Some features of Maxwell fluid along with observing the impact of Newtonian Heating and developed the fractional model using Prabhakar fractional approach is explored by Rehman et al. [20]. The ABC, CF and CPC comparative analysis of second-grade fluid under Newtonian heating effect, found the series solution, performed by Rehman et al. [26]. Saqib et al. [33] carried a study to analysed the natural movement of generalized convective Jeffrey fluid by employing the definition of CF fractional time derivative. Shehzad et al. [34] illustrated the problem that is considered three-dimensionally Jeffery fluid movement along with observing the impact of Newtonian Heating. Further, Hayat et al. [7] attained the series solutions of the problem related the flow of Jeffery fluid. Some of the contributions of the fractional calculus on the viscoelastic fluids are highlighted in [8, 21, 24].

In the most recent exploration, Talha Anwar et al. et al. [2] examined the classical version of Casson fluid model along with ramped boundary conditions by employing the method of Laplace transformation which is an efficiently applicable for boundary conditions that is non-uniform, but not considering the fractional behavior effect for the presented model. But it is noticed that the Casson fluid model with this innovative fractional operator namely Yang-Abdel-Cattani operator having non-local and singular kernel are not investigated together with ramped boundry conditions for valocity and energy distribution through the porous media, and not available in the previous literature related to fluid mechanics of fractional models. Inspired by the above literature, this article is devoted to studying the heat transfer analysis of the MHD fractional Casson fluid in achannel with ramped conditions. We have converted the integer-order derivative Casson fluid model with the non-integer order derivative YAC model. Laplace transform have been employed to get the analytical solutions of the current problem. The analytical expression for velocity and temperature are evaluated in a series form. Such exact solutions have never been noted in the literature before. Hence this article makes valuable contributions to the existing literature in view of pan-city of exact solutions ofCasson fluid with suitable boundary conditions.The influence of embedded parameters namely YAC fractional parameter α, porosity parameter K, Casson fluid parameter β, prandl parameter Pr, magnetic number M, heat injection/suction parameter Q, grashof number Gr and the radiation parameter Nr, on the velocity profile and heat distribution, are captured with the assistance of graphs.

2. Mathematical Model

The phenomenon of the heattransfer of the convective MHD Casson fluid flow overan infinite plate hanged vertically, is examined here. We have the coordinate axis in such a way that the plate is fixed in the direction of x-axis and ϕ-axis perpendicular the plate (as shown in Fig. 1). Initially, fortime $t = 0$ and $\phi = 0$ the fluid is not moving with ambient temprature T_∞. It is assumed that the fluid, fortime $t = 0^+$, ramped condition are taken for velocity, the wall temprature is T_w and $u(\phi, t)$ is supposed to be a velocity component that is taking alongx-axis while u_0 is considered as the charactaristic velocity and the fluid is flowing that

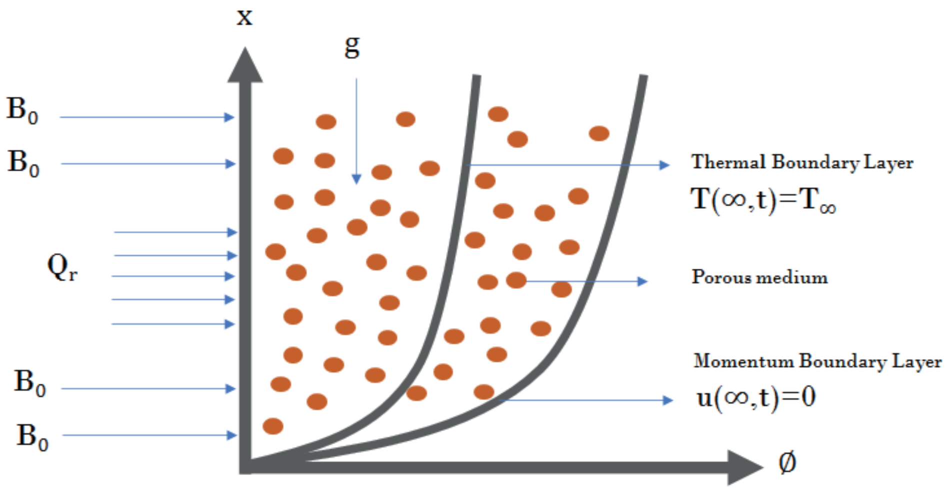

FIGURE 1. Schematic drawing of the flow model

is restrained to $\phi > 0$. Further, to govern the model assumptions are considered as a transverse magnetic force is introduced vertically to the fluid flow and the movement of the fluid is uni-directional. By neglecting the impacts of an induced magnatic field in which Reynolds parametre/number is supposed to be too small, viscous dissipation omit in energy equation, Joule heating, Q_r (radiative heat flux) and further assumes that fluid velocity in this present problem is the function of only two parameteres namely ϕ and t, the governing equation for the fluid flow description using the Boussinesq's approximation [10, 17] for velocity and energy equations will take the form:

The momentum and energy equations are given below:

$$\frac{\partial u(\phi,t)}{\partial t} = \upsilon\Big(1+\frac{1}{\beta}\Big)\frac{\partial^2 u(\phi,t)}{\partial \phi^2} + g\beta_T\Big(T(\phi,t)-T_\infty\Big) - \frac{\sigma}{\rho}{B_0}^2 u(\phi,t) - \upsilon\Big(1+\frac{1}{\beta}\Big)\frac{\zeta}{k_p}u(\phi,t), \tag{2.1}$$

$$\rho C_p\frac{\partial T(\phi,t)}{\partial t} = -\frac{\partial q(\phi,t)}{\partial \phi} - \frac{\partial Q_r}{\partial \phi} + Q_0\left(T(\phi,t)-T_\infty\right). \tag{2.2}$$
$$\left[Q_r = -\frac{4\sigma_1}{3k_1}\frac{\partial T^4}{\partial \phi}; T^4 \approx 4T_\infty^3 T - 3T_\infty^4\right]$$

The Fourier's Law of thermal flux are written as:

$$q(\phi,t) = -k\frac{\partial T(\phi,t)}{\partial \phi}. \tag{2.3}$$

with associated initial conditions together with ramped boundary conditions are displayed mathematically as:

$$u(\phi,0)=0, \quad T(\phi,0)=T_\infty, \quad \phi \geq 0,$$

$$u(0,t) = \begin{cases} u_0 \frac{t}{t_0}, & 0 < t \leq t_0; \\ u_0, & t > t_0 \end{cases},$$

$$T(0,t) = \begin{cases} T_\infty + (T_w - T_\infty)\frac{t}{t_0}, & 0 < t \leq t_0; \\ T_w, & t > t_0 \end{cases},$$

$$u(\phi,t) \to 0, \quad T(\phi,t) \to T_\infty, \quad as \quad \phi \to \infty \quad and \quad t > 0. \tag{2.4}$$

Introducing the new quantities used here to non-dimentionalize the equations:

$$t^* = \frac{u_0^2 t}{\upsilon}, \quad \phi^* = \frac{u_0 \phi}{\upsilon}, \quad u^* = \frac{u}{u_0}, \quad q^* = \frac{q}{q_0},$$

$$T^* = \frac{T - T_\infty}{T_w - T_\infty}, \quad q_0 = \frac{k(T_w - T_\infty)u_0}{\upsilon}. \tag{2.5}$$

when using the newly introduced entities as defined in the Eq. (2.5) into Eq. (2.1) and Eq.(2.2), and after that $*$ symbol ignoring, then finally the equations transformed in the following form:

$$\frac{\partial u(\phi,t)}{\partial t} = b\frac{\partial^2 u(\phi,t)}{\partial \phi^2} - \left[M + \frac{b}{K}\right] u(\phi,t) + GrT(\phi,t), \tag{2.6}$$

$$\frac{\partial T(\phi,t)}{\partial t} = -\left(\frac{1+Nr}{Pr}\right)\frac{\partial q(\phi,t)}{\partial \phi} + QT, \tag{2.7}$$

$$q(\phi,t) = -\frac{\partial T(\phi,t)}{\partial \phi}, \tag{2.8}$$

Along with the set of non-dimensional from of initial and boundry conditions becomes

$$u(\phi,0) = 0, \quad T(\phi,0) = 0, \quad for \quad \phi \geq 0, \tag{2.9}$$

$$u(0,t) = T(0,t) = \begin{cases} t & 0 < t \leq 1 \\ 1 & t > 1 \end{cases}, \tag{2.10}$$

$$u(\phi,t) \to 0, \quad T(\phi,t) \to 0 \quad as \quad \phi \to \infty \quad and \quad t > 0. \tag{2.11}$$

where

$$Q = \frac{Q_0 \upsilon}{\rho C_p u_0^2}, \quad Gr = \frac{g\beta_T(T_w - T_\infty)}{u_0^3},$$

$$Pr = \frac{\mu C_p}{k}, \quad M = \frac{\sigma B_0^2 \upsilon}{\rho u_0^2}, \quad N_r = \frac{16\sigma_1 T_\infty^3}{3kK_1},$$

$$\theta = \frac{P_r}{1+N_r}, \quad \frac{1}{K} = \frac{\upsilon^2 \zeta}{k_p u_0^2}, \quad a = b\theta Q + c,$$

$$b = 1 + \frac{1}{\beta}, \quad c = M + \frac{b}{K}, \quad d = 1 - b\theta.$$

here, Gr represents grash of number, Pr is denoted by Prandtl number, Nr is radiation parameter, M represents magnetic number, kp is permeability, k is thermal conductivity, k_1 is the coefficient of Rossland absorption, σ_1 Stefan-Boltzman constant, ζ is porosity, Qr is radioactive heat flux and K is defined as porosity.

3. Preliminaries

$$ {}^{YAC}D_t^{\alpha} f(t) = \int_0^t \Psi_\alpha(-\wp(t-\tau)^{\alpha}) f'(\tau) d\tau, \quad for \quad t>0, \quad 0<\alpha<1. \tag{3.1} $$

where

$$ \Psi(\wp z^{\alpha}) = \sum_{n=0}^{\infty} \frac{\wp^n z^{(n+1)(\alpha+1)-1}}{\Gamma(n+1)(\alpha+1)}, \quad z \in \mathbb{C}. $$

and Ψ_α is denoted the Robotnov exponentialfunction of order α.

Laplace transformation of this newly developed operator is defined as:

$$ \{{}^{YAC}D_t^{\alpha} f(t)\} = \frac{1}{\digamma^{\alpha+1}} \frac{\digamma\{f(t)\} - f(0)}{1+\wp \digamma^{-(\alpha+1)}}. \tag{3.2} $$

where $\digamma$ represnts Laplace transform parameter and α used as a fractional parameter.

4. Fractional formulation of governing equations and solutions

Replace the time deriavtive with the YAC fractional derivative into Eq. (2.6)-(2.8), then the time-fractional rate type fluid model for velocity and energy are written as:

$$ {}^{YAC}D_t^{\alpha} u(\phi,t) = b\frac{\partial^2 u(\phi,t)}{\partial \phi^2} - cu(\phi,t) + GrT(\phi,t), \tag{4.1} $$

$$ {}^{YAC}D_t^{\alpha} T(\phi,t) = \frac{1}{\theta}\frac{\partial^2 T(\phi,t)}{\partial \phi^2} + QT(\phi,t), \tag{4.2} $$

where ${}^{YAC}D_t^{\alpha}$ represents YAC fractional operator, for further properties about the Yang-Abdel-Cattani operator are discussed in [35].

4.1. **Investigation of exact solution for temprature profile.** Implement the Laplace transformation on Eq.(4.2) with transformed conditions as mentioned in Eqs. (2.9) − (2.11), we will get

$$ \frac{\digamma \bar{T}(\phi,\digamma) - \bar{T}(\phi,0)}{\digamma^{\alpha+1}+\wp} = \frac{1}{\theta}\frac{\partial^2 \bar{T}(\phi,\digamma)}{\partial \phi^2} + Q\bar{T}(\phi,\digamma). \tag{4.3} $$

with transformed boundary conditions

$$ \bar{T}(\phi,0)=0, \quad \bar{T}(0,\digamma) = \frac{1-e^{-\digamma}}{\digamma^2} \quad and \quad \bar{T}(\phi,\digamma)\to 0 \quad as \quad \phi\to\infty. \tag{4.4} $$

Employing the Laplace transformation of Eq.(4.3), then the energy solution is obtained as:

$$ \bar{T}(\phi,\digamma) = e_1 e^{-\phi\sqrt{\theta\left(\frac{\digamma}{\digamma^{\alpha+1}+\wp}-Q\right)}} + e_2 e^{\phi\sqrt{\theta\left(\frac{\digamma}{\digamma^{\alpha+1}+\wp}-Q\right)}}. \tag{4.5} $$

Applying the transformed boundary conditions, then the energy solution is written as:

$$
\begin{aligned}
\bar{T}(\phi,\digamma) &= \left(\frac{1-e^{-\digamma}}{\digamma^2}\right) e^{-\phi\sqrt{\theta\left(\frac{\digamma}{\digamma^{\alpha+1}+\wp}-Q\right)}}, \\
&= \bar{T}_1(\phi,\digamma) - e^{-\digamma}\bar{T}_1(\phi,\digamma). \qquad (4.6)
\end{aligned}
$$

To transform the solution in time variable again, we have to employ inverse Laplace transformation technique on Eq. (4.6)

$$
T(\phi,t) = T_1(\phi,t) - T_1(\phi,t)P(t-1). \qquad (4.7)
$$

In the above expression $P(t-1)$ represents a Heaviside function.
where

$$
T_1(\phi,t) = \left\{\bar{T}_1(\phi,\digamma)\right\} = \left\{\frac{1}{\digamma^2} e^{-\phi\sqrt{\theta\left(\frac{\digamma}{\digamma^{\alpha+1}+\wp}-Q\right)}}\right\}, \qquad (4.8)
$$

To compute Laplace inverse of Eq.(4.8), it seems to be difficult in the present form, we have to convert it in the series form, after that it takes the form:

$$
\begin{aligned}
T_1(\phi,t) &= \left\{\frac{1}{\digamma^2}\sum_{\chi=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-\phi)^{\chi}(\theta)^{\frac{\chi}{2}}(-Q)^{\frac{\chi}{2}-n}\Gamma(\frac{\chi}{2}+1)(\digamma)^n}{\chi!n!\Gamma(\frac{\chi}{2}-n+1)\left(\digamma^{\alpha+1}+\wp\right)^n}\right\} \\
&= \sum_{\chi=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-\phi)^{\chi}(\theta)^{\frac{\chi}{2}}(-Q)^{\frac{\chi}{2}-n}\Gamma(\frac{\chi}{2}+1)}{\chi!n!\Gamma(\frac{\chi}{2}-n+1)} t^{n\alpha+1} E^{n}_{\alpha+1,n\alpha+2}(-\wp t^{\alpha+1})
\end{aligned}
$$

by using $\left\{\frac{\digamma^{\alpha\gamma-\beta}}{(\digamma^{\alpha}-\wp)^{\gamma}}\right\} = t^{\beta-1}E^{\gamma}_{\alpha,\beta}(\wp t^{\alpha})$

4.2. **Investigation of exact solution for fluid velocity.** Applying the Laplace transformation into Eq. (4.1) with appropriate transformed conditions as defined in Eqs. (2.9) $-$ (2.11), we get

$$
\frac{\digamma\bar{u}(\phi,\digamma)-\bar{u}(\phi,0)}{\digamma^{\alpha+1}+\wp} = b\frac{d^2\bar{u}(\phi,\digamma)}{d\phi^2} - c\bar{u}(\phi,\digamma) + Gr\bar{T}(\phi,\digamma). \qquad (4.9)
$$

with conditions are:

$$
\bar{u}(\phi,0)=0, \quad \bar{u}(0,\digamma) = \frac{1-e^{-\digamma}}{\digamma^2} \quad \textit{and} \quad \bar{u}(\phi,\digamma)\to 0 \quad \textit{as} \quad \phi\to\infty. \qquad (4.10)
$$

Taking the computed temperature $\bar{T}(\varsigma,\digamma)$ from Eq. (4.6) and replacing in Eq.(4.9), then the solution after simplification are given in the form

$$
\begin{aligned}
\bar{u}(\phi,\digamma) =& e_5 e^{-\phi\sqrt{\frac{1}{b}\left(\frac{\digamma}{\digamma^{\alpha+1}+\wp}+c\right)}} + e_6 e^{\phi\sqrt{\frac{1}{b}\left(\frac{\digamma}{\digamma^{\alpha+1}+\wp}+c\right)}} \\
&+ Gr\left(\frac{1-e^{-\digamma}}{\digamma^2}\right)\frac{e^{-\phi\sqrt{\theta\left(\frac{\digamma}{\digamma^{\alpha+1}+\wp}-Q\right)}}}{a+\frac{d\digamma}{\digamma^{\alpha+1}+\wp}}. \qquad (4.11)
\end{aligned}
$$

With the help of the transformed boundary conditions, determine the unknown constant, the obtained velocity solution for Eq.(4.11) is given as

$$\bar{u}(\phi,\digamma)=\left(\frac{1-e^{-\digamma}}{\digamma^{2}}\right)e^{-\phi\sqrt{\frac{1}{b}\left(\frac{\digamma}{\digamma^{\alpha+1}+\wp}+c\right)}} + \frac{Gr\,(1-e^{-\digamma})}{\digamma^{2}\left(a+\frac{d\digamma}{\digamma^{\alpha+1}+\wp}\right)}\left[e^{-\phi\sqrt{\theta\left(\frac{\digamma}{\digamma^{\alpha+1}+\wp}-Q\right)}}-e^{-\phi\sqrt{\frac{1}{b}\left(\frac{\digamma}{\digamma^{\alpha+1}+\wp}+c\right)}}\right]. \tag{4.12}$$

To find Laplace inverse of the above Eq. (4.12), first we write it in the following form:

$$\bar{u}(\phi,\digamma)=\bar{\Omega}(\phi,\digamma)+Gr\bar{\Phi}(\phi,\digamma)\left[\bar{T}(\phi,\digamma)-\bar{\Omega}(\phi,\digamma)\right]. \tag{4.13}$$

and

$$\bar{\Omega}(\phi,\digamma)=\bar{\Omega}_1(\phi,\digamma)-e^{-\digamma}\bar{\Omega}_1(\phi,\digamma). \tag{4.14}$$

The inverse Laplace of the above Eq. (4.14), is obtained as:

$$\Omega(\phi,t)=\Omega_1(\phi,t)-\Omega_1(\phi,t)P(t-1). \tag{4.15}$$

where

$$\begin{aligned}
\Omega_1(\phi,t)&=\left\{\bar{\Omega}_1(\phi,\digamma)\right\}=\left\{\frac{1}{\digamma^{2}}e^{-\phi\sqrt{\frac{1}{b}\left(\frac{\digamma}{\digamma^{\alpha+1}+\wp}+c\right)}}\right\},\\
&=\left\{\frac{1}{\digamma^{2}}\sum_{\chi=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-\phi)^{\chi}(c)^{\frac{\chi}{2}-n}\Gamma(\frac{\chi}{2}+1)(\digamma)^{n}}{\chi!n!(b)^{\frac{\chi}{2}}\Gamma(\frac{\chi}{2}-n+1)\left(\digamma^{\alpha+1}+\wp\right)^{n}}\right\},\\
&=\sum_{\chi=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-\phi)^{\chi}(c)^{\frac{\chi}{2}-n}\Gamma(\frac{\chi}{2}+1)}{\chi!n!(b)^{\frac{\chi}{2}}\Gamma(\frac{\chi}{2}-n+1)}t^{n\alpha+1}E^{n}_{\alpha+1,n\alpha+2}(-\wp t^{\alpha+1})\\
\Phi(\phi,t)&=\left\{\bar{\Phi}(\phi,\digamma)\right\}=\left\{\frac{1}{a+\frac{d\digamma}{\digamma^{\alpha+1}+\wp}}\right\},\\
&=\left\{\sum_{m=0}^{\infty}\frac{(-1)^{m}(d)^{m}(\digamma)^{m}}{(a)^{m+1}\left(\digamma^{\alpha+1}+\wp\right)^{m}}\right\},\\
&=\sum_{m=0}^{\infty}\frac{(-1)^{m}(d)^{m}}{(a)^{m+1}}t^{m\alpha-1}E^{m}_{\alpha+1,m\alpha}(-\wp t^{\alpha+1})
\end{aligned}$$

The required velocity solution after employing the definition of inverse Laplace operator on the above Eq. (4.13) is

$$u(\phi,t)=\Omega(\phi,t)+Gr\Phi(\phi,t)*\left[T(\phi,t)-\Omega(\phi,t)\right]. \tag{4.16}$$

Limiting models. Some special cases are discussed here that derived from current problem to analyze the influence on solutions for different cases arises in theabsence of some physical parameters.

Solution in the absence of Casson parameter. In this case supposed that theCasson fluid parameter as denoted by β is taking as very large, i.e., $\frac{1}{\beta}\to 0$, after that the new transformed viscous fluid has velocity field solution derived from the already computed velocity Eq.(4.12) is transformed as:

$$\bar{u}(\phi, F) = \left(\frac{1-e^{-F}}{F^2}\right) e^{-\phi\sqrt{\left(\frac{F}{F^{\alpha+1}+\wp}+c_1\right)}} + \frac{Gr\,(1-e^{-F})}{F^2\left(a_1+\frac{d_1 F}{F^{\alpha+1}+\wp}\right)}\left[e^{-\phi\sqrt{\theta\left(\frac{F}{F^{\alpha+1}+\wp}-Q\right)}} - e^{-\phi\sqrt{\left(\frac{F}{F^{\alpha+1}+\wp}+c_1\right)}}\right]. \tag{4.17}$$

where $a_1 = \theta Q + c_1, \quad c_1 = M + \frac{1}{K}, \quad d_1 = 1-\theta$
Tofind Laplace inverse of the above Eq. (4.17), first we write it in the following form:

$$\bar{u}(\phi, F) = \bar{\omega}(\phi, F) + Gr\bar{\Psi}(\phi, F)\left[\bar{T}(\phi, F) - \bar{\omega}(\phi, F)\right]. \tag{4.18}$$

and

$$\bar{\omega}(\phi, F) = \bar{\omega_1}(\phi, F) - e^{-F}\bar{\omega_1}(\phi, F). \tag{4.19}$$

After the application of Laplace inverse operator, the above Eq. (4.19), is turn out again in the time variable as:

$$\omega(\phi, t) = \omega_1(\phi, t) - \omega_1(\phi, t)P(t-1). \tag{4.20}$$

where

$$\begin{aligned}
\omega_1(\phi, t) = \{\bar{\omega_1}(\phi, F)\} &= \left\{\frac{1}{F^2} e^{-\phi\sqrt{\left(\frac{F}{F^{\alpha+1}+\wp}+c_1\right)}}\right\}, \\
&= \left\{\frac{1}{F^2}\sum_{\chi=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-\phi)^{\chi}(c_1)^{\frac{\chi}{2}-n}\Gamma(\frac{\chi}{2}+1)(F)^n}{\chi! n!\Gamma(\frac{\chi}{2}-n+1)\left(F^{\alpha+1}+\wp\right)^n}\right\}, \\
&= \sum_{\chi=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-\phi)^{\chi}(c_1)^{\frac{\chi}{2}-n}\Gamma(\frac{\chi}{2}+1)}{\chi! n!\Gamma(\frac{\chi}{2}-n+1)} t^{n\alpha+1} E^n_{\alpha+1, n\alpha+2}(-\wp t^{\alpha+1}) \\
\Psi(\phi, t) = \left\{\bar{\Psi}(\phi, F)\right\} &= \left\{\frac{1}{a_1+\frac{d_1 F}{F^{\alpha+1}+\wp}}\right\}, \\
&= \left\{\sum_{m=0}^{\infty}\frac{(-1)^m(d_1)^m(F)^m}{(a_1)^{m+1}\left(F^{\alpha+1}+\wp\right)^m}\right\}, \\
&= \sum_{m=0}^{\infty}\frac{(-1)^m(d_1)^m}{(a_1)^{m+1}} t^{m\alpha-1} E^m_{\alpha+1, m\alpha}(-\wp t^{\alpha+1})
\end{aligned}$$

The inverse Laplace of the above Eq. (4.18), the required velocityfield solution, finally written as:

$$u(\phi, t) = \omega(\phi, t) + Gr\Psi(\phi, t) * \left[T(\phi, t) - \omega(\phi, t)\right]. \tag{4.21}$$

Solution in the absence of magnetic and porosity parameter. In this case supposed that $M = 0$ and $\frac{1}{K} = 0$ in the velocity Eq. (4.12) that reduced in the following form:

$$\bar{u}(\phi,\digamma)=\left(\frac{1-e^{-\digamma}}{\digamma^{2}}\right)e^{-\phi\sqrt{\frac{1}{b}\left(\frac{\digamma}{\digamma^{\alpha+1}+\wp}\right)}}$$

$$+\frac{Gr\,(1-e^{-\digamma})}{\digamma^{2}\left(b\theta Q+\frac{d\digamma}{\digamma^{\alpha+1}+\wp}\right)}\left[e^{-\phi\sqrt{\theta\left(\frac{\digamma}{\digamma^{\alpha+1}+\wp}-Q\right)}}-e^{-\phi\sqrt{\frac{1}{b}\left(\frac{\digamma}{\digamma^{\alpha+1}+\wp}\right)}}\right]. \tag{4.22}$$

Tofind Laplace inverse of the above Eq. (4.22), first we write it in the following form:

$$\bar{u}(\phi,\digamma)=\bar{\Upsilon}(\phi,\digamma)+Gr\bar{\varpi}(\phi,\digamma)\left[\bar{T}(\phi,\digamma)-\bar{\Upsilon}(\phi,\digamma)\right]. \tag{4.23}$$

and

$$\bar{\Upsilon}(\phi,\digamma)=\bar{\Upsilon}_{1}(\phi,\digamma)-e^{-\digamma}\bar{\Upsilon}_{1}(\phi,\digamma). \tag{4.24}$$

After the application of Laplace inverse operator, the above Eq. (4.24), is turn out again in the time variable as:

$$\Upsilon(\phi,t)=\Upsilon_{1}(\phi,t)-\Upsilon_{1}(\phi,t)P(t-1). \tag{4.25}$$

where

$$\begin{aligned}
\Upsilon_{1}(\phi,t)&=\left\{\bar{\Upsilon}_{1}(\phi,\digamma)\right\}=\left\{\frac{1}{\digamma^{2}}e^{-\phi\sqrt{\frac{1}{b}\left(\frac{\digamma}{\digamma^{\alpha+1}+\wp}\right)}}\right\},\\
&=\left\{\frac{1}{\digamma^{2}}\sum_{\chi=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-\phi)^{\chi}\Gamma(\frac{\chi}{2}+1)(\digamma)^{n}}{\chi!n!(b)^{\frac{\chi}{2}}\Gamma(\frac{\chi}{2}-n+1)\left(\digamma^{\alpha+1}+\wp\right)^{n}}\right\},\\
&=\sum_{\chi=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-\phi)^{\chi}\Gamma(\frac{\chi}{2}+1)}{\chi!n!(b)^{\frac{\chi}{2}}\Gamma(\frac{\chi}{2}-n+1)}t^{n\alpha+1}E^{n}_{\alpha+1,n\alpha+2}(-\wp t^{\alpha+1})\\
\varpi(\phi,t)&=\{\bar{\varpi}(\phi,\digamma)\}=\left\{\frac{1}{b\theta Q+\frac{d\digamma}{\digamma^{\alpha+1}+\wp}}\right\},\\
&=\left\{\sum_{m=0}^{\infty}\frac{(-1)^{m}(d)^{m}(\digamma)^{m}}{(b\theta Q)^{m+1}\left(\digamma^{\alpha+1}+\wp\right)^{m}}\right\},\\
&=\sum_{m=0}^{\infty}\frac{(-1)^{m}(d)^{m}}{(b\theta Q)^{m+1}}t^{m\alpha-1}E^{m}_{\alpha+1,m\alpha}(-\wp t^{\alpha+1})
\end{aligned}$$

The inverse Laplace of the above Eq. (4.23), the required velocityfield solution, finally written as:

$$u(\phi,t)=\Upsilon(\phi,t)+Gr\varpi(\phi,t)*\left[T(\phi,t)-\Upsilon(\phi,t)\right]. \tag{4.26}$$

5. Results and discussion

The heat transference analysis of MHD natural convective flow of the Casson fluid to derive analytical solutions via non-integer order derivative Yang-Abdel-Cattani (YAC) is elaborated here. The fluid flow is happened in the direction of ϕ-axis. The dimensionless system of equations representing the fluid flow phenomenon is solved by the integral LT. The obtained results are presented in a series form and also in the form of special functions. The graphical illustration is used to present the behaviour of embedded physical parameters such as memory parameter α, Pradtl number Pr, Casson parameter β, heat absorption parameter Q, thermalGrashof number Gr, magnetic parametr M, chemical reaction rate Nr and porosity paramter K on the Casson fluidvelocity and temprature distribution are portrayed graphically in Figs. 2–11 with the help of graphicall softwares.

Fig 2. manifests the memory parameter control on the temprature profile. The boundarylayer becomes thick with the increase in α which causes to decay in temprature. It is easy to valedate the result for $\alpha \to 1$.

The dominance of the mass diffusivity in fluid flow results in a reduction of the thermal boundary layer. The thermall boundry layer reduces, resulting in a decline in temperature. These are the effects of P_r which are elaborated in Fig. 3.

Fig. 4 illustrats the impact of Nr on temperature distribution of Casson fluid by assuming its various values. It is analysed from the graphs that energy profile elevated as enhancing the values of Nr. Physically, change of heat flux increases but k_1 reduces along the plate which is in normal direction, this impliesthat the more amount of heat radiation is absorbed to thefluid which cause to increase the temprature profile.

Fig. 5 exhibits the relationship between the amount of heat either sucked $(Q < 0)$ or injected $(Q > 0)$ and temperature. It is notedthat the energy profile increasing while rising the values of Q and it has much significance of heat suction/generation in cooling and heating processes. Next, Fig. 6 portray the behaviour of α on the fluid flow. Add up the value of the α result in descending velocity curves.

Fig. 7 elucidates the impact ofCasson fluid parameter β on thevelocity graphs for Casson fluid against ϕ, by choosing the distinct values of β for taking the values of various fluidic parameters.

Fig. 8 exhibits the impact of Prandlnumber P_r on Casson fluidvelocity against to ς, by taking distint valuesof P_r, at four different valuesof fractional paramter α. Enhancement in the distinct values of P_r, decay in the boundry layer of velocity noticed. To elaborate on the effects of G_r, Fig. 9 is plotted. Since Gr represents the fraction of buoyancyforce to viscousforce, as aresult, with an enhancing in Gr that cause to accelerate the fluid velocity, have appeared dueto boost in the value of Gr. Next, Fig. 10 elucidates the impact of permeability parameter K on thevelocity graphs for Maxwell fluid against ς, by choosing the distinct values of K for taking the values of α small andlarge. An increase in theporosity of medium cause to weak the resistive force and consequently, the flow regime enhances due to momentum development. It isdepicted that the elevation in the velocity profile with an increazing values of K under ramped conditions.

Fig. 11 interprets the impact of M on the momentum profile against ς, when assigned different values of M in the velocity expression to exemplify the physical behavior of Maxwell fluid velocity corresponding to distinct values of fractional paramter. It is established that the decline in both magnitude of boundarylayer thickness and velocity when the strong magnetic field applied. Subsequently, this explanation justifies the fluid gets slowed down corresponding to an increase in magnetic number because dragging forces cause to dominates theflow supporting forces. Eventually, decay in the velocity contour with an enhancement in valuesof magnetic number.

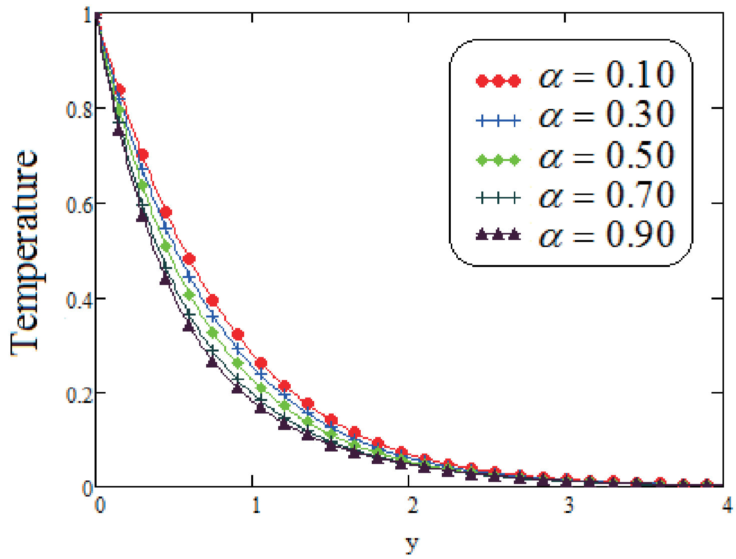

FIGURE 2. Influence of Casson fluid temperature and concentration against ϕ for multiple values of α.

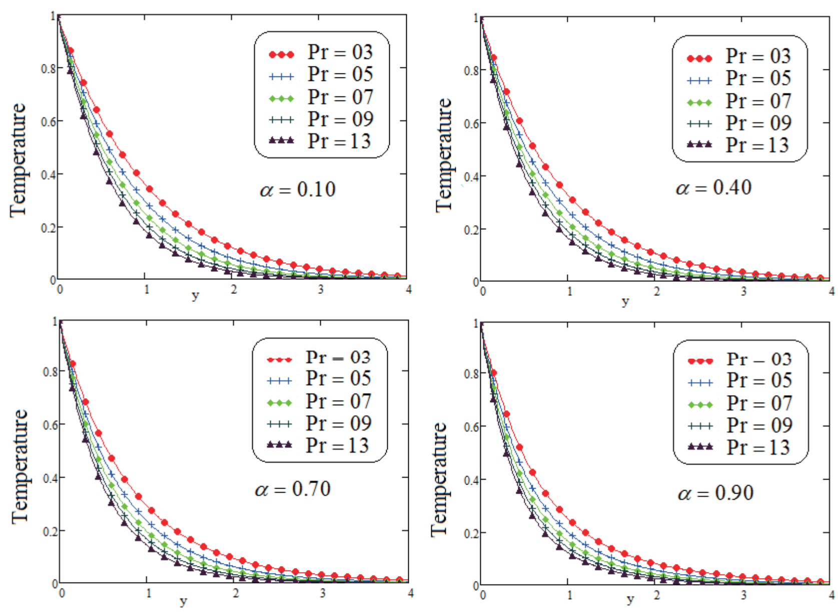

FIGURE 3. Representation of Casson fluid temperature against ϕ for multiple values of Pr.

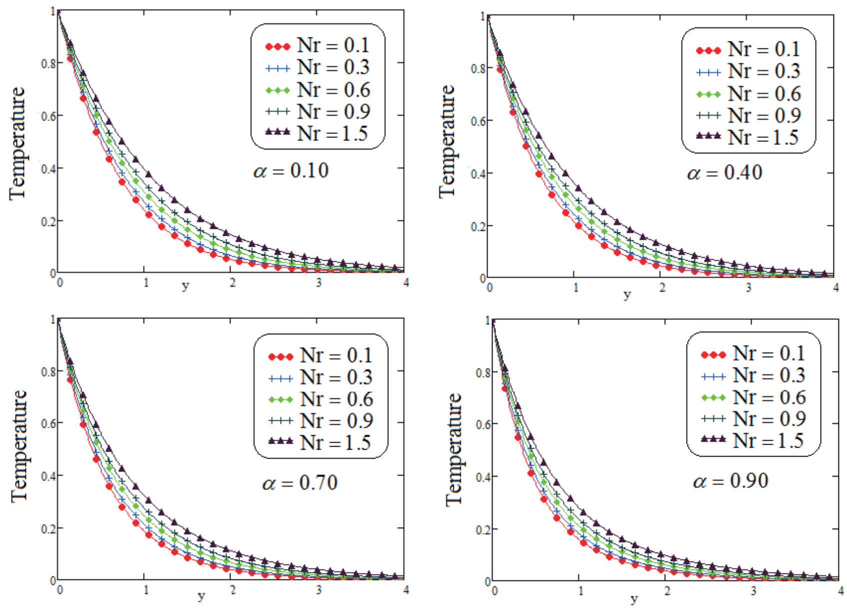

FIGURE 4. Representation of Casson fluid temperature against ϕ for distinct values of Nr.

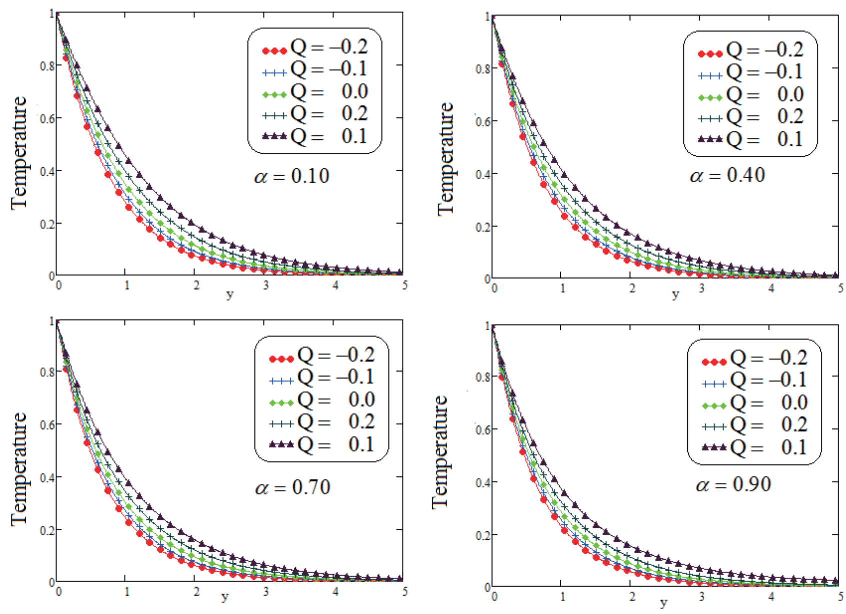

FIGURE 5. Representation of Casson fluid temperature against ϕ for distinct values of Q.

6. Conclusion

In this research article, MHD natural convective flow of the Casson fluid to derive analytical solutions with the non-integer order derivative Yang-Abdel-Cattani

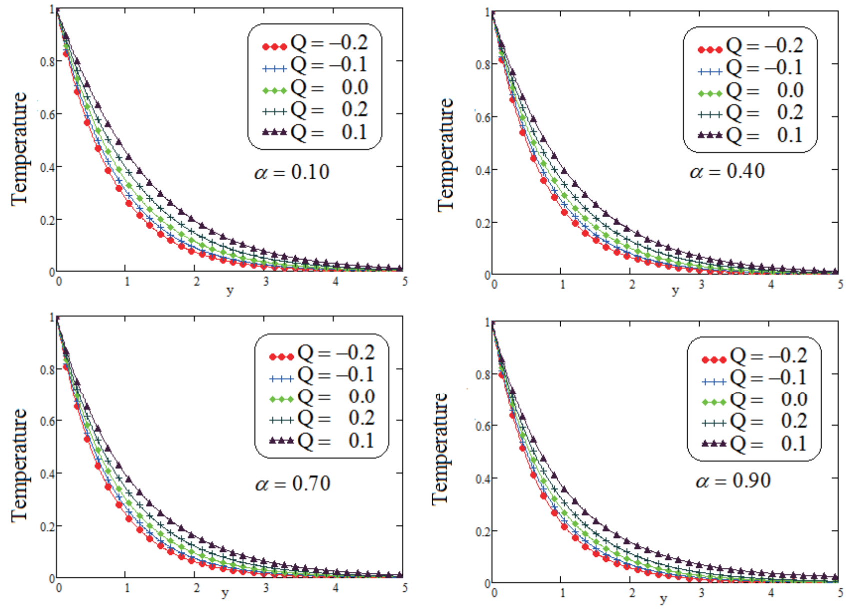

FIGURE 6. Representation of Casson fluid temperature against ϕ for distinct values of Q.

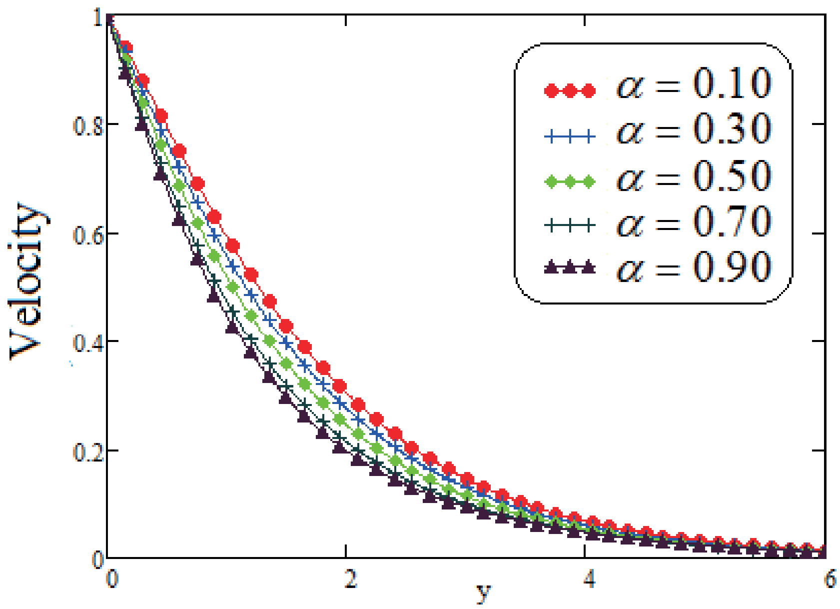

FIGURE 7. Velocity representation for multiple values of α

(YAC) is investigated. The fluid flow is elaborated near an infinitely verticalplate and Laplace transform(LT) is operated on the fractional system of equations and

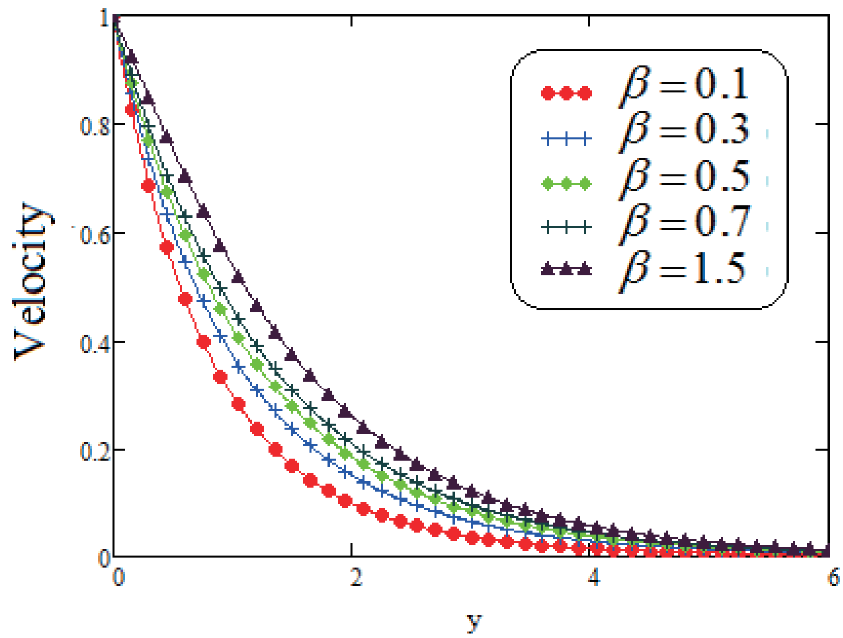

FIGURE 8. Velocity representation for multiple values of β

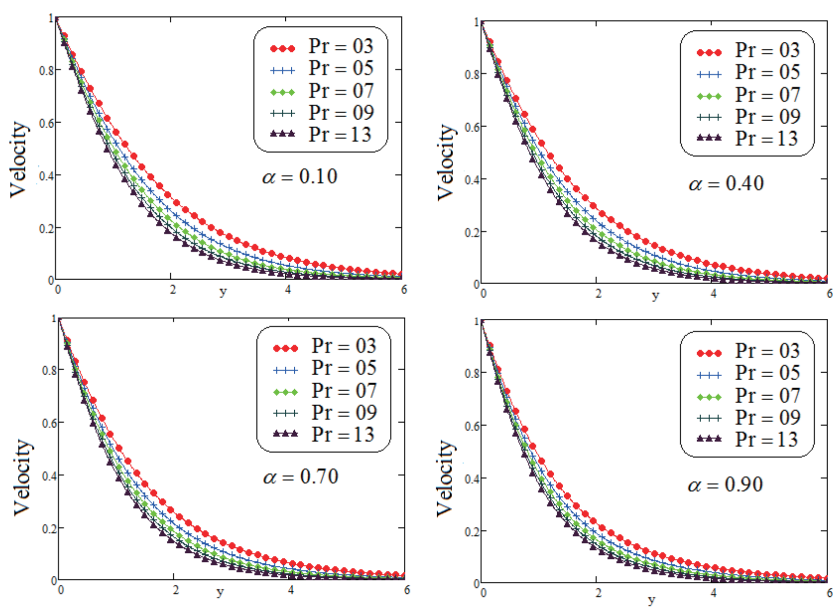

FIGURE 9. Velocity representation for multiple values of Pr.

results are presented in series form and also presented the solution in the form

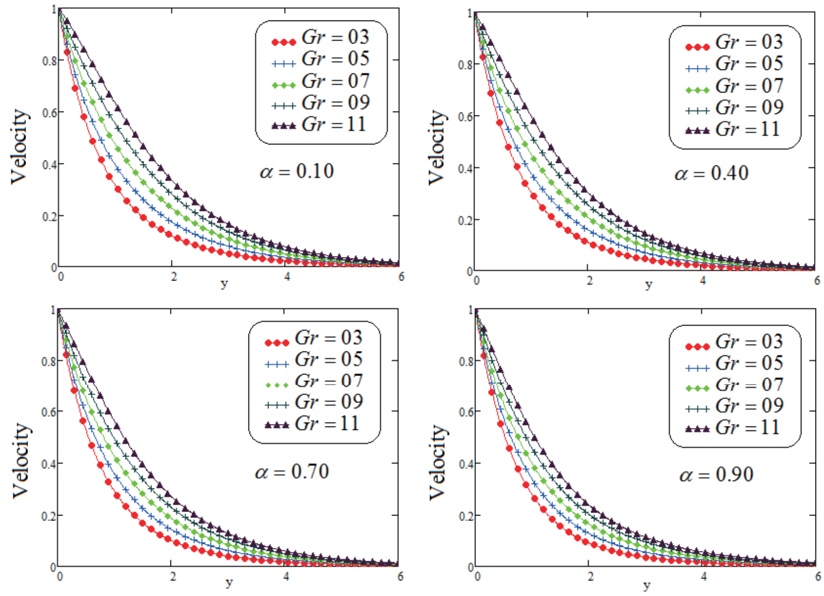

FIGURE 10. Velocity representation for multiple values of Gr.

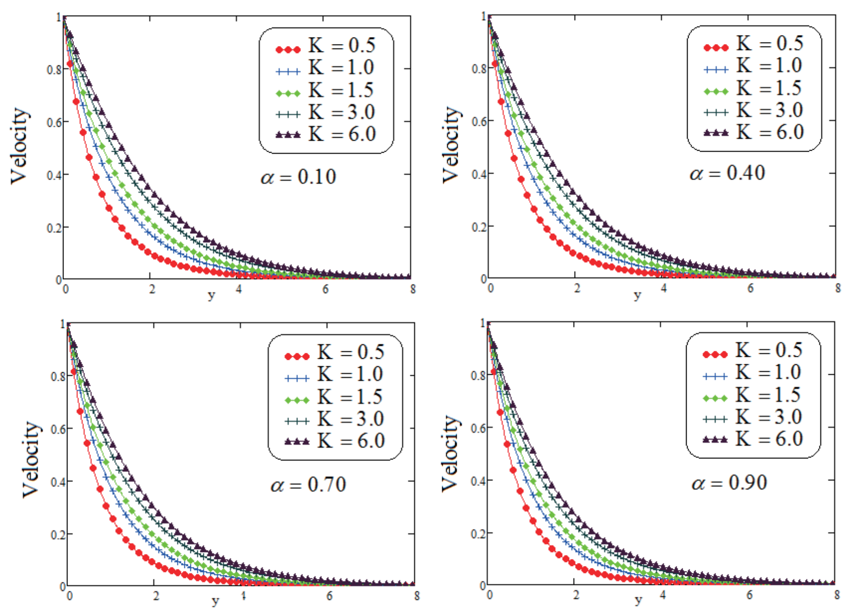

FIGURE 11. Velocity representation for multiple values of K.

of special functions. Some most essential key points noticed from the graphical behaviour are expressed as:

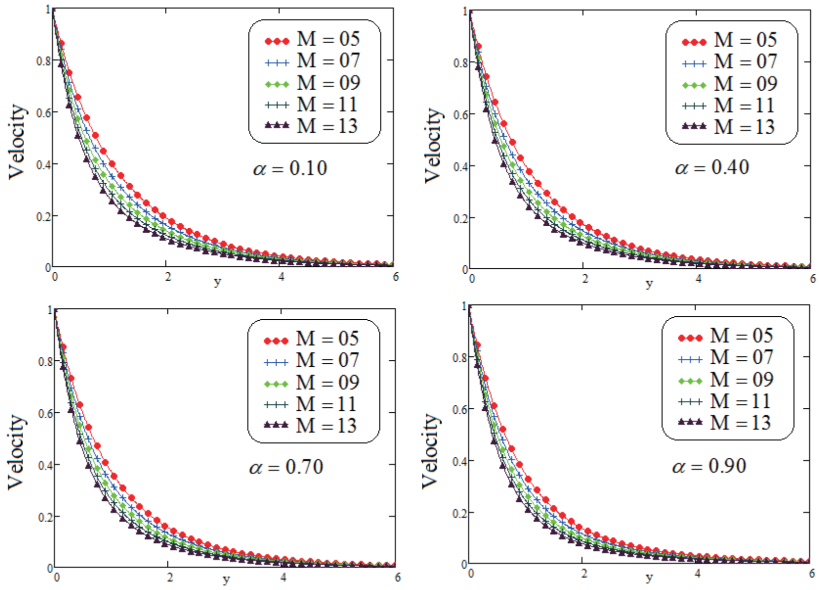

FIGURE 12. Velocity representation for multiple values of M.

- The velocity field and temprature decreases with rising values of memory parameter α.
- Temperature curves decay with the corresponding rise in P_r.
- The accumulative values of Nr and Q escalates the temperature graphs.
- Large values of Gr and K enhancing the Maxwell fluid velocity.
- The increasing variation of magnetic number M, decay in the velocity is observed.

In future work some new time fractional operators can be utilized on the same problem and compared with the previously computed results.

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Manuscript received November 1 2022
revised December 10 2022

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