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QUALITATIVE RESULTS FOR DIFFERENTIAL EQUATIONS OF FOURTH ORDER WITH MULTIPLE TIME DELAYS

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Abstract. In this paper, we consider a nonlinear delay differential equations (DDEs) of fourth order with multiple n-constant delays. Some new sufficient conditions to ensure uniformly boundedness (UB), ultimately uniformly boundedness (UUB), uniformly asymptotically stability (UAS) and existence of periodic solutions (EPSs) of certain nonlinear DDEs of fourth order with multiple n-constant time delays are presented. The technique used in the proofs depends on Lyapunov-Krasovskii functional (LKF) method and construction of a proper LKF. We give two theorems and a corollary on these properties of solutions. This paper has new results to the theory of DDEs of fourth order and these results include and generalize some earlier results on the mentioned concepts.

1. Introduction

In relative literature, qualitative behaviors of solutions, stability, boundedness, convergence, existence of periodic solutions, etc., nonlinear differential equations of fourth order with and without delay have been investigated by many researchers, see, the papers ([1–6, 8–11, 13–18, 20–33, 35–54, 56, 57]), [12], the book [34] and the references of these sources. In the mentioned research works, in generally, the second method of Lyapunov is used to investigate the qualitative properties of solutions of ODEs of fourth order. However, to investigate the qualitative properties of solutions of DDEs of fourth order, the LKF method used as basic tool. Indeed, the use of the LKF methods is a difficult task than the use of the second method of Lyapunov.

In fact, it is scientifically important to analyze the qualitative behaviors of functional differential equations, in particular, DDEs. In fact, DDEs, which are one of the types of functional differential equations, can be used as models to express many physical, biological systems and also may be encountered in mechanics, control theory, chemistry, medicine, economics, atomic energy, information theory, and so forth (see, in particular, the books of Burton [7], Hale [19], Yoshizawa [55]). Additionally, periodic solutions of nonlinear differential equations of fourth order can be used to characterize nonlinear oscillations, fluid mechanical and nonlinear elastic mechanical phenomena.

We now would like to outline the key paper for the motivation of this research work. Tejumola and Tchegnani [42] investigated stability, boundedness and existence of periodic solutions of the following nonlinear DDEs of fourth order

$$\begin{aligned} &x^{(4)} + \varphi(t, x, x', x'', x^{(3)})x^{(3)} + \psi(t, x'(t-\tau), x''(t-\tau)) \\ &\quad + \chi(t, x(t-\tau), x'(t-\tau)) + h(x(t-\tau)) = P_2(.), \end{aligned}$$

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where $\tau > 0$ is constant time delay, φ, ψ, χ, h and P_2 are real valued continuous functions in their arguments and

$$P_2(.) = P_2(t, x, x', x'', x''', x(t-\tau), x'(t-\tau), x''(t-\tau)).$$

In this research work, motivated by the results of Tejumola and Tchegnani [42], we consider the following nonlinear DDE of fourth order with multiple constant time delays $\tau_i > 0,\ i = 1, 2, ..., n$:

$$\begin{aligned} & x^{(4)} + f(t, x, x', x'', x''')x''' + \sum_{i=1}^{n} \psi_i(t, x'(t-\tau_i), x''(t-\tau_i)) \\ & \quad + \sum_{i=1}^{n} g_i(t, x(t-\tau_i), x'(t-\tau_i)) + \sum_{i=1}^{n} h_i(x(t-\tau_i) \\ & = P(t, x, x', x'', x''', x(t-\tau_1), ..., x(t-\tau_n), ..., x'''(t-\tau_1), ..., x'''(t-\tau_n)), \end{aligned} \tag{1.1}$$

where $t \in [0, \infty)$, $x \in \Re$, $\Re = (-\infty, \infty)$, f, g_i, h_i, ψ_i and P are continuous functions in their respective arguments. Also, the functions h_i are continuously differentiable.

The DDE (1.1) can be stated in the system form as follows:

$$\begin{aligned} x' &= y, \\ y' &= z, \\ z' &= w, \\ w' &= -aw - \sum_{i=1}^{n} b_i z - \sum_{i=1}^{n} c_i y - \sum_{i=1}^{n} h_i(x) + M(t), \end{aligned} \tag{1.2}$$

where

$$\begin{aligned} M(t) = & aw - f(t, x, y, z, w)w + \sum_{i=1}^{n} b_i \int_{-\tau_i}^{0} w(t+\theta)d\theta - \sum_{i=1}^{n} \psi_i(t, y(t-\tau_i), z(t-\tau_i)) \\ & + \sum_{i=1}^{n} b_i z(t-\tau_i) + \sum_{i=1}^{n} c_i \int_{-\tau_i}^{0} z(t+\theta)d\theta - \sum_{i=1}^{n} g_i(t, x(t-\tau_i), y(t-\tau_i)) \\ & + \sum_{i=1}^{n} c_i y(t-\tau_i) + \sum_{i=1}^{n} \int_{-\tau_i}^{0} h_i{}'(x(t+\theta))y(t+\theta)d\theta + P(.), \end{aligned} \tag{1.3}$$

a, b and c are real constants and $\sum_{i=1}^{n} c_i = c, \quad \sum_{i=1}^{n} b_i = b$.

In this research work, we establish new sufficient conditions, which make enable the UAS, the UB, the UUB of solutions and the EPSs of the DDE (1.1) by defining and using a suitable LKF. The results of this research work generalize some former results in the literature and have new the complementary inputs in relation to the qualitative theory of the DDEs of higher order.

2. Main results

The main results of this paper are given in the following theorems, Theorem 2.1 and Theorem 2.2, respectively.

Theorem 2.1. *Suppose that* $g_i(t,x,0) = 0 = \psi_i(t,y,0)$*, there exist positive constants* $a,\ b_i,\ c_i,\ d_i,\ \delta_i,\ K$ *such that* $ab - c > 0$, $s = abc - c^2 - a^2 d > 0$, $s^* = s + 2ad(ab-c)b^{-1}$ *and the following conditions hold for all* $t \in [0,\infty)$, $x,y,z,w \in \Re$*:*

(C1)

$$0 < f(t,x,y,z,w) \le a;$$

(C2)

$$0 < \frac{\psi_i(t,y,z)}{z} \le b_i, (z \neq 0);$$

(C3)

$$0 < \frac{g_i(t,x,y)}{y} \le c_i, (y \neq 0);$$

(C4)

$$\delta_i \le h_i{}'(x) \le d_i, \quad \sum_{i=1}^{n} d_i = d, \quad \sum_{i=1}^{n} \delta_i = \delta, \qquad d - 2asc^{-1} < \delta;$$

(C5)

$$d_i\left(1 - \frac{s_i c_i}{s_i^* a b_i}\right) < K_i < \frac{h_i(x)}{x}, (x \neq 0), B\sum_{i=1}^{n} K_i = K.$$

Then, the zero solution of the system (1.2) *is uniformly asymptotically stable for sufficiently small* τ_i, $i = 1,2,...,n$*, and* $P(.) \equiv 0$.

Theorem 2.2. *Suppose that (C1)-(C5) of Theorem 2.1 hold and there are positive constants* $\Delta > 0$ *and* $\Delta_1 > 0$ *such that*

$$|P(.)| \le \Delta + \Delta_1(|x| + |y| + |z| + |w|). \tag{2.1}$$

Then, every solution of the system (1.2) *is uniformly bounded and uniformly ultimately bounded for sufficiently small* τ_i, $i = 1,2,...,n$.

Corollary 2.3. *Subject to conditions (C1)-(C5) of Theorem 2.2, the system* (1.2) *admits of at least one* T*-periodic solution if* f, ψ_i, g_i *and* P *are periodic in* t *with the period* T*,* $T > \tau_i$ *(see, Tejumola and Tchegnani [42]).*

Remark 2.4. The results given above are new, they generalize the results of Tejumola and Tchegnani [42] and have new the complementary inputs in relation to the qualitative theory of the DDEs of fourth order, some of them are available in the references of this research work.

To prove Theorem 2.1 and Theorem 2.2, we define an LKF

$$V = V(x,y,z,w,x_t,y_t,z_t,w_t)$$

as the basic tool of the proofs by

$$V = V_1(x,y,z) + V_2(x,y,z) + V_3(x_t,y_t,z_t), \tag{2.2}$$

where

$$\begin{aligned} 2V_1 =& m_1(w + m_1 z + m_1 d c^{-1} y)^2 + c(z + m_1 y + m_1 c^{-1}\sum_{i=1}^{n} h_i(x))^2 \\ &+ m_1 d \sigma c^{-2}(y + c m_1 \alpha \sigma^{-1} z)^2 + m_2 z^2, \end{aligned}$$

$$
\begin{aligned}
2V_2 =& a(w+az+(ab-n_1)a^{-1}y+\beta x)^2+n_1(z+ay+a{n_1}^{-1}\sum_{i=1}^{n}h_i(x))^2\\
&+va^{-1}(y+an_1\beta v^{-1}x)^2+n_2x^2+2k_1\sum_{i=1}^{n}\int_0^x h_i(\xi)d\xi-dk_1x^2\\
&+(d^2x^2-\left(\sum_{i=1}^{n}h_i(x)\right)^2)(a^2{n_1}^{-1}+{m_1}^2c^{-1}),\\
V_3 =& \sum_{i=1}^{n}\frac{3\gamma_i}{\tau_i}\int_{-\tau_i}^{0}\int_s^0[y^2(t+\theta)+z^2(t+\theta)+w^2(t+\theta)]d\theta ds,
\end{aligned}
$$

where the constants $\gamma_i > 0$ are determined later in the proof.

According to the conditions $ab-c>0$, $s=abc-c^2-a^2d>0$, $s^*=s+2ad(ab-c)b^{-1}$, there exist constants $\alpha,\ \beta>0$ such that

$$
\begin{gathered}
0<\alpha<s(bc-ad)^{-1},\quad 0<\beta<s(ab-c)^{-1},\\
m_1=(a-\alpha)>0,\quad k_1=ab-n_1+d{m_1}^2c^{-1}>0,\quad n_1=c-\beta>0,\\
0<m_2=sc^{-1}+(ad-bc)\alpha c^{-1}+\alpha{m_1}^2m_3\sigma^{-1}+\alpha dm_1c^{-1},\\
m_3=\sigma-\alpha dm_1>0,\quad \sigma=m_1bc-{m_1}^2d-c^2>0,\\
n_3=v-\beta n_1,\quad v=abn_1-{n_1}^2-a^2d>0,\\
n_2=ds{n_1}^{-1}+a\beta n_1n_3v^{-1}-d\beta(ab-c){n_1}^{-1}+d\beta>0.
\end{gathered}
$$

In order to complete the proofs of Theorem 2.1 and Theorem 2.2, we need following lemma.

Lemma 2.5. *Suppose that (C1)-(C5) of Theorem 2.1 and* (2.1) *hold. Then, the LKF* $V=V(x,y,z,w,x_t,y_t,z_t,w_t)$ *defined above satisfies the following estimates:*

(C6) *There exist positive constants* d_1, d_2 *and* d_3 *such that*

$$d_1(x^2+y^2+z^2+w^2)^{\frac{1}{2}} \le V(x,y,z,w,x_t,y_t,z_t,w_t) \tag{2.3}$$

$$\le d_2(x^2+y^2+z^2+w^2)^{\frac{1}{2}}+d_3({x_t}^2+{y_t}^2+{z_t}^2+{w_t}^2)^{\frac{1}{2}}; \tag{2.4}$$

(C7) *for every solution* (x,y,z,w) *of the system* (1.2)*, there exist the positive constants* $d_i=d_i(a,b,c,d,\delta_1,\delta_2)>0$*,* $i=4,\ 5$*, such that*

$$V' \le -2d_4(x^2+y^2+z^2+w^2)+\Delta d_5(|x|+|y|+|z|+|w|). \tag{2.5}$$

Indeed, if the LKF V and its time derivative satisfy the estimates (2.3) *and* (2.5)*, then the trivial solution of the DDE* (1.1) *is uniformly asymptotically stable when* $P\equiv 0$*,* $(\Delta=0)$ *and every solution of the DDE* (1.1) *is uniformly bounded and uniformly ultimately bounded. Moreover, when the nonautonomous functions in the DDE* (1.1) *are T-periodic, then the DDE* (1.1)*) has at least one T-periodic solution.*

In the proofs of Theorem 2.1 and Theorem 2.2, we have the following derivative relation:

$$V'_i=\frac{\partial V_i}{\partial x}y+\frac{\partial V_i}{\partial y}z+\frac{\partial V_i}{\partial y}w+\frac{\partial V_i}{\partial z}w'.$$

Hence, the derivatives of the components V_1, V_2, V_3 of (2.2) along solutions of the system (1.2) are calculated as the following, respectively:

$$\begin{aligned}
V'_1 =& m_1 y \sum_{i=1}^{n} h_i'(x) \left(z + m_1 y + m_1 c^{-1} \sum_{i=1}^{n} h_i(x) \right) \\
&+ {m_1}^2 d c^{-1} z \left(w + m_1 z + m_1 d c^{-1} y \right) \\
&+ m_1 c z \left(z + m_1 y + m_1 c^{-1} \sum_{i=1}^{n} h_i(x) \right) + m_1 d \sigma c^{-2} z \left(y + m_1 c \alpha \sigma^{-1} z \right) \\
&+ {m_1}^2 w \left(w + m_1 z + m_1 d c^{-1} y \right) + c w \left(z + m_1 y + m_1 c^{-1} \sum_{i=1}^{n} h_i(x) \right) \\
&+ {m_1}^2 c^{-1} d \alpha w \left(y + m_1 c \alpha \sigma^{-1} z \right) + m_2 z w + \left(m_1 w + {m_1}^2 z + {m_1}^2 d c^{-1} y \right) \dot{w} \ ; \\
V'_2 =& a \beta y \left(w + a z + (ab - n_1) a^{-1} y + \beta x \right) \\
&+ a y \sum_{i=1}^{n} h_i'(x) \left(z + a y + a {n_1}^{-1} \sum_{i=1}^{n} h_i(x) \right) \\
&+ n_1 \beta y \left(y + a n_1 \beta \upsilon^{-1} x \right) + n_2 x y + k_1 y \sum_{i=1}^{n} h_i(x) - d k_1 x y \\
&+ \left(a^2 {n_1}^{-1} + {m_1}^2 c^{-1} \right) \left(d^2 x y - y \sum_{i=1}^{n} h_i(x) \sum_{i=1}^{n} h_i'(x) \right) \\
&+ z (ab - n_1) \left(w + a z + (ab - n_1) a^{-1} y + \beta x \right) \\
&+ a n_1 z \left(z + a y + a {n_1}^{-1} \sum_{i=1}^{n} h_i(x) \right) \\
&+ \upsilon a^{-1} z \left(y + a n_1 \beta \upsilon^{-1} x \right) + a^2 w \left(w + a z + (ab - n_1) a^{-1} y + \beta x \right) \\
&+ n_1 w \left(z + a y + a {n_1}^{-1} \sum_{i=1}^{n} h_i(x) \right) + \left(a w + a^2 z + (ab - n_1) y + a \beta x \right) \dot{w} \ ; \\
V'_3 =& \sum_{i=1}^{n} \frac{3\gamma_i}{\tau_i} \int_{-\tau_i}^{0} [y^2 + z^2 + w^2 - y^2(t+\theta) - z^2(t+\theta) - w^2(t+\theta)] d\theta.
\end{aligned}$$

Proceeding some elementary calculations, we obtain

$$\begin{aligned}
V' =& V'_1 + V'_2 + V'_3 = - U(x, y, z, w) \\
&+ [(m_1 + a) w + ({m_1}^2 + a^2) z + k_1 y + a \beta x] M(t) \\
&+ \sum_{i=1}^{n} \frac{3\gamma_i}{\tau_i} \int_{-\tau_i}^{0} [y^2 + z^2 + w^2 - y^2(t+\theta) - z^2(t+\theta) - w^2(t+\theta)] d\theta,
\end{aligned} \tag{2.6}$$

$$U(x, y, z, w) = m_1 \alpha w^2 + n_3 y^2 + a \beta \sum_{i=1}^{n} \frac{h_i(x)}{x} x^2 + T(x, y, z),$$

$$\begin{aligned}T(x,y,z) =& m_1 c^{-1} m_3 z^2 + ({m_1}^2 + a^2)\left[\sum_{i=1}^{n}(d_i - {h_i}'(x))\right] y^2\\ &+ (m_1 + a)\left[\sum_{i=1}^{n}(d_i - {h_i}'(x))\right] yz\\ =&\left[d - \sum_{i=1}^{n} {h_i}'(x)\right]\left\{\left(m_1 y + \frac{z}{2}\right)^2 + \left(ay + \frac{z}{2}\right)^2\right\}\\ &+\left\{m_1 c^{-1} m_3 - \frac{1}{2}\left[d - \sum_{i=1}^{n} {h_i}'(x)\right]\right\} z^2.\end{aligned}$$

Using the conditions $\delta_i \le {h_i}'(x) \le d_i$ *and* $\sum_{i=1}^{n} d_i = d$*, it can be easily shown that*

$$T(x,y,z) \ge \frac{1}{2}\left[\sum_{i=1}^{n} {h_i}'(x) - (d - 2m_1 m_3 c^{-1})\right] z^2.$$

By substituting the last inequality in $U(x,y,z,w)$ *and choosing*

$$D_3 = \min\{m_1\alpha, n_3, a\beta K, \frac{1}{2}[\delta - (d - 2m_1 m_3 c^{-1})]\} > 0$$

and

$$D_4 = \max\{(m_1 + a), ({m_1}^2 + a^2), k_1, a\beta\} > 0$$

then it follows that

$$U(x,y,z,w) \ge D_3(x^2 + y^2 + z^2 + w^2).$$

By substituting the above inequalities into (2.6)*, we have*

$$\begin{aligned}V' \le& - D_3(x^2 + y^2 + z^2 + w^2) + D_4(|x| + |y| + |z| + |w|)\\ &\times [a\,|w| + |f(t,x,y,z,w)|\,|w| + \sum_{i=1}^{n} b_i \int_{-\tau_i}^{0} |w(t+\theta)|\,d\theta\\ &+ \sum_{i=1}^{n} |\psi_i(t, y(t-\tau_i), z(t-\tau_i))| + \sum_{i=1}^{n} b_i\,|z(t-\tau_i)|\\ &+ \sum_{i=1}^{n} c_i \int_{-\tau_i}^{0} |z(t+\theta)|\,d\theta + \sum_{i=1}^{n} |g_i(t, x(t-\tau_i), y(t-\tau_i))|\\ &+ \sum_{i=1}^{n} c_i\,|y(t-\tau_i)| + \sum_{i=1}^{n} \int_{-\tau_i}^{0} \left|{h_i}'(x(t+\theta))\right| |y(t+\theta)| d\theta + |P_2|]\end{aligned}$$

$$+ \sum_{i=1}^{n} \frac{3\gamma_i}{\tau_i} \int_{-\tau_i}^{0} [y^2 + z^2 + w^2 - y^2(t+\theta) - z^2(t+\theta) - w^2(t+\theta)] d\theta. \tag{2.7}$$

Using (C2) and (C3) of Theorem 2.1, we obtain

$$\sum_{i=1}^{n} |\psi_i(t, y(t-\tau_i), z(t-\tau_i))| + \sum_{i=1}^{n} b_i\,|z(t-\tau_i)| \le 2b \sum_{i=1}^{n} |z(t-\tau_i)|,$$

$$\sum_{i=1}^{n}|g_i(t,x(t-\tau_i),y(t-\tau_i))|+\sum_{i=1}^{n}c_i\,|y(t-\tau_i)|\le 2c\sum_{i=1}^{n}|y(t-\tau_i)|.$$

From the last inequalities, (C4) of Theorem 2.1 and (2.7), *we derive*

$$\begin{aligned}V'\le &-D_3(x^2+y^2+z^2+w^2)+D_4(|x|+|y|+|z|+|w|)\\ &\times[(a+|f(t,x,y,z,w)|)\,|w|+\sum_{i=1}^{n}b_i\int_{-\tau_i}^{0}|w(t+\theta)|\,d\theta+2b\sum_{i=1}^{n}|z(t-\tau_i)|\\ &+2c\sum_{i=1}^{n}|y(t-\tau_i)|+\sum_{i=1}^{n}c_i\int_{-\tau_i}^{0}|z(t+\theta)|\,d\theta+d\sum_{i=1}^{n}\int_{-\tau_i}^{0}|y(t+\theta)|\,d\theta+|P(.)|]\\ &+\sum_{i=1}^{n}\frac{3\gamma_i}{\tau_i}\int_{-\tau_i}^{0}[y^2+z^2+w^2-y^2(t+\theta)-z^2(t+\theta)-w^2(t+\theta)]d\theta.\end{aligned}\tag{2.8}$$

Consider the terms $2b\sum_{i=1}^{n}|z(t-\tau_i)|$ *and* $2c\sum_{i=1}^{n}|y(t-\tau_i)|$. *From the equalities*

$$|z(t-\tau_i)|=|z(t)|-\int_{-\tau_i}^{0}|w(t+\theta)|\,d\theta,$$

$$|y(t-\tau_i)|=|y(t)|-\int_{-\tau_i}^{0}|z(t+\theta)|\,d\theta\tag{2.9}$$

we obtain

$$2\sum_{i=1}^{n}b_i\,|z(t-\tau_i)|\le 2b\,|z(t)|+2b\sum_{i=1}^{n}\int_{-\tau_i}^{0}|w(t+\theta)|\,d\theta,\tag{2.10}$$

$$2\sum_{i=1}^{n}c_i\,|y(t-\tau_i)|\le 2c\,|y(t)|+2c\sum_{i=1}^{n}\int_{-\tau_i}^{0}|z(t+\theta)|\,d\theta.\tag{2.11}$$

Using (C1) of Theorem 2.1, (2.5), (2.10) *and the inequality* (2.1) *into* (2.8), *it follows that*

$$\begin{aligned}V'\le &-D_3(x^2+y^2+z^2+w^2)+\Delta D_4(|x|+|y|+|z|+|w|)\\ &+D_4(|x|+|y|+|z|+|w|)\left[3b\sum_{i=1}^{n}\int_{-\tau_i}^{0}|w(t+\theta)|\,d\theta\right.\\ &+3c\sum_{i=1}^{n}\int_{-\tau_i|z(t+\theta)|d\theta}^{0}+d\sum_{i=1}^{n}\int_{-\tau_i^0|y(t+\theta)|d\theta}]\\ &+D_4(|x|+|y|+|z|+|w|)[\Delta_1|x|+(\Delta_1+2c)|y|\\ &+(\Delta_1+2b)|z|+(\Delta_1+2a)|w|]\\ &+\sum_{i=1}^{n}\frac{3\gamma_i}{\tau_i}\int_{-\tau_i}^{0}[y^2+z^2+w^2-y^2(t+\theta)-z^2(t+\theta)-w^2(t+\theta)]d\theta.\end{aligned}\tag{2.12}$$

If we choose

$$\Delta_2=\max\{\Delta_1,\Delta_1+2c,\Delta_1+2b,\Delta_1+2a\},$$
$$k=D_4\max\{3b,3c,d,\Delta_2\},$$

then from (2.12) *we get*

$$
\begin{aligned}
V' \leq & - D_3(x^2+y^2+z^2+w^2) + \Delta D_4(|x|+|y|+|z|+|w|) \\
& + k(|x|+|y|+|z|+|w|) \\
& \times \left[\sum_{i=1}^{n}\int_{-\tau_i}^{0}\{|y(t+\theta)|+|z(t+\theta)|+|w(t+\theta)|\}\,d\theta + (|x|+|y|+|z|+|w|)\right] \\
& + \sum_{i=1}^{n}\frac{3\gamma_i}{\tau_i}\int_{-\tau_i}^{0}[y^2+z^2+w^2-y^2(t+\theta)-z^2(t+\theta)-w^2(t+\theta)]d\theta. \qquad (2.13)
\end{aligned}
$$

Using the inequality $2\,|x|\,|y| \leq x^2+y^2$*, it follows that*

$$k(|x|+|y|+|z|+|w|)^2 \leq 4k(x^2+y^2+z^2+w^2).$$

By substituting the last inequality into (2.13)*, we obtain*

$$
\begin{aligned}
V' \leq & - D_3(x^2+y^2+z^2+w^2) + \Delta D_4(|x|+|y|+|z|+|w|) + 4k(x^2+y^2+z^2+w^2) \\
& + k\sum_{i=1}^{n}\int_{-\tau_i}^{0}(|x|+|y|+|z|+|w|)\{|y(t+\theta)|+|z(t+\theta)|+|w(t+\theta)|\}\,d\theta \\
& + \sum_{i=1}^{n}\frac{3\gamma_i}{\tau_i}\int_{-\tau_i}^{0}[y^2+z^2+w^2-y^2(t+\theta)-z^2(t+\theta)-w^2(t+\theta)]d\theta. \qquad (2.14)
\end{aligned}
$$

If we take $D_3 - 4k = 5D > 0$ *and choose* γ_i, μ_i *as*

$$\gamma_i = \frac{D}{2n} - \frac{\left(D^2 - 4k^2{\tau_i}^2\right)^{1/2}}{2n} \geq 0,$$

$$\mu_i = \frac{D}{2n} + \frac{\left(D^2 - 4k^2{\tau_i}^2\right)^{1/2}}{2n} \geq 0,$$

then, from (2.14) *we find*

$$
\begin{aligned}
V' \leq & - 2D(x^2+y^2+z^2+w^2) + \Delta D_4(|x|+|y|+|z|+|w|) \\
& - \sum_{i=1}^{n}\frac{1}{\tau_i}\int_{-\tau_i}^{0}\left[\mu_i x^2 + \frac{3}{4}\gamma_i y^2(t+\theta) - k\tau_i\,|x|\,|y(t+\theta)|\right]d\theta \\
& - \sum_{i=1}^{n}\frac{1}{\tau_i}\int_{-\tau_i}^{0}\left[\mu_i x^2 + \frac{3}{4}\gamma_i z^2(t+\theta) - k\tau_i\,|x|\,|z(t+\theta)|\right]d\theta \\
& - \sum_{i=1}^{n}\frac{1}{\tau_i}\int_{-\tau_i}^{0}\left[\mu_i x^2 + \frac{3}{4}\gamma_i w^2(t+\theta) - k\tau_i\,|x|\,|w(t+\theta)|\right]d\theta \\
& - \sum_{i=1}^{n}\frac{1}{\tau_i}\int_{-\tau_i}^{0}\left[\mu_i y^2 + \frac{3}{4}\gamma_i y^2(t+\theta) - k\tau_i\,|y|\,|y(t+\theta)|\right]d\theta \\
& - \sum_{i=1}^{n}\frac{1}{\tau_i}\int_{-\tau_i}^{0}\left[\mu_i y^2 + \frac{3}{4}\gamma_i z^2(t+\theta) - k\tau_i\,|y|\,|z(t+\theta)|\right]d\theta
\end{aligned}
$$

$$-\sum_{i=1}^{n}\frac{1}{\tau_i}\int_{-\tau_i}^{0}\left[\mu_i y^2+\frac{3}{4}\gamma_i w^2(t+\theta)-k\tau_i\,|y|\,|w(t+\theta)|\right]d\theta$$
$$-\sum_{i=1}^{n}\frac{1}{\tau_i}\int_{-\tau_i}^{0}\left[\mu_i z^2+\frac{3}{4}\gamma_i y^2(t+\theta)-k\tau_i\,|z|\,|y(t+\theta)|\right]d\theta$$
$$-\sum_{i=1}^{n}\frac{1}{\tau_i}\int_{-\tau_i}^{0}\left[\mu_i z^2+\frac{3}{4}\gamma_i z^2(t+\theta)-k\tau_i\,|z|\,|z(t+\theta)|\right]d\theta$$

$$-\sum_{i=1}^{n}\frac{1}{\tau_i}\int_{-\tau_i}^{0}\left[\mu_i z^2+\frac{3}{4}\gamma_i w^2(t+\theta)-k\tau_i\,|z|\,|w(t+\theta)|\right]d\theta$$
$$-\sum_{i=1}^{n}\frac{1}{\tau_i}\int_{-\tau_i}^{0}\left[\mu_i w^2+\frac{3}{4}\gamma_i y^2(t+\theta)-k\tau_i\,|w|\,|y(t+\theta)|\right]d\theta$$
$$-\sum_{i=1}^{n}\frac{1}{\tau_i}\int_{-\tau_i}^{0}\left[\mu_i w^2+\frac{3}{4}\gamma_i z^2(t+\theta)-k\tau_i\,|w|\,|z(t+\theta)|\right]d\theta$$
$$-\sum_{i=1}^{n}\frac{1}{\tau_i}\int_{-\tau_i}^{0}\left[\mu_i w^2+\frac{3}{4}\gamma_i w^2(t+\theta)-k\tau_i\,|w|\,|w(t+\theta)|\right]d\theta. \tag{2.15}$$

Each of the integrals above in the inequality (2.15) *is positive definite (because the discriminant of each integral is* $-2k^2{\tau_i}^2<0$*). Therefore, we can conclude*

$$V'\le -2D(x^2+y^2+z^2+w^2)+\Delta D_4(|x|+|y|+|z|+|w|)$$

and hence the inequality (2.5) *holds. From the results obtained above, it can be said that the zero solution of the system* (1.2) *is uniformly asymptotically stable. From Chukwu [11, Theorem 1.1, Theorem 1.2]), it can be concluded that the DDE* (1.1) *has at least one periodic solution and its solutions are bounded. We omit the details of the mathematical calculations.*

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