Applied Analysis and Optimization
Volume 7, Number 3, 2023, 45–61

Yokohama Publishers
ISSN 2189-1664 Online Journal

FLOQUET THEORY OF TIME-VARYING DIFFERENTIAL ALGEBRAIC EQUATIONS TRANSFERABLE INTO STANDARD CANONICAL FORM

AWAIS YOUNUS AND NUDRAT YASMEEN

Abstract. This paper deals with time-varying periodic differential algebraic equations (DAEs) transferable into standard canonical form (SCF). The solution theory of linear differential algebraic equations is considered in detail and well known concept of ordinary algebraic equations is generalized for DAEs. We note that Floquet theory of DAEs demands a strict constant rank condition on the singular coefficient matrix which is not required by solution theory of periodic DAEs transferable into SCF. Moreover, we obtain the Floquet type result for periodic DAEs which is transferable into SCF.

1. Introduction

Differential algebraic equations (DAEs) are a combination of the differential equations along with algebraic constraints. DAEs were invented by Gantmacher which were discussed in the book, "The Theory of Matrices" about 60 years ago and laid the foundation of DA system [6]. Later, this system was also discussed by Ruth, Weierstrass, and Kronecker in the terms of matrix pencils. In the past few decades, DAEs have become a valuable tool for the modeling and simulation of dynamical systems with constraints because of its showed applications in many diverse and widespread fields of engineering and science such as control theory [4], electrical circuit simulation [2,13], mechanical multibody systems [11], fluid dynamics [5] and chemical engineering [8].

Now we consider the subclass; Time-varying (TV) differential-algebraic equations transferable into SCF. Almost 37 years ago, this subclass was introduced by Campbell [3]. Later many researchers construct various techniques to handle this particular subclass.

Moreover, this precise subclass is very interesting due to the following properties:

- The SCF of the subclass is a normal form in an equivalence class.
- All entries of the system remain real-valued in an equivalence class.
- This subclass works with continuous or C^pcoefficients.
- New results are constructed for DAEs as generalizations of previously known ODEs results without their derivation.
- If the DAE has analytic coefficients then the transformation matrices can be determined by using an algorithm.

2020 *Mathematics Subject Classification.* 34A09, 65L80.

Key words and phrases. Time-varying linear DAEs, equivalence of DAEs, standard canonical form, generalized transition matrix.

We will study the solution structure of first order linear TV differential-algebraic equations of the following form

$$L(t)\dot{z} = M\,(t)\,z + g\,(t)\,, \tag{1.1}$$

where $(L, M) \in C\,(J; \mathbb{R}^{p\times p})^2\,, g \in C\,(J; \mathbb{R}^p)$ for finite $p \in \mathbb{N}$. Here $t \in J \subseteq \mathbb{R}$ is an open interval and $L(t)$ is a singular matrix. For simplicity, the tuples (L, M, g) and $(L, M, 0) = (L, M)$ denote the homogeneous and non-homogeneous systems (1.1) respectively.

A function $z : K \to \mathbb{R}^p$ is called a solution of DAE (1.1) if and only if z is a continuously differentiable function on $K \subseteq J$, and solves equation (1.1) for all $t \in K$. It is called global solution if and only if $K = J$.

Evaluating the solution structure of DAEs with periodic coefficients is of great importance from theoretical point of view as well as against the background of applicability. Solution structure of periodic DAE was studied by René et. al. in [10]. We note that Floquet theory of index-1-tractable DAEs demands a strict constant rank condition on the singular coefficient matrix which is not required by solution theory of periodic DAEs transferable into SCF and derives a result about the periodic solution. We also obtain a Floquet type result for periodic DAEs which is transferable into SCF.

This paper has been organized as follows: Section 2 presents a collection of basic definitions on linear DAEs. Section 3 presents the uniqueness of SCF for a particular system, the various properties of the generalized transition matrix, and an algorithm for computing the SCF with the help of transformation matrices. Section 4 contains the Floquet theory of index-1-tractable DAEs and a result about their periodic solution. Section 5 shows that Floquet theory demands strict constant rank condition on singular coefficient matrix which is not required for the solution of periodic DAEs transferable into SCF. Moreover, we obtain Floquet type result for periodic DAEs transferable into SCF.

2. Preliminaries

Nomenclature

$\mathbb{N},\ \mathbb{N}_0$	The set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
$\ker L,\ imL$	The kernel, image, of the matrix $L \in \mathbb{R}^{p\times p}$, respectively.
$Gl_p\,(\mathbb{R})$	General linear group of degree p i.e. set of all invertible $p \times p$ matrices over $\mathbb{R}$.
$C\,(J; V)$	The set of continuously differentiable functions $g : J \to V$ from an open interval $J \subseteq \mathbb{R}$ to a vector space V.
$C^k\,(J; V)$	The set of k-times continuously differentiable functions $g : J \to V$ from an open interval $J \subseteq \mathbb{R}$ to a vector space V.
$dom\ g$	The domain of the function g.
$g\mid_A$	The restriction of the function g on a set $A \subseteq\ dom\ g$.

To understand the solution structure of DEAs, first, we review some basic definitions with the help of suitable examples.

The DAEs (L_1, M_1), $(L_2, M_2) \in C\left(J; \mathbb{R}^{p\times p}\right)^2$ are called equivalent if, and only if, there exists $(P, Q) \in C(J; Gl_p(\mathbb{R})) \times C^1(J; Gl_p(\mathbb{R}))$ such that

$$L_2 = PL_1Q,\ M_2 = PM_1Q - PL_1\dot{Q}.$$

Then we write

$$(L_1, M_1) \overset{P,Q}{\sim} (L_2, M_2). \tag{2.1}$$

A square matrix which has all the entries on and above the main diagonal are zero for all $t \in J$, is called pointwise strictly lower triangular [3,9]. Regular pencil is a pair of matrices (L, M) such that $(Lc - M)^{-1}$ exists, where $c \in \mathbb{C}$. If $(Lc - M)^{-1}$ does not exists, then pair of matrices (L, M) will be called a singular pencil.

The DAE $(L, M) \in C\left(J; \mathbb{R}^{p\times p}\right)^2$ for which transformation matrices $(P, Q) \in C\left(J; Gl_p(\mathbb{R})\right) \times C^1(J; Gl_p(\mathbb{R}))$ exist such that

$$(L, M) \overset{P,Q}{\sim} \left(\begin{bmatrix} I_{p_1} & 0 \\ 0 & A \end{bmatrix}, \begin{bmatrix} B & 0 \\ 0 & I_{p_2} \end{bmatrix}\right), \tag{2.2}$$

where A is a square matrix of size p_2 and pointwise strictly lower triangular and B is a square matrix of size p_1. Then DAE (L, M) is called transferable into SCF.

Example 2.1. Consider DAE with $(E, F) \in C\left((0,\infty); \mathbb{R}^{2\times 2}\right)^2$ and $\left(P', Q'\right) \in C\left((0,\infty); Gl_2(\mathbb{R})\right) \times C^1\left((0,\infty); Gl_2(\mathbb{R})\right)$, where

$$E(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ F(t) = \begin{bmatrix} -1 & 0 \\ 0 & e^t \end{bmatrix},\ t \in \mathbb{R},$$

and $P'(t) = \begin{bmatrix} 1 & 0 \\ 0 & e^t \end{bmatrix}$, $Q'(t) = I_2$, $t \in \mathbb{R}$ such that

$$A(t) = 0,\ B(t) = -1.$$

By using equation (2.2), we can write

$$(E, F) \overset{P',Q'}{\sim} \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right). \tag{2.3}$$

Thus DAE (L, M) is transferred into SCF.

The set which consist of all possible pairs of consistent initial values of $(L, M) \in C\left(J; \mathbb{R}^{p\times p}\right)^2$ is denoted by

$$W_{L,M} := \{\left(t^0, z^0\right) \in \mathbb{R} \times \mathbb{R}^p \mid \exists\ t^0 \in dom z(\cdot),\ z(t^0) = z^0\},$$

where $z(\cdot)$ is (local) solution of homogeneous DAE. Then the subspace of initial values which are consistent at time $t^0 \in J$ is called linear subspace of initial values and represented by

$$W_{L,M}\left(t^0\right) := \{z^0 \in \mathbb{R}^p \mid \left(t^0, z^0\right) \in W_{L,M}\}.$$

Note that $z : K \longrightarrow \mathbb{R}^p$ is a solution of equation (1.1), then $z(t) \in W_{L,M}$ for all $t \in K$.

Example 2.2. Suppose $(L, M) \in C\left(J; \mathbb{R}^{p\times p}\right)^2$, which is interchangeable into SCF for some $(P, Q) \in C\left(J; Gl_p\left(\mathbb{R}\right)\right) \times C^1(J; Gl_p(\mathbb{R}))$. Then generalized transition matrix $Y\left(\cdot, \cdot\right)$ of the system is defined by

$$Y\left(t, t^0\right) := Q\left(t\right) \begin{bmatrix} \Phi_B\left(t, t^0\right) & 0 \\ 0 & 0 \end{bmatrix} Q\left(t^0\right)^{-1}, \; t, t^0 \in J, \tag{2.4}$$

where $\Phi_B\left(t, t^0\right)$ denotes the transition matrix of inherent ODE $\dot{x} = B\left(t\right)x$.

Consider DAE $(E, F) \in C\left((0, \infty); \mathbb{R}^{2\times 2}\right)^2$, which is transferable into SCF, as expressed in equation (2.3) with same $P'\left(t\right)$ and $Q'\left(t\right)$. Then equation (2.4) gives $Y\left(t, t^0\right) = I_2 \begin{bmatrix} e^{-\left(t-t^0\right)} & 0 \\ 0 & 0 \end{bmatrix} I_2^{-1} = \begin{bmatrix} e^{-\left(t-t^0\right)} & 0 \\ 0 & 0 \end{bmatrix}$.

A square matrix function $L\left(t\right)$ is called T-periodic if $\exists$ a constant $T > 0$ such that

$$L\left(t + T\right) = A\left(t\right), t \in \mathbb{R}.$$

3. SCF form of time-varying linear DAEs

3.1. **Standard Canonical Form.** We know that the SCF in equation (2.2) is unique in the sense that the dimensions of the ODE and the pure DAE are unique and that the ODE and pure DAE are unique up to some equivalence as given in equation (2.1) [1]. Consider homogeneous time-varying linear differential-algebraic equations of the following form:

$$L(t)\dot{z} = M\left(t\right)z, \tag{3.1}$$

where $(L, M) \in C\left(J; \mathbb{R}^{p\times p}\right)^2$ for some finite $p \in N$, $t \in J$ and $J \subseteq R$.

Lemma 3.1. *Let $A \in C\left(J; \mathbb{R}^{p\times p}\right)$ be a pointwise strictly lower triangular. Then the pure DAE*

$$A\left(t\right)\dot{z} = z, \tag{3.2}$$

has the unique global solution $z\left(\cdot\right) = 0$ and every local solution $y : K \longrightarrow \mathbb{R}^p$ of equation (3.2) satisfies $y\left(t\right) = 0$, $(\forall)$ $t \in K$.

Theorem 3.2. *Suppose $p_1, p_2, \widetilde{p}_1, \widetilde{p}_2 \in \mathbb{N}_0, B_1 \in C\left(J; \mathbb{R}^{p_1\times p_1}\right), B_2 \in C\left(J; \mathbb{R}^{\widetilde{p}_1\times\widetilde{p}_1}\right)$ and $A_1 \in C\left(J; \mathbb{R}^{p_2\times p_2}\right)$, $A_2 \in C\left(J; \mathbb{R}^{\widetilde{p}_2\times\widetilde{p}_2}\right)$, where A_1, A_2 are pointwise strictly lower triangular. If for some $(P, Q) \in C\left(J; Gl_p\left(\mathbb{R}\right)\right) \times C^1\left(J; Gl_p\left(\mathbb{R}\right)\right)$, we have*

$$\left(\begin{bmatrix} I_{p_1} & 0 \\ 0 & A_1 \end{bmatrix}, \begin{bmatrix} B_1 & 0 \\ 0 & I_{p_2} \end{bmatrix}\right) \overset{P,Q}{\sim} \left(\begin{bmatrix} I_{\widetilde{p}_1} & 0 \\ 0 & A \end{bmatrix}, \begin{bmatrix} B & 0 \\ 0 & I_{\widetilde{p}_2} \end{bmatrix}\right),$$

then we obtain

(1) $p_1 = \widetilde{p}_1, \; p_2 = \widetilde{p}_2$.
(2) $P = \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix}, Q = \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix}, Q_{11} = P_{11}^{-1}$.
(3) $\left(I_{P_1}, B_1\right) \overset{Q_{11}^{-1}, Q_{11}}{\sim} \left(I_{P_1}, B_2\right), \left(A_1, I_{P_2}\right) \overset{P_{11}, Q_{22}}{\sim} \left(A_2, I_{P_2}\right)$.

Proposition 3.3. *Consider* $(L,M)\in C\left(J;\mathbb{R}^{p\times p}\right)^2$ *, then we have* (L,M) *is transferable into SCF if and only if* (L,M) *is regular.*

3.2. **Transition Matrix and Variation of Constants Formula.** In this section, we define generalized transition matrix by using the transformation matrix and SCF and discuss the solution of DAEs by using the generalized transition matrix.

Proposition 3.4. *Suppose the DAE* $(L,M)\in C\left(J;\mathbb{R}^{p\times p}\right)^2$ *, which is transferable into SCF as described in equation (2.2). Then we have*

$$\text{(3.3)}\qquad \textbf{(1)}\ \left(t^0,z^0\right)\in W_{L,M} \text{ if and only if } z^0\in imQ\left(t^0\right)\begin{bmatrix} I_{p_1}\\ 0\end{bmatrix},$$

(**2**) Any solution of the IVP (3.1), $z\left(t^0\right)=z^0$, where $\left(t^0,z^0\right)\in W_{L,M}$, extends uniquely to a global solution $z(\cdot)$, and this solution satisfies

$$z(t)=Y\left(t,t^0\right)z^0,\ \ (\forall)\ \ t\in J, \tag{3.4}$$

where $Y\left(t,t^0\right)$ is the generalized transition matrix of (3.1) as defined in (3.1).

Proposition 3.5. *Suppose* $(L,M)\in C\left(J;\mathbb{R}^{p\times p}\right)^2$ *is transferable into SCF. Then* $Y(\cdot,\cdot)$ *defined in (2.4) is independent of the choice of* (P,Q) *in (2.2).*

Proposition 3.6. *Suppose* $(L,M)\in C\left(J;\mathbb{R}^{p\times p}\right)^2$ *which is transferable into SCF and has generalized transition matrix* $Y(\cdot,\cdot)$. *Then* $Y(\cdot,\cdot)$ *has the following properties for all* $t,s,r\in J$,

(1) $L(t)\frac{d}{dt}Y(t,r)=M(t)Y(t,r)$,
(2) $imY(t,r)=W_{L,M}(t)$,
(3) $Y(t,s)Y(s,r)=Y(t,r)$,
(4) $Y(t,r)^2=Y(t,r)$,
(5) $Y(t,r)z=z,\ \forall z\in W_{L,M}(t)$,
(6) $\frac{d}{dt}Y(r,t)=-Y(r,t)Q(t)P(t)M(t)$.

Theorem 3.7. *Suppose* $(L,M)\in C\left(J;\mathbb{R}^{p\times p}\right)^2$ *is transferable into SCF and* $t^0\in J$, *then the linear map*

$$\psi: W_{L,M}\left(t^o\right)\longrightarrow\{z: J\longrightarrow\mathbb{R}^p \mid z(\cdot)\ \textit{ is a global solution of (3.1)}\},$$

such that

$$\psi(z^0)=Y\left(\cdot,t^0\right)z^0,$$

is a vector space isomorphism.

Corollary 3.8. *If* $(L,M)\in C\left(J;\mathbb{R}^{p\times p}\right)^2$ *is transferable into SCF then* $\dim W_{L,M}(\cdot)$ *is constant.*

Consider inhomogeneous time-varying linear differential-algebraic equations of the form

$$L(t)\dot{z}=M(t)z+g(t),\ z\left(t^0\right)=z^0, \tag{3.5}$$

where $(L,M)\in C\left(J;\mathbb{R}^{p\times p}\right)^2$, $g\in C\left(J;\mathbb{R}^p\right)$, $(t^0,z^0)\subseteq\mathbb{R}\times\mathbb{R}^p$ for finite $p\in\mathbb{N}$ and $J\subseteq\mathbb{R}$.

Theorem 3.9. *Suppose that the DAE $(L,M) \in C^p\left(J;\mathbb{R}^{p\times p}\right)^2$ is transferable into SCF by some transformation matrices $(P,Q) \in C^p\left(J;Gl_p\left(\mathbb{R}\right)\right)$. Then the statements described in following holds for $g \in C^{p_2}\left(J;\mathbb{R}^p\right)$:*

(1) *The IVB (3.5) has a solution if, and only if,*

$$z^0 + Q\left(t^0\right)\begin{bmatrix} 0 \\ I_{p_2}\end{bmatrix}\left(\sum_{k=0}^{p_2-1}\left(A\left(\cdot\right)\tfrac{d}{dt}\right)^k \begin{bmatrix} 0 & I_{p_2}\end{bmatrix} P\left(\cdot\right)g\left(\cdot\right)\right)\Bigg|_{t=t^0} \in imQ\left(t^0\right)\begin{bmatrix} I_{p1} \\ 0\end{bmatrix}. \tag{3.6}$$

(2) *Any solution of equation (3.5) such that equation (3.6) holds can be uniquely extended to a global solution $z\left(\cdot\right)$, and this solution satisfies, for the generalized transition matrix $Y\left(\cdot,\cdot\right)$ of (L,M) and all $t \in J$,*

$$\begin{aligned} z\left(t\right) = {} & Y\left(t,t^0\right)z^0 + \int_{t^0}^{t} Y\left(t,r\right)Q\left(r\right)P(r)g\left(r\right)dr \\ & - Q\left(t\right)\begin{bmatrix} 0 \\ I_{p_2}\end{bmatrix}\sum_{k=0}^{p_2-1}\left(A\left(\cdot\right)\frac{d}{dt}\right)^k\begin{bmatrix} 0 & I_{p_2}\end{bmatrix}P\left(t\right)g\left(t\right). \end{aligned} \tag{3.7}$$

Theorem 3.10. *Let $A \in C^p\left(J;\mathbb{R}^{p\times p}\right)$ be pointwise strictly lower triangular, $g \in C^p\left(J;\mathbb{R}^p\right)$ and $\left(t^0,z^0\right) \in J\times\mathbb{R}^p$. Then the IVP*

$$A(t)\dot{z} = z + g(t),\ z\left(t\right) = z^0, \tag{3.8}$$

has a solution iff

$$-\sum_{k=0}^{p-1}\left(A\left(\cdot\right)\frac{d}{dt}\right)^k g\left(t\right)\Bigg|_{t=t^0} = z^0.$$

Any solution of (3.8) can be uniquely extended to a global solution $z\left(\cdot\right)$, and this solution

$$z\left(t\right) = -\sum_{k=0}^{p-1}\left(A\left(\cdot\right)\frac{d}{dt}\right)^k g\left(t\right),\ t \in J. \tag{3.9}$$

3.3. **Computing SCF.** In this section, three main algorithms are presented in **quasi-MATLAB code** for computing the SCF of DAEs (L,M) with real analytic coefficients and this algorithm also decides whether (L,M) is transferable into SCF or not.

Algorithm 3.11. ***(Function transfSCF)***

(1) *function* $[P,Q,A,B] =$ *transfSCF* (L,M)
(2) *reached SCF*$= 0$; *% initial value for global variable*
(3) $[P_1,Q_1,A_1,B_2] =$ *getSCF* (L,M);
(4) $n :=$*size*(B);
(5) $P := \begin{bmatrix} I_n & 0 \\ 0 & \begin{bmatrix} & & 1 \\ & ⁄ & \\ 1 & & \end{bmatrix}\end{bmatrix} P_1;\ Q := Q_1\begin{bmatrix} I_n & 0 \\ 0 & \begin{bmatrix} & & 1 \\ & ⁄ & \\ 1 & & \end{bmatrix}\end{bmatrix}$;

(6) $A := \begin{bmatrix} & & 1 \\ & \diagup & \\ 1 & & \end{bmatrix} A_1 \begin{bmatrix} & & 1 \\ & \diagup & \\ 1 & & \end{bmatrix}$; $B = B_1$.

Algorithm 3.12. ***(Function get SCF)***

(1) *function* $[P, Q, A, B]$ =*getSCF* (L, M)
(2) $[L_1, L_2, M_1, M_2, H, E, F]$:=*reduce* (L, M);
(3) *if reachedSCF*= 0 *then*
(4) $[P_1, Q_1, A_1, B_1]$ =*getSCF* (L_1, M_1);
(5) *else if* $L \equiv 0$, *then*
(6) $A_1 := 0$; $B_1 := \emptyset$, $P_1 := Q_1 := I$; % *set* $B = \emptyset$ *if the matrix* B *is absent*
(7) *else*
(8) $A_1 := \emptyset$; $B_1 := L_1^{-1} M$, $P_1 := L_1^{-1}$, $Q_1 := I$;
(9) *end if*
(10) n_1 :=*size*(B_1); n_2 :=*size*(A_1); % *the size of an empty matrix is* 0
(11) $\begin{bmatrix} \widetilde{L}_1 \\ \widetilde{L}_2 \end{bmatrix} := P_1 L_2$ *s.t.* $\widetilde{L}_i$ *has* n_i *rows,* $i = 1, 2$;
(12) $\begin{bmatrix} \widetilde{M}_1 \\ \widetilde{M}_2 \end{bmatrix} := P_1 M_2$ *s.t.* $\widetilde{M}_i$ *has* n_i *rows,* $i = 1, 2$;
(13) $$P := \begin{bmatrix} I_{n_1} & 0 & \frac{d}{dt}\widetilde{L}_1 + B_1 \widetilde{L}_1 - \widetilde{M}_1 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & I_{size(L)-n_1-n_2} \end{bmatrix} \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & I_{n_2} & -\widetilde{M}_2 H^{-1} \\ 0 & 0 & H^{-1} \end{bmatrix} \times \begin{bmatrix} P_1 & 0 \\ 0 & I_{size(L)-n_1-n_2} \end{bmatrix} E;$$
(14) $$Q := F \begin{bmatrix} Q_1 & 0 \\ 0 & I_{size(L)-n_1-n_2} \end{bmatrix} \begin{bmatrix} I_{n_1} & 0 & -\widetilde{L}_1 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & I_{size(L)-n_1-n_2} \end{bmatrix};$$
(15) $B := B_1$, $A := \begin{bmatrix} A_1 & \widetilde{L}_2 \\ 0 & 0 \end{bmatrix}$, *such that size*$(A)$+*size*$(B)$ =*size*(L).

Algorithm 3.13. ***(Function reduce)***

(1) *function* $[L_1, L_2, M_1, M_2, H, E, F]$ =*reduce* (L, M),
(2) *if* $L \equiv 0$ *or* $(\forall t \in I : \det L(t) \neq 0)$ *then*
(3) $L_1 := L$; $M_1 := M$; $L_2 := M_2 := H := \emptyset$; $E := F := I$;
(4) *reached SCF*:= 1;
(5) *else if not*$(\forall t \in I : \det L(t) = 0)$, *then*
(6) ***print*** *"Not Transferable into SCF"* ***STOP***
(7) *else*
(8) *determine (minimal)* $n < p :=$ *size* (L) *such that* $rkL(t) \leq n < p$ *for all* $t \in I$ *and* $E : J \to \mathbb{R}^{p \times p}$ *real analytic and pointwise nonsingular such that* $EL = \begin{bmatrix} \hat{L}_1 & \hat{L}_2 \\ 0 & 0 \end{bmatrix}$ *where* $\hat{L}_1 \in \mathbb{R}^{p \times p}$;

(9) $\begin{bmatrix} \hat{M}_{11} & \hat{M}_{12} \\ \hat{M}_{21} & \hat{M}_{22} \end{bmatrix} := EM,$ *where* $\hat{M}_{11} \in \mathbb{R}^{p\times p}$;

(10) *if not*$\left(\forall t \in I:\ rk \begin{bmatrix} \hat{M}_{21} & \hat{M}_{22} \end{bmatrix} = p - n = \max\right)$ *then*

(11) ***print*** *"Not Transferable into SCF"* ***STOP***

(12) *else*

(13) *choose* $F : J \to \mathbb{R}^{p\times p}$ *real analytic and pointwise nonsingular such that* $[\hat{M}_{21}, \hat{M}_{21}]Q = \begin{bmatrix} 0_{(p-n)\times n} & H \end{bmatrix}$, $\det H(t) \neq 0\ \forall t \in J$;

(14) $\begin{bmatrix} L_1 & L_2 \end{bmatrix} := \begin{bmatrix} \hat{L}_1 & \hat{L}_2 \end{bmatrix} F$;

(15) $\begin{bmatrix} M_1 & M_2 \end{bmatrix} := \begin{bmatrix} \hat{M}_{11} & \hat{M}_{12} \end{bmatrix} F - \begin{bmatrix} \hat{L}_1 & \hat{L}_2 \end{bmatrix} \dot{F}$;

(16) *end if*

(17) *end if*

4. Floquet theory of index one tractable DAEs

Consider linear homogenous time-varying DAE with continuous coefficients

$$L(t)\,\dot{z} + M(t)z = 0, \tag{4.1}$$

where $(L, M) \in C(\mathbb{R}, \mathbb{R}^{p\times p})^2$ for finite $p \in \mathbb{N}$.and $t \in\mathbb{R}$.

Consider $F(t) := \ker L(t)$ is the the null-space. Suppose $F(t)$ to be smooth that is to be spanned by continuously differential basis functions. In particular, $F(t)$ then has constant rank. Obviously, all solutions of equation (4.1) belong to the subspace,

$$R(t) := \{z \in \mathbb{R}^p : M(t)z \in imL(t)\} \subset \mathbb{R}^p.$$

Assume that equation (4.1) is index-1-tractable, that is

$$R(t) \cap F(t) = \{0\}.$$

Then exactly one solution passes through each point of $R(t)$ at time t [7]. Using any C^1 projector function $G(t)$ onto $F(t)$ and $H(t) := I - G(t)$, initial value problems (IVPs) are properly stated with the initial condition

$$G(t)\left(z(0) - z^0\right) = 0. \tag{4.2}$$

IVPs (4.1) and (4.2 are uniquely solvable for all $z^0 \in \mathbb{R}^P$. The solution of the DAE (4.1) should belong to the function space,

$$C_F^1 := \left\{z \in C : Gz \in C^1\right\}.$$

By using the fundamental matrix $U(t, 0)$, which is the solution of the matrix-valued IVP, we have

$$L(t)\,\dot{U}(t,0) + M(t)U(t,0) = 0,$$
$$G(t)\left(U(0,0) - U^0\right) = 0.$$

We write the solutions of equation (4.1) and equation (4.2) as

$$z(t) = U(t,0)z^{\cdot 0}.$$

In the above equation U represents the fundamental matrix of the DAE as we know that V is the fundamental matrix of the inherent ODE [7].

$$\dot{V} + \left[-GG_{can} + G(L+MG)^{-1}M\right]V = 0,\ V(0) = I \in \mathbb{R}^{p\times p}, \tag{4.3}$$

where G_{can} represents the canonical projector along $F(t)$ onto $R(t)$. Then

$$U(t,0) = G_{can}(t)\,V(t)G(t)\,. \tag{4.4}$$

Consider linear homogenous ODE with periodic coefficient,

$$\dot{z} + B(t)z = 0, \tag{4.5}$$

where $B \in C(\mathbb{R}, \mathbb{R}^{p\times p})$, $B(t) = B(t+T)$, $\forall t \in \mathbb{R}$ and T is a positive constant. Its fundamental matrix is $V(t)$ with

$$\dot{V}(t) + B(t)V(t) = 0,\ V(t) = I.$$

Floquet describes the decomposition of fundamental matrix of system (4.5).

Theorem 4.1. *The fundamental matrix $V(t)$ can be written in the form*

$$V(t) = E(t)\,e^{tB_0},$$

where $B \in \mathbb{C}^{p\times p}$, $E \in C^1(\mathbb{R}, \mathbb{C}^{p\times p})$ is nonsingular, $E(t) = E(t+T)$, $\forall t \in \mathbb{R}$ and T is positive constant.

Now we consider linear homogenous time-varying DAEs with periodic coefficients

$$L(t)\,\dot{z} + M(t)z = 0, \tag{4.6}$$

where $(L, M) \in C(\mathbb{R}, \mathbb{R}^{p\times p})^2$, $L(t) = L(t+T)$, $M(t) = M(t+T)$, $\forall t \in \mathbb{R}$ and $T > 0$. Since $L(\cdot)$ are $M(\cdot)$ are T-periodic coefficients then $F(\cdot)$ and $R(\cdot)$, are also T-periodic. We use the natural splitting of the following form

$$\mathbb{R}^p = F(t) \oplus R(t)\,,$$

for index-1 tractable DAEs. We span $F(\cdot)$ by T-periodic C^1 function,

$$F(t) = span\{\eta_{s+1}(t), \cdots, \eta_p(t)\},\ s = rkL(t)\,,$$

and $R(\cdot)$ may be only continuous, let $R(\cdot)$ be spanned by T-periodic continuous function,

$$R(t) = span\{r_1(t), \cdots, r_s(t)\}\,.$$

Since G_{can} projects onto R along F, we have the representation

$$G_{can}(t) = Q(t)\begin{bmatrix} I & \\ & 0 \end{bmatrix} Q^{-1}(0)\,, \tag{4.7}$$

where

$$Q(t) := [r_1(t), \cdots, r_s(t), n_{s+1}(t), \cdots, n_n(t)] \in \mathbb{R}^{p\times p}. \tag{4.8}$$

We choose a projector $G(\cdot)$ along $F(\cdot)$, so that G is not only smooth but also periodic and

$$G(0) = G_{can}(t)\,. \tag{4.9}$$

From equation (4.4), we have

$$U(t,0) = G_{can}(t)\,V(t)G(0)\,.$$

By using equation (4.9), we have

$$\begin{aligned} U(t,0) &= G_{can}(t)\,V(t)G_{can}(0) \\ &= Q(t)\begin{bmatrix} I & \\ & 0 \end{bmatrix} Q^{-1}(t)\,V(t)Q(0)\begin{bmatrix} I & \\ & 0 \end{bmatrix} Q^{-1}(0) \\ &= Q(t)\begin{bmatrix} X(t) & \\ & 0 \end{bmatrix} Q^{-1}(0), \end{aligned}$$

where $X \in C(\mathbb{R}, \mathbb{R}^{s\times s})$, $X(0) = I$ and the Monodromy matrix $U(T,0)$ is of the following form

$$U(T,0) = Q(T)\begin{bmatrix} X(T) & \\ & 0 \end{bmatrix} Q^{-1}(0) = Q(0)\begin{bmatrix} X(T) & \\ & 0 \end{bmatrix} Q^{-1}(0). \tag{4.10}$$

Since rank $X(t) = s$ is constant so $X(t) \in \mathbb{R}^{s\times s}$ is nonsingular for all $t \in \mathbb{R}$.

From the theory of linear algebra [12], it is known that every nonsingular matrix $D \in \mathbb{R}^{s\times s}$ can written in the form

$$D = e^{B} \text{ with } B \in \mathbb{C}^{s\times s},$$

and

$$D^2 = e^{\bar{B}} \text{ with } \bar{B} \in \mathbb{R}^{s\times s}.$$

Now, let

$$X(T) = e^{TB_0},\ B_0 \in \mathbb{C}^{s\times s},$$

and

$$\begin{aligned} X(2T) &= e^{2TW_0},\ W_0 \in \mathbb{C}^{s}, \\ &= X(T)^2. \end{aligned}$$

respectively. Here

$$X(2T) = X(T)^2,$$

from the corresponding property of X and the relation $W(2T) = W(T)^2 = W(0)$.

By using the Theorem of Floquet for ODEs 4.1, we set

$$\begin{aligned} E_k(t) &:= Q(t)\begin{bmatrix} X(t)e^{-tB_0} & \\ & I \end{bmatrix} \\ &= U(t,0)Q(0)\begin{bmatrix} e^{-tB_0} & \\ & 0 \end{bmatrix} + Q(t)\begin{bmatrix} 0 & \\ & 1 \end{bmatrix}. \end{aligned}$$

Now we state and proof the Floquet decomposition of fundamental matrix of periodic DAEs.

Theorem 4.2. *The fundamental matrix $U(t,0)$ of (L, M) can be written as*

$$U(t,0) = E(t)\begin{bmatrix} e^{tB_0} & \\ & 0 \end{bmatrix} E(0)^{-1},$$

where $E \in C^1_F(\mathbb{R}, \mathbb{C}^{p\times p})$ is nonsingular and T-periodic.

Now, we prove a result for DAEs as a generalization of known ODEs result.

Theorem 4.3. *A solution $z(\cdot)$ of T-periodic time varying linear DAE*

$$L(t)\dot{z}(t) = M(t)z(t) + g(t), \; z\left(t^0\right) = z^0, \tag{4.11}$$

is T-periodic if and only if, $z\left(t^0 + T\right) = z^0$.

Proof. If $z(\cdot)$ is T-periodic, then clearly $z\left(t^0 + T\right) = z^0$. Conversely, suppose z^0 is such that the corresponding solution of equation (4.11) satisfies $z\left(t^0 + T\right) = z^0$. Letting $x(t) := z(t+T) - z(t)$, it follows that $x\left(t^0\right) = 0$ and

$$\dot{x}(t) = \dot{z}(t+T) - \dot{z}(t).$$

Multiplying the above equation by $L(\cdot)$, we get

$$L(t)\dot{x}(t) = L(t)\dot{z}(t+T) - L(t)\dot{z}(t).$$

Since L, M and g are T-periodic. Therefore, the above equation becomes

$$\begin{aligned} L(t)\dot{x}(t) &= L(t+T)\dot{z}(t+T) - L(t)\dot{z}(t), \\ &= [M(t+T)z(t+T) + g(t+T)] - [M(t)z(t) + g(t)], \\ &= [M(t)z(t+T) + g(t)] - [M(t)z(t) + g(t)], \\ &= M(t)[z(t+T) - z(t)], \\ &= M(t)x(t), \end{aligned}$$

but uniqueness of solution implies that $x(t) = 0$ for all t, that is

$$z(t+T) = z(t),$$

which shows $z(\cdot)$ is T-periodic. This completes the proof of theorem. ☐

5. Floquet theory of systems with standard canonical form

Consider linear homogenous time-varying periodic DAE transferable into SCF,

$$L(t)\dot{z}(t) + M(t)z(t) = 0, \; z(t) = z^0, \tag{5.1}$$

where $(L, M) \in C\left(J, \mathbb{R}^{p\times p}\right)^2$, $L(t) = L(t+T)$, $M(t) = M(t+T)$, $\forall\, t \in J \subseteq \mathbb{R}$ and T is a positive constant. The Floquet theory requires some strict conditions for the solution of periodic DAE. The Floquet theory demands constant rank condition on the singular coefficient matrix for the solution of periodic DAEs but there is no such condition in the solution theory of periodic DAEs transferable into SCF. This fact is obvious from the following example.

Example 5.1. Consider the time-varying periodic DAE

$$L(t)\dot{z} = M(t)z,$$

where

$$L(t) = \begin{bmatrix} \sin(t) & \cos(t) & 0 \\ 0 & 0 & 0 \\ -\cos(t)\sin(t) & \sin^2(t) & 0 \end{bmatrix}, \; t \in \mathbb{R},$$

$$M(t) = \begin{bmatrix} \sin(t) - \cos(t) & \cos(t) & 0 \\ -\cos(t) & \sin(t) & 0 \\ \sin^2(t) & -\sin(t)\cos(t) & t^2 + 1 \end{bmatrix}, \; t \in \mathbb{R}.$$

Note that rank of L is not constant. We show that (L, M) is transferable into SCF by applying Algorithm 3.11.

transf SCF(L, M)
reached SCF= 0, $[P_1, Q_1, A_1, B_2]$=get SCF(L, M),
get SCF(L, M) (First Instance):
$[L_1^1, L_2^1, M_1^1, M_2^1, H^1, E^1, F^1]$ =reduce (L, M),
reduce(L, M) : Conditions in rows 2 and 5 of Algorithm 3.13 not fulfilled, go to rows 8 to 16,
for $n = 2$

$$\hat{L}_1 = \begin{bmatrix} \sin(t) & \cos(t) \\ -\cos(t)\sin(t) & \sin^2(t) \end{bmatrix}, \; \hat{L}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Choose $E^1(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, such that

$$E^1(t)\, L(t) = \begin{bmatrix} \sin(t) & \cos(t) & 0 \\ -\cos(t)\sin(t) & \sin^2(t) & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{L}_1 & \hat{L}_2 \\ 0 & 0 \end{bmatrix},$$

and evaluate

$$\begin{bmatrix} \hat{M}_{11} & \hat{M}_{12} \\ \hat{M}_{21} & \hat{M}_{22} \end{bmatrix} = E^1(t)\, M(t) = \begin{bmatrix} \sin(t) - \cos(t) & \cos(t) + \sin(t) & 0 \\ \sin^2(t) & -\sin(t)\cos(t) & t^2+1 \\ -\cos(t) & \sin(t) & 0 \end{bmatrix}.$$

Condition in row 10 of Algorithm 3.13 not fulfilled, go to rows 13 to 15,

choose $F^1(t) = \begin{bmatrix} 0 & \sin(t) & -\cos(t) \\ 0 & \cos(t) & \sin(t) \\ 1 & 0 & 0 \end{bmatrix}$, such that

$$\begin{bmatrix} \hat{M}_{21} & \hat{M}_{22} \end{bmatrix} F^1(t) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

Then we have

$$\begin{bmatrix} L_1^1(t) & L_2^1(t) \end{bmatrix} = \begin{bmatrix} \hat{L}_1 & \hat{L}_2 \end{bmatrix} F^1 = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & \sin(t) \end{array}\right].$$

$$\begin{aligned} \begin{bmatrix} M_1^1(t) & M_2^1(t) \end{bmatrix} &= \begin{bmatrix} \hat{M}_{11} & \hat{M}_{12} \end{bmatrix} F^1 - \begin{bmatrix} \hat{L}_1 & \hat{L}_2 \end{bmatrix} \dot{F}^1 \\ &= \left[\begin{array}{cc|c} 0 & 1 & 0 \\ t^2+1 & 0 & 0 \end{array}\right]. \end{aligned}$$

Condition in row 3 of Algorithm 3.12 fulfilled, go to row 4,
$[P_1^2, Q_1^2, A_1^2, B_1^2]$=get SCF$\left(L_1^1, M_1^1\right)$,
get SCF$\left(L_1^1, M_1^1\right)$ (**second instance**):
$[L_1^2, L_2^2, M_1^2, M_2^2, H^2, E^2, F^2]$ =reduce $\left(L_1^1, M_1^1\right)$,
reduce$\left(L_1^1, M_1^1\right)$: Conditions in rows 2 and 5 of Algorithm 3.13 not fulfilled, go to rows 8 to 16, for $n = 1$

$$\begin{bmatrix} \hat{L}_1 & \hat{L}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Choose $E^2(t) = I_2$, such that

$$E^2(t)\,L^1(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{L}_1 & \hat{L}_2 \\ 0 & 0 \end{bmatrix}.$$

and evaluate

$$\begin{bmatrix} \hat{M}_{11} & \hat{M}_{12} \\ \hat{M}_{21} & \hat{M}_{22} \end{bmatrix} = E^2(t)\,M_1^1(t) = \begin{bmatrix} 0 & 1 \\ t^2+1 & 0 \end{bmatrix}.$$

Condition in rows 10 of Algorithm 3.13 not fulfilled, go to rows 13 to 15, choose $F^2(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, such that

$$\begin{bmatrix} \hat{M}_{21} & \hat{M}_{22} \end{bmatrix} F^2(t) = \begin{bmatrix} 0 & t^2+1 \end{bmatrix}.$$

Then we have

$$\begin{bmatrix} L_1^2(t) & L_2^2(t) \end{bmatrix} = [0\ |1].$$

and

$$\begin{bmatrix} M_1^2(t) & M_2^2(t) \end{bmatrix} = [1\ |0].$$

get SCF $\left(L_1^2, M_1^2\right)$. (**Third Instance**):

$[L_1^3, L_2^3, M_1^3, M_2^3, H^3, E^3, F^3] =$ reduce $\left(L_1^2, M_1^2\right)$,

reduce$\left(L_1^2, M_1^2\right)$: Conditions in rows 2 and 5 of Algorithm 3.13 fulfilled, go to rows 3 and 4, $L_1^3 = L_1^2$, $M_1^3 = M_1^2$, $L_2^3 = M_2^3 = H^3 = \emptyset$, $E^3 := F^3 = I$, reached SCF=1, conditions in row 3 and 5 of Algorithm 3.12 not fulfilled, go to row 8,

$$A_1^4 = \emptyset,\ B_1^4 = (L_1^3)^{-1}M_1^3 = 1,\ P_1^4 = L_1^{-1} = 1,\ Q_1^4 = 1.$$

(**Third Instance**) n_1 =size$\left(B_1^4\right)$, $\Rightarrow n_1 = 1$. n_2 =size$\left(A_1^4\right)$, $\Rightarrow n_2 = 1$.

$$\begin{bmatrix} \tilde{L}_1 \\ \tilde{L}_2 \end{bmatrix} = P_1^4 L_2^3 = \begin{bmatrix} \emptyset \\ \emptyset \end{bmatrix},\quad \begin{bmatrix} \tilde{M}_1 \\ \tilde{M}_2 \end{bmatrix} = P_1 M_2 = \begin{bmatrix} \emptyset \\ \emptyset \end{bmatrix}.$$

$$\Rightarrow\ \tilde{L}_1 = \tilde{L}_2 = \tilde{M}_1 = \tilde{M}_2 = \emptyset.$$

Then we have

$$P_1^3 = P_1^4 E^3 = 1,\ Q_1^3 = F^3 Q_1^4 = 1,$$

Also

$$B_1^3 = B_1^4 = 1,\ A_1^3 = \begin{bmatrix} A_1^4 & \tilde{L}_2 \\ 0 & 0 \end{bmatrix} = \emptyset,$$

such that $size\left(A_1^3\right) + size\left(B_1^3\right) = size\left(L_1^3\right)$.

(Second Instance) n_1 =size$\left(B_1^3\right)$, $\Rightarrow n_1 = 1$. n_2 =size$\left(A_1^3\right)$, $\Rightarrow n_2 = 0$.

$$\begin{bmatrix} \tilde{L}_1 \\ \tilde{L}_2 \end{bmatrix} = P_1^3 L_2^2 = \begin{bmatrix} 0 \\ \emptyset \end{bmatrix},\quad \begin{bmatrix} \tilde{M}_1 \\ \tilde{M}_2 \end{bmatrix} = P_1^3 M_2^2 = \begin{bmatrix} 0 \\ \emptyset \end{bmatrix},$$

$$\Rightarrow\ \tilde{L}_1 = \tilde{M}_1 = 0,\ \tilde{L}_2 = \tilde{M}_2 = \emptyset.$$

Then we obtain

$$P_1^2 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{t^2+1} \end{bmatrix}, \; Q_1^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and

$$B_1^2 = B_1^3 = 1, \; A_1^2 = \begin{bmatrix} A_1^3 & \widetilde{L}_2 \\ 0 & 0 \end{bmatrix} = 0.$$

(First Instance) $n_1 = \text{size}(B_1^2)$, $\Rightarrow n_1 = 1$. $n_2 = \text{size}(A_1^2)$, $\Rightarrow n_2 = 0$.

$$\begin{bmatrix} \widetilde{L}_1 \\ \widetilde{L}_2 \end{bmatrix} = P_1^2 L_2^1 = \begin{bmatrix} 0 \\ \frac{\sin(t)}{t^2+1} \end{bmatrix},$$

$$\Rightarrow \; \widetilde{L}_1 = 0, \; \widetilde{L}_2 = \frac{\sin(t)}{t^2+1}.$$

$$\begin{bmatrix} \widetilde{M}_1 \\ \widetilde{M}_2 \end{bmatrix} = P_1^2 M_2^1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\Rightarrow \; \widetilde{M}_1 = \widetilde{M}_2 = 0.$$

Then we obtain

$$P_1^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{t^2+1} \\ 0 & 1 & 0 \end{bmatrix}. \; Q_1^2 = \begin{bmatrix} \sin(t) & 0 & -\cos(t) \\ \cos(t) & 0 & \sin(t) \\ 0 & 1 & 0 \end{bmatrix},$$

and

$$B_1^1 = B_1^2 = 1, \; A_1^1 = \begin{bmatrix} A_1^2 & \widetilde{L}_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{\sin(t)}{t^2+1} \\ 0 & 0 \end{bmatrix},$$

back in trans SCF $r = \text{size}(B)$, $\Rightarrow r = 1$. We have

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{t^2+1} \end{bmatrix},$$

$$Q = Q_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \sin(t) & -\cos(t) & 0 \\ \cos(t) & \sin(t) & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{\sin(t)}{t^2+1} & 0 \end{bmatrix}, \; B = B_1 = 1.$$

Thus time-varying periodic DAE (L, M) is transferable into SCF in the following form

$$(L, M) \overset{P,Q}{\sim} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{\sin(t)}{t^2+1} & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right), \; t \in \mathbb{R}.$$

So we have

$$P(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{t^2+1} \end{bmatrix}, \ Q(t) = \begin{bmatrix} \sin(t) & -\cos(t) & 0 \\ \cos(t) & \sin(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ t \in \mathbb{R}.$$

Definition 5.2. Suppose $(L, M) \in C(J; \mathbb{R}^{p\times p})^2$, which is interchageable into SCF for some $(P, Q) \in C(J; Gl_p(\mathbb{R})) \times C^1(J; Gl_p(\mathbb{R}))$ and $Q(t)$ is periodic. Then the generalized transition matrix $Y(\cdot, 0)$ of system (5.1) is given as

$$Y(t,0) := Q(t) \begin{bmatrix} \Phi_B(t,0) & \\ & 0 \end{bmatrix} Q(0)^{-1}, \ t, \in J,$$

where $\Phi_B(t,0)$ denotes the transition matrix of inherent ODE $\dot{x} = Bx$.

We will decompose the generalized transition matrix of system (5.1). This Floquet type decomposition can be used to investigate solution properties of periodic DAEs transferable into SCF.

Theorem 5.3. *The generalized transition matrix $Y(t,0)$ of periodic DAEs transferable into SCF (5.1) can be written in the form*

$$Y(t,0) = E(t) \begin{bmatrix} e^{tB_0} & \\ & 0 \end{bmatrix} E(0)^{-1},$$

where $E(t) \in C^1(J, Gl_p(\mathbb{R}))$ and T-periodic.

Proof. Assume that $Q(t)$ is periodic and

$$E(t) = Q(t) \begin{bmatrix} \Phi_B(t,0)e^{-tB_0} & \\ & I \end{bmatrix}. \tag{5.2}$$

Since, we know that

$$Y(t,0) = Q(t) \begin{bmatrix} \Phi_B(t,0) & \\ & 0 \end{bmatrix} Q(0)^{-1}. \tag{5.3}$$

From equation (5.2), we have

$$Q(t) = E(t) \begin{bmatrix} e^{tB_0}\Phi_B(t,0)^{-1} & \\ & I \end{bmatrix}, \tag{5.4}$$

and

$$Q(0) = E(0) \begin{bmatrix} \Phi_B(0,0)^{-1} & \\ & I \end{bmatrix}. \tag{5.5}$$

Since

$$\Phi_B(0,0) = I.$$

By using the equation (5.5), we have

$$Q(0)^{-1} = E(0)^{-1}. \tag{5.6}$$

By substituting equations (5.4) and (5.6) in equation (5.3), we have

$$Y(t,0) = E(t)^{-1} \begin{bmatrix} e^{tB_0}\Phi_B(0,0)^{-1} & \\ & I \end{bmatrix} \begin{bmatrix} \Phi_B(0,0) & \\ & 0 \end{bmatrix} E(0)^{-1},$$

$$= E(t)^{-1} \begin{bmatrix} e^{tB_0} & \\ & 0 \end{bmatrix} E(0)^{-1}.$$

This completes the proof. □

6. Conclusion

In this paper, we have presented a collection of basic definitions on linear DAEs. We have showed DAEs can be transformed into SCF. This class is a time-varying generalization of time invariants DAEs, where the corresponding matrix pencil is regular. It has also been discussed that in which sense the SCF is a canonical form, that allows for a transition matrix similar to the one for ODEs, and how this can be exploited to derive a variation of constants formula. At the end, we have presented an algorithm which determines the transformation matrices which put a DAE into SCF. We also showed that Floquet theory of DAEs demands a strict constant rank condition on the singular coefficient matrix which is not required by solution theory of DAEs transferable into SCF. Moreover, we have obtained the Floquet type result for periodic DAEs which are transferable into SCF.

References

[1] T. Berger and A. Ilchmann, *On the standard canonical form of time-varying linear DAEs*, Q. Appl. Math. **71** (2013), 69–87.
[2] S. L. V. Campbell, *Singular systems of differential equations II*, vol. 61, Pitman, London, 1982.
[3] V. Campbell, *One canonical form for higher-index linear time-varying singular systems*, Circuits Syst. Signal Process **2** (1983), 311–326.
[4] G. Denk and R. Winkler, *Modelling and simulation of transient noise in circuit simulation*, Math. Comput. Model. Dyn. Syst. **13** (2007), 383–394.
[5] E. Emmrich and V. Mehrmann, *Operator differential-algebraic equations arising in fluid dynamics*. Comput. Methods Appl. Math. **13** (2013), 443–470.
[6] F. R. Gantmacher *The Theory of Matrices*, Chelsea Pub. Co., New York, 1959.
[7] E. Griepentrog and R. März, *Differential-Algebraic Equations and Their Numerical Treatment*, (vol.88), BSB Teubner, 1986.
[8] A. Kumar and P. Daoutidis, *Feedback control of nonlinear differential-algebraic-equation systems*, AIChE J. **41** (1995), 619–636.
[9] P. Kunkel and V. Mehrmann, *Differential-Algebraic Equations: Analysis and Numerical Solution*, (vol. 2), European Mathematical Society, 2006.
[10] R. Lamour, R. März, and R. Winkler, *How Floquet theory applies to index 1 differential algebraic equations*, J. Math. Anal. Appl. **217** (1998), 372–394.
[11] P. C. Müller, *Linear mechanical descriptor systems: identification, analysis and design*, IFAC Proc. **30** (1997), 375–380.
[12] L. S. Pontryagin, *On some differential games*, J. SIAM Control Ser. A, **3** (1965), 49–52.
[13] R. Riaza, *Differential-Algebraic Systems: Analytical Aspects and Circuit Applications*, World Scientific, 2008.

Manuscript received August 5 2022
revised September 27 2002

A. Younus
CASPAM, BZU, Multan, Pakistan
E-mail address: awais@bzu.edu.pk

N. Yasmeen
CASPAM, BZU, Multan, Pakistan
E-mail address: nudratyasmeen789@gmail.com