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ASYMPTOTICAL STABILITY OF NONLINEAR FRACTIONAL NEUTRAL SYSTEMS WITH UNBOUNDED DELAY

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Abstract. In this paper, we investigate the asymptotical stability of zero solution of a nonlinear fractional neutral system (NFNS) with unbounded delay. We define two new Lyapunov-Krasovskii functionals (LKFs) and use some well-known inequalities to prove the results of this paper. By using MATLAB-Simulink sofware, we give two numerical examples to show applications our results.

1. Introduction

The concept of delay is widely encountered in many different systems and models such as biological systems, chemical engineering systems, software systems, economic systems, nuclear reactors, transportation systems, population dynamic models, financial systems and more. Hence, stability problems functional differential equations and systems have an importance place in the scientific world and they have been intensively studied in the literature (see, for example, [1-20] and the references therein).

Factional calculus has also a very old history that has existed since the day of the regular calculus came into existence. It should be noted that neutral delay systems are more complex and more general than other delayed systems.

We would now like to outline some papers on the stability of neutral and some other fractional differential systems.

By using LKFs, asymptotic robust stability of neutral type of fractional order systems have been discussed in [1, 2].

In [3], the authors investigate delay-dependent asymptotic stability of a differential and Riemann-Liouville fractional differential neutral system with constant delays and nonlinear perturbation.

In [7], asymptotic stability of linear and interval linear fractional-order neutral systems with time delay is discussed.

In [8], applying Lyapunov direct method, asymptotical stability of Riemann-Liouville fractional neutral systems is studied.

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In Liu et al. [9], applying Lyapunov direct method, certain sufficient conditions on asymptotical stability of nonlinear fractional systems without and with unbounded delays are given.

Motivated the works mentioned and that in the references of this paper, we consider an NFNS with unbounded delay:

$$\begin{aligned} {}_{t_0}D_t^q x(t) - A_{t_0}D_t^q x(t-\kappa(t)) &= Bx(t) + Cx(t-\kappa(t)) + H_1(t,x(t)) \\ &+ H_2(t,x(t-\kappa(t))) + H_3(t,{}_{t_0}D_t^q x(t-\kappa(t))), \end{aligned} \tag{1.1}$$

where $x(t) \in R^n, A, B, C \in R^{n\times n}$ are known real constant matrices with suitable dimensions, $H_j \in R^{n\times n}$ are continuous matrices functions and satisfy $H_j(t,0) = 0, j = 1,2,3$. The variable $\kappa(t) \geq 0$ is a differentiable variable delay and

$$\dot{\kappa}(t) \leq h_d < 1, \tag{1.2}$$

where h_d is positive constant. We also assume that the nonlinear terms $H_j(t,x)$ are the higher terms in (t,x), that is,

$$\lim_{||x||\to 0} \frac{||H_j(t,x)||}{||x||} = 0, j = 1,2,3. \tag{1.3}$$

Now, we give some basic definitions and lemmas before the main results and numerical examples.

Definition 1.1 ([10]). The Riemann-Liouville fractional integral and derivative are defined by

$$\begin{aligned} {}_{t_0}D_t^{-q} x(t) &= \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} x(s) ds, (q > 0), \\ {}_{t_0}D_t^{q} x(t) &= \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_{t_0}^t \frac{x(s)}{(t-s)^{q+1-n}} ds, (n-1 \leq q < n), \end{aligned}$$

respectively, where Γ is the Gamma function.

Property 1.2 ([6]). If $p > q > 0$, then the equality

$${}_{t_0}D_t^q({}_{t_0}D_t^{-p} x(t)) = {}_{t_0} D_t^{q-p} x(t).$$

holds for "sufficiently good" functions $x(t)$. In particular, this relation holds if $x(t)$ is integrable.

Lemma 1.3 ([9]). *Suppose $x(t) \in R^n$ is a differentiable vector. Then, the following inequality holds:*

$${}_{t_0}D_t^q (x^T(t) N x(t)) \leq 2x^T(t) N_{t_0} D_t^q x(t), \forall q \in (0,1), \forall t \geq t_0,$$

where $N \in R^{n\times n}, N = N^T \geq 0$ is a constant matrix.

Lemma 1.4 ([12]). *For any $x, y \in R^n, \alpha > 0$, the following inequality holds:*

$$2x^T y \leq \alpha x^T x + \alpha^{-1} y^T y.$$

Lemma 1.5 ([12]). *Let $S > 0$ and $M \geq 0$ are real symmetric matrices and ξ is a positive constant. Then ,*

$$\xi S > M \Leftrightarrow \lambda_{max}(MS^{-1}) < \xi \Leftrightarrow \lambda_{max}(S^{\frac{-1}{2}} M S^{\frac{-1}{2}}) < \xi.$$

2. Main results and numerical applications

We give some assumptions to prove the asymptotic stability of zero solution of NFNS (1.1) under consideration.

A. Assumptions

(A1) Let $P = P^T > 0, R = R^T > 0, Q = Q^T > 0$ and $Z = Z^T > 0$ are symmetric matrices with suitable dimensions such that

$$B^T P + PB + 2Q + [\alpha(1 - h_d) + 1]Z = 0, \tag{2.1}$$

$$||PC + PA|| < [\lambda_{min}(Q) + \lambda_{min}(Z)]\sqrt{1 - h_d}, \tag{2.2}$$

$$Z = R, \tag{2.3}$$

$$||RB + RC + RA|| < \lambda_{min}(Z)\sqrt{1 - h_d}. \tag{2.4}$$

(A2) Let C and A are regular matrices, $P = P^T > 0, Q = Q^T > 0$ and $Z = Z^T > 0$ are symmetric matrices with suitable dimensions such that

$$\begin{aligned} &B^T P + PB + \mu C^T PC + \alpha\mu(1 - h_d)B^T PB \\ &+ \frac{2}{\mu(1 - h_d)}P + Q + Z = 0, \end{aligned} \tag{2.5}$$

$$\mu A^T PA - 2P + \frac{3}{\mu(1 - h_d)}P + Z < 0, \tag{2.6}$$

where μ is a positive number.

Theorem 2.1. *If conditions (A1) and (1.2) are satisfied, then the zero solution of NFNS (1.1) is asymptotically stable.*

Proof. We define the LKF

$$\begin{aligned} V(t, x) =& {}_{t_0}D_t^{q-1}(x^T(t)Px(t)) + \int_{t-\kappa(t)}^{t} x^T(s)Qx(s)ds \\ &+ \int_{t-\kappa(t)}^{t} ({}_{t_0}D_t^q x(s))^T Z({}_{t_0}D_t^q x(s))ds. \end{aligned} \tag{2.7}$$

It can be easily shown that the LKF (2.7) is positive definite. In light of Property 1.2, Lemma 1.3 and condition (1.2), by the time-derivative of the LKF (2.7) along the solutions of NFNS (1.1), we obtain

$$\begin{aligned} \dot{V}(t, x) \leq& x^T(t)[B^T P + PB + Q]x(t) + 2x^T(t)PCx(t - \kappa(t)) \\ &+ 2x^T(t)PA({}_{t_0}D_t^q x(t - \kappa(t))) + 2x^T(t)PH_1(t, x(t)) \\ &+ 2x^T(t)PH_2(t, x(t - \kappa(t))) + 2x^T(t)PH_3(t, {}_{t_0}D_t^q x(t - \kappa(t))) \\ &- (1 - h_d)x^T(t - \kappa(t))Qx(t - \kappa(t)) + ({}_{t_0}D_t^q x(t))^T Z({}_{t_0}D_t^q x(t)) \end{aligned}$$

$$- (1-h_d)({}_{t_0}D_t^q x(t-\kappa(t)))^T Z({}_{t_0}D_t^q x(t-\kappa(t))). \tag{2.8}$$

Using Lemma 1.4, for some terms included in (2.8), we have

$$\begin{aligned}2x^T(t)PCx(t-\kappa(t)) =& 2x^T(t)PCQ^{-\frac{1}{2}}Q^{\frac{1}{2}}x(t-\kappa(t))\\ \le& \frac{1}{\alpha(1-h_d)}x^T(t)PCQ^{-1}C^TPx(t)\\ &+\alpha(1-h_d)x^T(t-\kappa(t))Qx(t-\kappa(t)),\end{aligned} \tag{2.9}$$

$$\begin{aligned}2x^T(t)PA({}_{t_0}D_t^q x(t-\kappa(t))) =& 2x^T(t)PAZ^{-\frac{1}{2}}Z^{\frac{1}{2}}({}_{t_0}D_t^q x(t-\kappa(t)))\\ \le& \frac{1}{\alpha(1-h_d)}x^T(t)PAZ^{-1}A^TPx(t)\\ &+\alpha(1-h_d)({}_{t_0}D_t^q x(t-\kappa(t)))^T Z({}_{t_0}D_t^q x(t-\kappa(t))),\end{aligned} \tag{2.10}$$

$$\begin{aligned}2x^T(t)PH_1(t,x(t)) \le& \beta^{-1}x^T(t)P^2x(t)\\ &+\beta H_1^T(t,x(t))H_1(t,x(t)),\end{aligned} \tag{2.11}$$

$$\begin{aligned}2x^T(t)PH_2(t,x(t-\kappa(t))) \le& \gamma^{-1}x^T(t)P^2x(t)\\ &+\gamma H_2^T(t,x(t-\kappa(t)))H_2(t,x(t-\kappa(t))),\end{aligned} \tag{2.12}$$

$$\begin{aligned}2x^T(t)PH_3(t,{}_{t_0}D_t^q x(t-\kappa(t))) \le& \xi^{-1}x^T(t)P^2x(t)\\ &+\xi H_3^T(t,({}_{t_0}D_t^q x(t-\kappa(t))))H_3(t,({}_{t_0}D_t^q x(t-\kappa(t)))).\end{aligned} \tag{2.13}$$

Using the equality

$$\begin{aligned}&-{}_{t_0}D_t^q x(t)+Bx(t)+Cx(t-\kappa(t))+A{}_{t_0}D_t^q x(t-\kappa(t))+H_1(t,x(t))\\ &\quad+H_2(t,x(t-\kappa(t)))+H_3(t,{}_{t_0}D_t^q x(t-\kappa(t)))=0,\end{aligned}$$

We have

$$\begin{aligned}&2{}_{t_0}D_t^q x(t)R[-{}_{t_0}D_t^q x(t)+Bx(t)+Cx(t-\kappa(t))\\ &+A{}_{t_0}D_t^q x(t-\kappa(t))+H_1(t,x(t))+H_2(t,x(t-\kappa(t)))\\ &+H_3(t,{}_{t_0}D_t^q x(t-\kappa(t)))]=-2({}_{t_0}D_t^q x(t))^TR({}_{t_0}D_t^q x(t))\\ &+2({}_{t_0}D_t^q x(t))^TRBx(t)+2({}_{t_0}D_t^q x(t))^TRCx(t-\kappa(t))\\ &+2({}_{t_0}D_t^q x(t))^TRA({}_{t_0}D_t^q x(t-\kappa(t)))\\ &+2({}_{t_0}D_t^q x(t))^TRH_1(t,x(t))+2({}_{t_0}D_t^q x(t))^TRH_2(t,x(t-\kappa(t)))\\ &+2({}_{t_0}D_t^q x(t))^TRH_3(t,{}_{t_0}D_t^q x(t-\kappa(t)))=0.\end{aligned} \tag{2.14}$$

Using Lemma 1.4, for some terms included in (2.14), we obtain the following inequalities

$$\begin{aligned}2({}_{t_0}D_t^q x(t))^TRBx(t) =& 2({}_{t_0}D_t^q x(t))^TRBZ^{-\frac{1}{2}}Z^{\frac{1}{2}}x(t)\\ \le& \frac{1}{\alpha(1-h_d)}({}_{t_0}D_t^q x(t))^TRBZ^{-1}B^TR({}_{t_0}D_t^q x(t))\\ &+\alpha(1-h_d)x^T(t)Zx(t),\end{aligned} \tag{2.15}$$

$$\begin{aligned}2({}_{t_0}D_t^q x(t))^T RCx(t-\kappa(t)) =& 2({}_{t_0}D_t^q x(t))^T RCZ^{-\frac{1}{2}}Z^{\frac{1}{2}}x(t-\kappa(t))\\ \le& \frac{1}{\alpha(1-h_d)}({}_{t_0}D_t^q x(t))^T RCZ^{-1}C^T R({}_{t_0}D_t^q x(t))\\ &+\alpha(1-h_d)x^T(t-\kappa(t))Zx(t-\kappa(t)),\end{aligned} \tag{2.16}$$

$$\begin{aligned}2({}_{t_0}D_t^q x(t))^T RA({}_{t_0}D_t^q x(t-\kappa(t))) =& 2({}_{t_0}D_t^q x(t))^T RAZ^{-\frac{1}{2}}Z^{\frac{1}{2}}({}_{t_0}D_t^q x(t-\kappa(t)))\\ \le& \frac{1}{\alpha(1-h_d)}({}_{t_0}D_t^q x(t))^T RAZ^{-1}A^T R({}_{t_0}D_t^q x(t))\\ &+\alpha(1-h_d)({}_{t_0}D_t^q x(t-\kappa(t)))^T Z({}_{t_0}D_t^q x(t-\kappa(t))),\end{aligned} \tag{2.17}$$

$$\begin{aligned}2({}_{t_0}D_t^q x(t))^T RH_1(t,x(t)) \le& a^{-1}({}_{t_0}D_t^q x(t))^T R^2({}_{t_0}D_t^q x(t))^T\\ &+aH_1^T(t,x(t))H_1(t,x(t)),\end{aligned} \tag{2.18}$$

$$\begin{aligned}2({}_{t_0}D_t^q x(t))^T RH_2(t,x(t-\kappa(t))) \le& b^{-1}({}_{t_0}D_t^q x(t))^T R^2({}_{t_0}D_t^q x(t))\\ &+bH_2^T(t,x(t-\kappa(t)))H_2(t,x(t-\kappa(t))),\end{aligned} \tag{2.19}$$

$$\begin{aligned}2({}_{t_0}D_t^q x(t))^T RH_3(t,({}_{t_0}D_t^q x(t-\kappa(t)))) \le& c^{-1}({}_{t_0}D_t^q x(t))^T R^2({}_{t_0}D_t^q x(t))\\ &+cH_3^T(t,({}_{t_0}D_t^q x(t-\kappa(t))))H_3(t,({}_{t_0}D_t^q x(t-\kappa(t)))).\end{aligned} \tag{2.20}$$

where $a, \gamma, b, \beta, c, \xi$ and α are some positive constants.

Combining (2.8)-(2.20) and using (2.1) and (2.3), we have

$$\begin{aligned}\dot V(t,x) \le& x^T(t)\Big[\frac{1}{\alpha(1-h_d)}PCQ^{-1}C^TP+\frac{1}{\alpha(1-h_d)}PAZ^{-1}A^TP-Q-Z\\ &+\Big(\frac{1}{\beta}+\frac{1}{\xi}+\frac{1}{\gamma}\Big)P^2\Big]x(t)+({}_{t_0}D_t^q x(t))^T\Big[\frac{1}{\alpha(1-h_d)}RBZ^{-1}B^TR\\ &+\frac{1}{\alpha(1-h_d)}RCZ^{-1}C^TR+\frac{1}{\alpha(1-h_d)}RAZ^{-1}A^TR-Z\\ &+\Big(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\Big)R^2\Big]({}_{t_0}D_t^q x(t))+x^T(t-\kappa(t))[(1-h_d)(\alpha-1)Q\\ &+\alpha(1-h_d)Z]x(t-\kappa(t))+({}_{t_0}D_t^q x(t-\kappa(t)))^T[(\alpha-1)(1-h_d)Z\\ &+\alpha(1-h_d)Z]({}_{t_0}D_t^q x(t-\kappa(t)))+(\beta+a)H_1^T(t,x(t))H_1(t,x(t))\\ &+(\gamma+b)H_2^T(t,x(t-\kappa(t)))H_2(t,x(t-\kappa(t)))\\ &+(\xi+c)H_3^T(t,{}_{t_0}D_t^q x(t-\kappa(t)))H_3(t,{}_{t_0}D_t^q x(t-\kappa(t))).\end{aligned} \tag{2.21}$$

Moreover, for some terms of (2.21), in the light of definition of spectral norm, we write the following inequalities, respectively:

$$\begin{aligned}\Big[\lambda_{max}\Big(\frac{1}{1-h_d}Q^{\frac{-1}{2}}PCQ^{-1}C^TPQ^{\frac{-1}{2}}\Big)\Big]^{\frac{1}{2}} =& \frac{1}{\sqrt{1-h_d}}[\lambda_{max}(Q^{\frac{-1}{2}}PCQ^{-1}C^TPQ^{\frac{-1}{2}})]^{\frac{1}{2}}\\ =& \frac{1}{\sqrt{1-h_d}}||Q^{\frac{-1}{2}}PCQ^{\frac{-1}{2}}||\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\sqrt{1-h_d}}||Q^{\frac{-1}{2}}||^2||PC|| \\
&= \frac{1}{\sqrt{1-h_d}\lambda_{min}(Q)}||PC||, \qquad (2.22)
\end{aligned}
$$

$$
\begin{aligned}
\Big[\lambda_{max}\Big(\frac{1}{1-h_d}Z^{\frac{-1}{2}}PAZ^{-1}A^TPZ^{\frac{-1}{2}}\Big)\Big]^{\frac{1}{2}} &= \frac{1}{\sqrt{1-h_d}}[\lambda_{max}(Z^{\frac{-1}{2}}PAZ^{-1}A^TPZ^{\frac{-1}{2}})]^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{1-h_d}}||Z^{\frac{-1}{2}}PAZ^{\frac{-1}{2}}|| \\
&\leq \frac{1}{\sqrt{1-h_d}}||Z^{\frac{-1}{2}}||^2||PA|| \\
&= \frac{1}{\sqrt{1-h_d}\lambda_{min}(Z)}||PA||, \qquad (2.23)
\end{aligned}
$$

$$
\begin{aligned}
\Big[\lambda_{max}\Big(\frac{1}{1-h_d}Z^{\frac{-1}{2}}RBZ^{-1}B^TRZ^{\frac{-1}{2}}\Big)\Big]^{\frac{1}{2}} &= \frac{1}{\sqrt{1-h_d}}[\lambda_{max}(Z^{\frac{-1}{2}}RBZ^{-1}B^TRZ^{\frac{-1}{2}})]^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{1-h_d}}||Z^{\frac{-1}{2}}RBZ^{\frac{-1}{2}}|| \\
&\leq \frac{1}{\sqrt{1-h_d}}||Z^{\frac{-1}{2}}||^2||RB|| \\
&= \frac{1}{\sqrt{1-h_d}\lambda_{min}(Z)}||RB||, \qquad (2.24)
\end{aligned}
$$

$$
\begin{aligned}
\Big[\lambda_{max}\Big(\frac{1}{1-h_d}Z^{\frac{-1}{2}}RCZ^{-1}C^TRZ^{\frac{-1}{2}}\Big)\Big]^{\frac{1}{2}} &= \frac{1}{\sqrt{1-h_d}}[\lambda_{max}(Z^{\frac{-1}{2}}RCZ^{-1}C^TRZ^{\frac{-1}{2}})]^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{1-h_d}}||Z^{\frac{-1}{2}}RCZ^{\frac{-1}{2}}|| \\
&\leq \frac{1}{\sqrt{1-h_d}}||Z^{\frac{-1}{2}}||^2||RC|| \\
&= \frac{1}{\sqrt{1-h_d}\lambda_{min}(Z)}||RC||, \qquad (2.25)
\end{aligned}
$$

$$
\begin{aligned}
\Big[\lambda_{max}\Big(\frac{1}{1-h_d}Z^{\frac{-1}{2}}RAZ^{-1}A^TRZ^{\frac{-1}{2}}\Big)\Big]^{\frac{1}{2}} &= \frac{1}{\sqrt{1-h_d}}[\lambda_{max}(Z^{\frac{-1}{2}}RAZ^{-1}A^TRZ^{\frac{-1}{2}})]^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{1-h_d}}||Z^{\frac{-1}{2}}RAZ^{\frac{-1}{2}}|| \\
&\leq \frac{1}{\sqrt{1-h_d}}||Z^{\frac{-1}{2}}||^2||RA|| \\
&= \frac{1}{\sqrt{1-h_d}\lambda_{min}(Z)}||RA||. \qquad (2.26)
\end{aligned}
$$

From (2.22), (2.23) and the inequality (2.2), we obtain

$$\lambda_{max}\Big(\frac{1}{1-h_d}Q^{\frac{-1}{2}}PCQ^{-1}C^TPQ^{\frac{-1}{2}}\Big) < 1.$$

Hence, there exist a constant $\eta > 0$ such that

$$\lambda_{max}\Big(\frac{1}{1-h_d}Q^{\frac{-1}{2}}PCQ^{-1}C^TPQ^{\frac{-1}{2}}\Big) < \eta < 1.$$

Since $P > 0$ and $CQ^{-1}C^T \geq 0$, then it follows from Lemma 1.5 that

$$\frac{1}{1-h_d}PCQ^{-1}C^TP < \eta Q.$$

Let $\alpha < \frac{1}{2}$ such that $0 < \frac{\eta}{\alpha} < 1$. Hence, we get

$$\frac{1}{\alpha(1-h_d)}PCQ^{-1}C^TP - Q < (\frac{\eta}{\alpha} - 1)Q < 0 \tag{2.27}$$

and

$$\lambda_{max}\Big(\frac{1}{1-h_d}Z^{\frac{-1}{2}}PAZ^{-1}A^TPZ^{\frac{-1}{2}}\Big) < 1.$$

Next, there exist a constant $\eta > 0$ such that

$$\lambda_{max}\Big(\frac{1}{1-h_d}Z^{\frac{-1}{2}}PAZ^{-1}A^TPZ^{\frac{-1}{2}}\Big) < \eta < 1.$$

Since $P > 0$ and $AZ^{-1}A^T \geq 0$, in the light of Lemma 1.5, we find

$$\frac{1}{1-h_d}PAZ^{-1}A^TP < \eta Z.$$

Let $\alpha < \frac{1}{2}$ such that $0 < \frac{\eta}{\alpha} < 1$. Thus, in view of the last inequality, we have

$$\frac{1}{\alpha(1-h_d)}PAZ^{-1}A^TP - Z < \Big(\frac{\eta}{\alpha} - 1\Big)Z < 0 \tag{2.28}$$

From (2.27) and (2.28), we have

$$\frac{1}{\alpha(1-h_d)}(PCQ^{-1}C^TP + PAZ^{-1}A^TP) - Q - Z < \Big(\frac{\eta}{\alpha} - 1\Big)(Q+Z) < 0.$$

Using (2.24)-(2.26) and (2.4), we arrive at

$$\frac{1}{\alpha(1-h_d)}(RBZ^{-1}B^TR + RCZ^{-1}C^TR + RAZ^{-1}A^TR) - Z < \Big(\frac{\eta}{\alpha} - 1\Big)Z < 0.$$

Let $a, \gamma, b, \beta, c, \xi$ and α be positive numbers such that the following inequalities hold:

(2.29)

$$\Theta_1 = \frac{1}{\alpha(1-h_d)}\{PCQ^{-1}C^TP + PAZ^{-1}A^TP\} - Q - Z + \Big(\frac{1}{\beta} + \frac{1}{\xi} + \frac{1}{\gamma}\Big)P^2 < 0,$$

$$\begin{aligned}\Theta_2 = {} & \frac{1}{\alpha(1-h_d)}\{RBZ^{-1}B^TR + RCZ^{-1}C^TR + RAZ^{-1}A^TR\} \\ & - Z + \Big(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\Big)R^2 \\ & < 0.\end{aligned} \tag{2.30}$$

Since $0 < h_d < 1, \alpha < \frac{1}{2}, Q = Q^T > 0$ and $Z = Z^T > 0$, we obtain

$$\Theta_3 = (\alpha - 1)(1 - h_d)Q + \alpha(1 - h_d)Z < 0, \tag{2.31}$$

$$\Theta_4 = (2\alpha - 1)(1 - h_d)Z < 0. \tag{2.32}$$

Hence, we have

$$\begin{aligned}\dot{V}(t,x) \leq & x^T(t)\Theta_1 x(t) + ({}_{t_0}D_t^q x(t))^T \Theta_2 ({}_{t_0}D_t^q x(t)) + x^T(t - \kappa(t))\Theta_3 \\ & \times x(t - \kappa(t)) + ({}_{t_0}D_t^q x(t - \kappa(t)))^T \Theta_4 ({}_{t_0}D_t^q x(t - \kappa(t))) \\ & + (\beta + a)||H_1(t, x(t))||^2 + (\gamma + b)||H_2(t, x(t - \kappa(t)))||^2 \\ & + (\xi + c)||H_3(t, {}_{t_0}D_t^q x(t - \kappa(t)))||^2. \end{aligned} \tag{2.33}$$

In view of the inequalities (2.29)-(2.32), we choose a positive constant ρ such that

$$\Theta_j + \rho I < 0, (j = 1, 2, 3, 4). \tag{2.34}$$

From (1.3), there exist a positive number δ such that when $||x(t)|| < \delta, t \geq t_0$, the following inequalities hold:

$$\begin{aligned} ||H_1(t, x(t))||^2 &\leq \frac{\rho}{\beta + a}||x(t)||^2, \\ ||H_2(t, x(t - \kappa(t)))||^2 &\leq \frac{\rho}{\gamma + b}||x(t - \kappa(t))||^2, \\ ||H_3(t, {}_{t_0}D_t^q x(t - \kappa(t)))||^2 &\leq \frac{\rho}{\xi + c}||{}_{t_0}D_t^q x(t - \kappa(t))||^2 \end{aligned}$$

Using these inequalities in (2.33), we obtain

$$\begin{aligned}\dot{V}(t,x) = & x^T(t)(\Theta_1 + \rho I)x(t) + ({}_{t_0}D_t^q x(t))^T(\Theta_2 + \rho I)({}_{t_0}D_t^q x(t)) \\ & + x^T(t - \kappa(t))(\Theta_3 + \rho I)x(t - \kappa(t)) + ({}_{t_0}D_t^q x(t - \kappa(t)))^T \\ & \times (\Theta_4 + \rho I)({}_{t_0}D_t^q x(t - \kappa(t))), \end{aligned}$$

where I is $n \times n$ - identity matrix.

Considering (2.34), we can write $\dot{V}(t,x) < 0$. Thus, the zero solution of NFNS (1.1) is asymptotically stable. This result completes the proof of Theorem 2.1. □

Example 2.2. Consider the following NFNS with unbounded delays, which is a special case of (1.1):

$$\begin{aligned} {}_{t_0}D_t^q x(t) - A {}_{t_0}D_t^q x(t - \kappa(t)) = & Bx(t) + Cx(t - \kappa(t)) + H_1(t, x(t)) \\ & + H_2(t, x(t - \kappa(t))) + H_3(t, {}_{t_0}D_t^q x(t - \kappa(t))), \end{aligned}$$

where

$$0 < q < 1, x(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T, \alpha = 0.3, \kappa(t) = 0.5t, \dot{\kappa}(t) = 0.5 = h_d.$$

$$A = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.3 \end{bmatrix}, B = \begin{bmatrix} -0.8 & 0 \\ 0 & -0.6 \end{bmatrix}, C = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.02 \end{bmatrix},$$

and

$H_1(t, x(t)) = \begin{bmatrix} x_1(t)sin(x_1(t)) & x_2(t)sin(x_2(t)) \end{bmatrix}^T,$

$H_2(t, x(t - \kappa(t))) = \begin{bmatrix} x_1(t - \kappa(t))cos(x_1(t - \kappa(t))) & x_2(t - \kappa(t))cos(x_2(t - \kappa(t))) \end{bmatrix}^T,$

$H_3(t,_{t_0} D_t^q x(t-\kappa(t))) = \begin{bmatrix} {}_{t_0}D_t^q x_1(t-\kappa(t))cos(t) & {}_{t_0}D_t^q x_2(t-\kappa(t))cos(t) \end{bmatrix}^T,$

Let $P = diag(25, 30)$ and $Q = diag(10, 9)$. Then, it follows from (A1) that

$$R = Z = \begin{bmatrix} 17.3913 & 0 \\ 0 & 15.6522 \end{bmatrix}.$$

Next, we obtain $||PC + PA|| = 12.75, ||RB + RC + RA|| = 5.0435, [\lambda_{min}(Q) + \lambda_{min}(Z)]\sqrt{1-h_d} = 17.4317$ and $\lambda_{min}(Z)\sqrt{1-h_d} = 11.0678$. Thus, assumption (A1) hold. Hence, all conditions of Theorem 2.1 are satisfied. According to Theorem 2.1, the zero solution of NFNS of Example 2.2 is asymptotically stable.

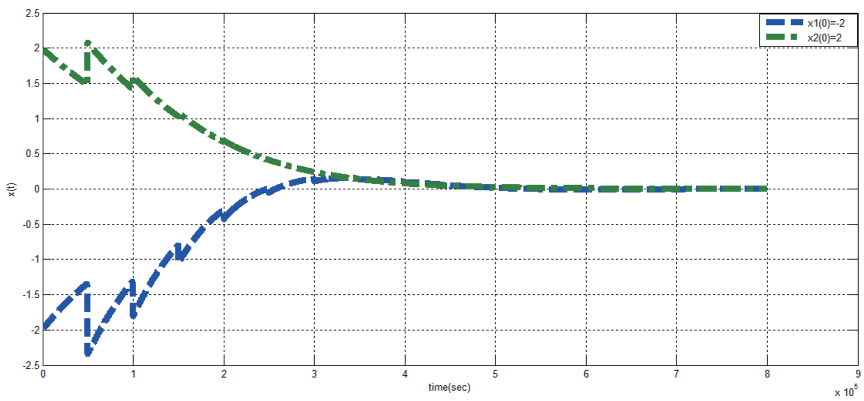

FIGURE 1. The numerical simulation of the NFNS of Example 2.2 for $\kappa(t) = 0.5t$.

Theorem 2.3. *If the conditions (2.5) and (2.6), that is, assumption (A2) and (1.2) are satisfied, then the zero solution of NFNS (1.1) is asymptotically stable.*

Proof. We define the LKF

$$V(t,x) =_{t_0} D_t^{q-1}(x^T(t)Px(t)) + \mu \int_{t-\kappa(t)}^t x^T(s)C^TPCx(s)ds$$
$$+ \mu \int_{t-\kappa(t)}^t ({}_{t_0}D_t^q x(s))^T A^T PA({}_{t_0}D_t^q x(s))ds. \tag{2.35}$$

Since the matrices C, A are regular and $P = P^T > 0$, then $C^TPC > 0$ and $A^TPA > 0$. Hence, it can be shown that the LKF (2.35) is positive definite. By the time-derivative of the LKF (2.35) along the trajectories of NFNS (1.1), we obtain

$$\dot{V}(t,x) \leq x^T(t)[B^TP + PB + \mu C^TPC]x(t) + \mu({}_{t_0}D_t^q x(t))^T A^T PA({}_{t_0}D_t^q x(t))$$
$$+ 2x^T(t)PCx(t-\kappa(t)) + 2x^T(t)PA({}_{t_0}D_t^q x(t-\kappa(t)))$$
$$+ 2x^T(t)PH_1(t,x(t)) + 2x^T(t)PH_2(t,x(t-\kappa(t))) + 2x^T(t)P$$
$$\times H_3(t,_{t_0} D_t^q x(t-\kappa(t))) - \mu(1-h_d)x^T(t-\kappa(t))C^TPCx(t-\kappa(t))$$
$$- \mu(1-h_d)({}_{t_0}D_t^q x(t-\kappa(t)))^T A^T PA({}_{t_0}D_t^q x(t-\kappa(t))). \tag{2.36}$$

Since $P = P^T > 0, P$ has a decomposition such as $P = L^T L$, where L is any nonsingular matrix with appropriate dimension. In view of Lemma 1.4, we have

$$\begin{aligned} 2x^T(t)PCx(t-\kappa(t)) =& 2x^T(t)L^T LCx(t-\kappa(t)) \\ \le& \frac{1}{\alpha\mu(1-h_d)} x^T(t)L^T Lx(t) \\ &+ \alpha\mu(1-h_d)x^T(t-\kappa(t))C^T L^T LCx(t-\kappa(t)) \\ =& \frac{1}{\alpha\mu(1-h_d)} x^T(t)Px(t) \\ &+ \alpha\mu(1-h_d)x^T(t-\kappa(t))C^T PCx(t-\kappa(t)), \end{aligned} \tag{2.37}$$

$$\begin{aligned} 2x^T(t)PA({}_{t_0}D_t^q x(t-\kappa(t))) &= 2x^T(t)L^T LA({}_{t_0}D_t^q x(t-\kappa(t))) \\ \le& \frac{1}{\alpha\mu(1-h_d)} x^T(t)L^T Lx(t) \\ &+ \alpha\mu(1-h_d)({}_{t_0}D_t^q x(t-\kappa(t)))^T A^T L^T LA({}_{t_0}D_t^q x(t-\kappa(t))) \\ =& \frac{1}{\alpha\mu(1-h_d)} x^T(t)Px(t) \\ &+ \alpha\mu(1-h_d)({}_{t_0}D_t^q x(t-\kappa(t)))^T A^T PA({}_{t_0}D_t^q x(t-\kappa(t))), \end{aligned} \tag{2.38}$$

$$\begin{aligned} 2x^T(t)PH_1(t,x(t)) \le& \beta^{-1}x^T(t)P^2x(t) \\ &+ \beta H_1^T(t,x(t))H_1(t,x(t)), \end{aligned} \tag{2.39}$$

$$\begin{aligned} 2x^T(t)PH_2(t,x(t-\kappa(t))) \le& \gamma^{-1}x^T(t)P^2x(t) \\ &+ \gamma H_2^T(t,x(t-\kappa(t)))H_2(t,x(t-\kappa(t))), \end{aligned} \tag{2.40}$$

$$\begin{aligned} 2x^T(t)PH_3(t,{}_{t_0}D_t^q x(t-\kappa(t))) \le& \xi^{-1}x^T(t)P^2x(t) \\ &+ \xi H_3^T(t,{}_{t_0}D_t^q x(t-\kappa(t)))H_3(t,{}_{t_0}D_t^q x(t-\kappa(t))). \end{aligned} \tag{2.41}$$

Considering the equality

$$\begin{aligned} -{}_{t_0}D_t^q x(t) + Bx(t) + Cx(t-\kappa(t)) + A{}_{t_0}D_t^q x(t-\kappa(t)) + H_1(t,x(t)) \\ + H_2(t,x(t-\kappa(t))) + H_3(t,{}_{t_0}D_t^q x(t-\kappa(t))) = 0, \end{aligned}$$

We have

$$\begin{aligned} &2{}_{t_0}D_t^q x(t)P[-{}_{t_0}D_t^q x(t) + Bx(t) + Cx(t-\kappa(t)) \\ &+ A{}_{t_0}D_t^q x(t-\kappa(t)) + H_1(t,x(t)) + H_2(t,x(t-\kappa(t))) \\ &+ H_3(t,{}_{t_0}D_t^q x(t-\kappa(t)))] = -2({}_{t_0}D_t^q x(t))^T P({}_{t_0}D_t^q x(t)) \\ &+ 2({}_{t_0}D_t^q x(t))^T PBx(t) + 2({}_{t_0}D_t^q x(t))^T PCx(t-\kappa(t)) \\ &+ 2({}_{t_0}D_t^q x(t))^T PA({}_{t_0}D_t^q x(t-\kappa(t))) \\ &+ 2({}_{t_0}D_t^q x(t))^T PH_1(t,x(t)) + 2({}_{t_0}D_t^q x(t))^T PH_2(t,x(t-\kappa(t))) \\ &+ 2({}_{t_0}D_t^q x(t))^T PH_3(t,{}_{t_0}D_t^q x(t-\kappa(t))) = 0. \end{aligned} \tag{2.42}$$

By using Lemma 1.4, for some terms of (2.42), we obtain the following inequalities, respectively:

$$\begin{aligned}2({}_{t_0}D_t^q x(t))^T PBx(t) &= 2({}_{t_0}D_t^q x(t))^T L^T LBx(t \\ &\leq \frac{1}{\alpha\mu(1-h_d)}({}_{t_0}D_t^q x(t))^T L^T L({}_{t_0}D_t^q x(t)) \\ &\quad + \alpha\mu(1-h_d)x^T(t)B^T L^T LBx(t) \\ &= \frac{1}{\alpha\mu(1-h_d)}({}_{t_0}D_t^q x(t))^T P({}_{t_0}D_t^q x(t)) \\ &\quad + \alpha\mu(1-h_d)x^T(t)B^T PBx(t),\end{aligned} \tag{2.43}$$

$$\begin{aligned}2({}_{t_0}D_t^q x(t))^T PCx(t-\kappa(t)) &= 2({}_{t_0}D_t^q x(t))^T L^T LCx(t-\kappa(t)) \\ &\leq \frac{1}{\alpha\mu(1-h_d)}({}_{t_0}D_t^q x(t))^T L^T L({}_{t_0}D_t^q x(t)) \\ &\quad + \alpha\mu(1-h_d)x^T(t-\kappa(t))C^T L^T LCx(t-\kappa(t)) \\ &= \frac{1}{\alpha\mu(1-h_d)}({}_{t_0}D_t^q x(t))^T P({}_{t_0}D_t^q x(t)) \\ &\quad + \alpha\mu(1-h_d)x^T(t-\kappa(t))C^T PCx(t-\kappa(t)),\end{aligned} \tag{2.44}$$

$$\begin{aligned}2({}_{t_0}D_t^q x(t))^T PA({}_{t_0}D_t^q x(t-\kappa(t))) &= 2({}_{t_0}D_t^q x(t))^T L^T LA({}_{t_0}D_t^q x(t-\kappa(t))) \\ &\leq \frac{1}{\alpha\mu(1-h_d)}({}_{t_0}D_t^q x(t))^T L^T L({}_{t_0}D_t^q x(t)) \\ &\quad + \alpha\mu(1-h_d)({}_{t_0}D_t^q x(t-\kappa(t)))^T A^T L^T LA({}_{t_0}D_t^q x(t-\kappa(t))) \\ &= \frac{1}{\alpha\mu(1-h_d)}({}_{t_0}D_t^q x(t))^T P({}_{t_0}D_t^q x(t)) \\ &\quad + \alpha\mu(1-h_d)({}_{t_0}D_t^q x(t-\kappa(t)))^T A^T PAx({}_{t_0}D_t^q x(t-\kappa(t))),\end{aligned} \tag{2.45}$$

$$\begin{aligned}2({}_{t_0}D_t^q x(t))^T PH_1(t,x(t)) &\leq a^{-1}({}_{t_0}D_t^q x(t))^T P^2({}_{t_0}D_t^q x(t))^T \\ &\quad + aH_1^T(t,x(t))H_1(t,x(t)),\end{aligned} \tag{2.46}$$

$$\begin{aligned}2({}_{t_0}D_t^q x(t))^T PH_2(t,x(t-\kappa(t))) &\leq b^{-1}({}_{t_0}D_t^q x(t))^T P^2({}_{t_0}D_t^q x(t)) \\ &\quad + bH_2^T(t,x(t-\kappa(t)))H_2(t,x(t-\kappa(t))),\end{aligned} \tag{2.47}$$

$$\begin{aligned}2({}_{t_0}D_t^q x(t))^T PH_3(t,({}_{t_0}D_t^q x(t-\kappa(t)))) &\leq c^{-1}({}_{t_0}D_t^q x(t))^T P^2({}_{t_0}D_t^q x(t)) \\ &\quad + cH_3^T(t,({}_{t_0}D_t^q x(t-\kappa(t))))H_3(t,({}_{t_0}D_t^q x(t-\kappa(t)))).\end{aligned} \tag{2.48}$$

where $a, \gamma, b, \beta, c, \xi$ and α are some positive constants.

Combining (2.36)-(2.48) and using (2.5) and (2.6) of (A2), we arrive at

$$\begin{aligned}\dot{V}(t,x) \leq & x^T(t)\left[\frac{2}{\mu(1-h_d)}\Big(\frac{1}{\alpha}-1\Big)P + \Big(\frac{1}{\beta}+\frac{1}{\xi}+\frac{1}{\gamma}\Big)P^2 - Q - Z\right]x(t) \\ &+({}_{t_0}D_t^q x(t))^T\left[\frac{3}{\mu(1-h_d)}\Big(\frac{1}{\alpha}-1\Big)P + \Big(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\Big)P^2 - Z\right]({}_{t_0}D_t^q x(t))\end{aligned}$$

$$+x^T(t-\kappa(t))[(2\alpha-1)\mu(1-h_d)C^TPC]x(t-\kappa(t))$$
$$+({}_{t_0}D_t^q x(t-\kappa(t)))^T[\mu(2\alpha-1)(1-h_d)A^TPA]({}_{t_0}D_t^q x(t-\kappa(t)))$$
$$+(\beta+a)H_1^T(t,x(t))H_1(t,x(t))$$
$$+(\gamma+b)H_2^T(t,x(t-\kappa(t)))H_2(t,x(t-\kappa(t)))$$
$$+(\xi+c)H_3^T(t,{}_{t_0}D_t^q x(t-\kappa(t)))H_3(t,{}_{t_0}D_t^q x(t-\kappa(t))).$$

Since $P=P^T>0$ and A,C are regular matrices, it follows that $A^TPA>0$ and $C^TPC>0$. Let $\alpha<\frac{1}{2}$, then

$$\begin{aligned}\dot V(t,x)\leq & x^T(t)\left[\frac{2}{\mu(1-h_d)}\Big(\frac{1}{\alpha}-1\Big)P+\Big(\frac{1}{\beta}+\frac{1}{\xi}+\frac{1}{\gamma}\Big)P^2-Q-Z\right]x(t)\\ &+({}_{t_0}D_t^q x(t))^T\left[\frac{3}{\mu(1-h_d)}\Big(\frac{1}{\alpha}-1\Big)P+\Big(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\Big)P^2-Z\right]({}_{t_0}D_t^q x(t))\\ &+x^T(t-\kappa(t))[(2\alpha-1)\mu(1-h_d)C^TPC]x(t-\kappa(t))\\ &+({}_{t_0}D_t^q x(t-\kappa(t)))^T[(2\alpha-1)\mu(1-h_d)A^TPA]({}_{t_0}D_t^q x(t-\kappa(t)))\\ &+(\beta+a)||H_1(t,x(t))||^2+(\gamma+b)||H_2(t,x(t-\kappa(t)))||^2\\ &+(\xi+c)||H_3(t,{}_{t_0}D_t^q x(t-\kappa(t)))||^2.\end{aligned}\tag{2.49}$$

We choose positive constants a,γ,b,β,c,ξ sufficiently large and $\alpha<\frac{1}{2}$ with $1-\alpha$ sufficiently small such that

$$\Theta_1=\frac{2}{\mu(1-h_d)}\Big(\frac{1}{\alpha}-1\Big)P+\Big(\frac{1}{\beta}+\frac{1}{\xi}+\frac{1}{\gamma}\Big)P^2-Q-Z<0,\tag{2.50}$$

$$\Theta_2=\frac{3}{\mu(1-h_d)}\Big(\frac{1}{\alpha}-1\Big)P+\Big(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\Big)P^2-Z<0.\tag{2.51}$$

Since $\alpha<\frac{1}{2}$ and A,C are regular matrices, we have

$$\Theta_3=(2\alpha-1)\mu(1-h_d)C^TPC<0,\tag{2.52}$$

$$\Theta_4=(2\alpha-1)\mu(1-h_d)A^TPA<0.\tag{2.53}$$

Using (2.49)-(2.53), we find

$$\begin{aligned}\dot V(t,x)\leq & x^T(t)\Theta_1x(t)+({}_{t_0}D_t^q x(t))^T\Theta_2({}_{t_0}D_t^q x(t))+x^T(t-\kappa(t))\Theta_3\\ &\times x(t-\kappa(t))+({}_{t_0}D_t^q x(t-\kappa(t)))^T\Theta_4({}_{t_0}D_t^q x(t-\kappa(t)))\\ &+(\beta+a)||H_1(t,x(t))||^2+(\gamma+b)||H_2(t,x(t-\kappa(t)))||^2\\ &+(\xi+c)||H_3(t,{}_{t_0}D_t^q x(t-\kappa(t)))||^2.\end{aligned}\tag{2.54}$$

From the inequalities (2.50)-(2.53), we choose a positive constant ρ such that

$$\Theta_j+\rho I<0,(j=1,2,3,4).\tag{2.55}$$

From (1.3), there exist a positive number δ such that when $||x(t)||<\delta, t\geq t_0$, the following inequalities hold:

$$||H_1(t,x(t))||^2\leq\frac{\rho}{\beta+a}||x(t)||^2,$$

$$||H_2(t,x(t-\kappa(t)))||^2 \leq \frac{\rho}{\gamma+b}||x(t-\kappa(t))||^2,$$
$$||H_3(t,{}_{t_0}D_t^q x(t-\kappa(t)))||^2 \leq \frac{\rho}{\xi+c}||{}_{t_0}D_t^q x(t-\kappa(t))||^2.$$

Substituting last three inequalities into (2.54), we obtain

$$\begin{aligned}\dot{V}(t,x) \leq & x^T(t)(\Theta_1+\rho I)x(t) + ({}_{t_0}D_t^q x(t))^T(\Theta_2+\rho I)({}_{t_0}D_t^q x(t)) \\ & + x^T(t-\kappa(t))(\Theta_3+\rho I)x(t-\kappa(t)) + ({}_{t_0}D_t^q x(t-\kappa(t)))^T \\ & \times (\Theta_4+\rho I)({}_{t_0}D_t^q x(t-\kappa(t))),\end{aligned}$$

where I is $n\times n$ - identity matrix.

In the light of inequality (2.55), we conclude that $\dot{V}(t,x) < 0$. Thus, the zero solution of NFNS (1.1) is asymptotically stable. This result is the end of the proof. □

Example 2.4. Consider the NFNS with unbounded delay, which is a special case of NFNS (1.1):

$$\begin{aligned}{}_{t_0}D_t^q x(t) - A_{t_0}D_t^q x(t-\kappa(t)) = & Bx(t) + Cx(t-\kappa(t)) + H_1(t,x(t)) \\ & + H_2(t,x(t-\kappa(t))) + H_3(t,{}_{t_0}D_t^q x(t-\kappa(t))),\end{aligned}$$

where

$$0<q<1, x(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) \end{bmatrix}^T,$$
$$\alpha = 0.3, \mu = 6, \kappa(t) = 0.5t, \dot{\kappa}(t) = 0.5 = h_d.$$

$$A = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -0.6 & 0 \\ 0 & 0 & -0.5 \end{bmatrix}, C = \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0.02 & 0 \\ 0 & 0 & 0.03 \end{bmatrix}$$

and

$$H_1(t,x(t)) = \begin{bmatrix} x_1(t)e^{-x_1^2(t)} & x_2(t)e^{-x_2^2(t)} & x_3(t)e^{-x_3^2(t)} \end{bmatrix}^T,$$
$$H_2(t,x(\Xi)) = \begin{bmatrix} x_1(\Xi)e^{-x_1^2(\Xi)} & x_2(\Xi)e^{-x_2^2(\Xi)} & x_3(\Xi)e^{-x_3^2(\Xi)} \end{bmatrix}^T,$$
$$H_3(t,{}_{t_0}D_t^q x(\Xi)) \begin{bmatrix} {}_{t_0}D_t^q x_1(\Xi)e^{-{}_{t_0}D_t^q x_1^2(\Xi)} & {}_{t_0}D_t^q x_2(\Xi)e_{t_0}^{-{}_{t_0}D_t^q x_2^2(\Xi)} D_t^q x_3(\Xi)e^{-{}_{t_0}D_t^q x_3^2(\Xi)} \end{bmatrix}^T,$$

here $\Xi = t-\kappa(t)$. Let

$$P = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 35 \end{bmatrix}, Q = \begin{bmatrix} 7.9 & 0 & 0 \\ 0 & 3.8 & 0 \\ 0 & 0 & 2.1 \end{bmatrix}.$$

Then, it follows from (2.5) and (2.6) of (A2) that

$$Z = \begin{bmatrix} 2.9183 & 0 & 0 \\ 0 & 2.4080 & 0 \\ 0 & 0 & 1.5027 \end{bmatrix}.$$

This shows that (2.5) and (2.6) of (A2) are satisfied. Thus, all conditions of Theorem 2.3 hold. According to Theorem 2.3, the zero solution of the NFNS of Example 2.4 is asymptotically stable.

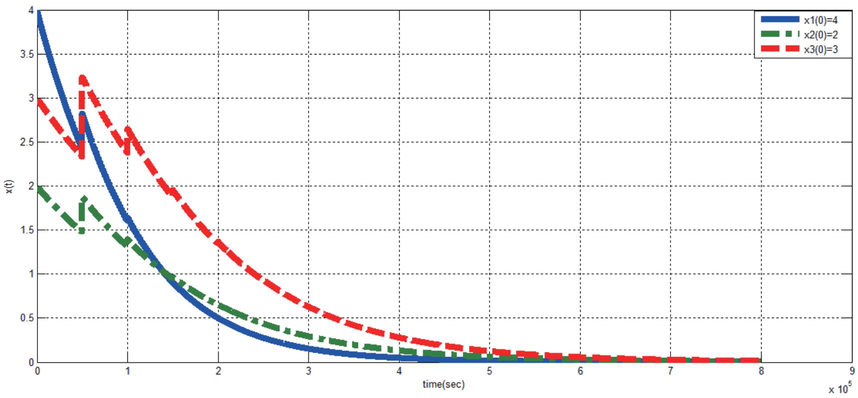

FIGURE 2. The numerical simulation of the system given by Example 2.4 for $\kappa(t) = 0.5t$.

3. Conclusion

In this article, using two different LKFs, we prove two theorems, which include some sufficient conditions, on the asymptotic stability of zero solution of an NFNS with an unbounded delay. We also provide two new examples with their graphs to show that the given conditions are applicable. Our results have contributions to the relevant literature.

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