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VARIATIONAL PROBLEMS WITH GENERALIZED FRACTAL DERIVATIVE OPERATOR

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Abstract. In this paper, the fractal Frechet derivative is defined. the fractal generalized Euler-Lagrange equation and the fractal Dubois-Reymond optimality condition are given by utilizing fractal calculus and comparing its results with the fractional space approach which shows the effect of fractal space with fractional dimension on a given problem in variational calculus.

1. Introduction

Fractals are often shapes with fractional dimensions and self-similar properties and many processes in physics have fractal structures [6, 7, 16]. Analysis on fractal has been formulated by many researchers such as fractional space, measure theory, harmonic analysis, fractional calculus, and probability theory [1,2,6,7,13,16,20–23]. In the seminal paper fractal calculus was formulated [17–19]. Fractal calculus was used to model physical processes in fractional space and time [9]. Stability and method of solving fractal differential equations were defined such as fractal Laplace, Fourier transforms and fractal Euler numerical method [9–12]. The Calculus of variations and optimal conditions have found many applications in physics and engineering [3, 4, 8, 14, 15]. The local fractal derivative operator was introduced and applied implications in classical systems through the Lagrangian and Hamiltonian formalisms [5].In Section 2 we fractalize variational calculus and corresponding dynamics and results through the paper using fractal calculus.

2. Variational problems with generalized fractal derivative operator

In this section, fractal calculus is formulated base of ordinary calculus which includes functions with fractal support [9].

Definition 2.1. Fractal derivative of $q(t)$ at t is defined by

$$D_F^{\nu} q(t) = F - \lim_{t_1 \to t} \frac{g(t_1) - g(t)}{S_F^{\nu}(t_1) - S_F^{\nu}(t)},\ t \in F \tag{2.1}$$

where F is fractal time set (see for details in [9]). Consider fractal action as

$$J^{\nu}(q(.)) = \int_a^b L^{\nu}(S_F^{\nu}(t), q, D^{\nu}q) d_F^{\nu} t, \tag{2.2}$$

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where $q(.) \in C^{\infty}(F, R^n)$ and

$$D^{\nu} = D_F^{\nu} + \alpha \left(D_F^{\nu} q + \beta \frac{q}{S_F^{\nu}(t)} \right) \frac{\partial}{\partial q} +$$

$$\gamma \left(D_F^{2\nu} q + \delta \frac{D_F^{\nu} q}{S_F^{\nu}(t)} + \lambda \frac{q}{S_F^{\nu}(t)^2} \right) \frac{\partial}{\partial (D_F^{\nu} q)}, \tag{2.3}$$

we suppose $(S_F^{\nu}(t), u, v) \to L(S_F^{\nu}(t), u, v)$ to be a $C^{\nu}(F)$-function [9].

Definition 2.2. The fractal Frechet derivative is defined by

$$D^{\nu} J^{\nu}[q](h) = F - \lim_{\epsilon \to 0} \frac{J^{\nu}(q + \epsilon h) - J^{\nu}(q)}{S_F^{\nu}(\epsilon)}$$

$$= \int_a^b \left[\left(1 + \frac{\alpha\beta}{S_F^{\nu}(t)} \right) \frac{\partial L^{\nu}}{\partial q} h + (1 + \alpha) \frac{\partial L^{\nu}}{\partial D^{\nu} q} h \right] d_F^{\nu} t, \tag{2.4}$$

where $h \in C^{\infty}(F, R^n)$.

Theorem 2.3. *The fractal Euler-Lagrange equation is given by*

$$\left(1 + \frac{\alpha\beta}{S_F^{\nu}(t)} \right) \frac{\partial L^{\nu}}{\partial q} - (1 + \alpha) D_F^{\nu} \left(\frac{\partial L^{\nu}}{\partial D_F^{\nu} q} \right) = 0, \; q \in C^2(F, R^n) \tag{2.5}$$

Theorem 2.4. *The fractal Dubois-Reymond optimality condition is given by*

$$D_F^{\nu} \left[L^{\nu}(S_F^{\nu}(t), q, D^{\nu} q) - (1 + \alpha) D_F^{\nu} q \cdot \frac{\partial L^{\nu}(S_F^{\nu}(t), q, D^{\nu} q)}{\partial D_F^{\nu} q} \right]$$

$$= \left(1 - \frac{\alpha\beta}{S_F^{\nu}(t)^2} q \right) D_{F,t}^{\nu} L^{\nu}(S_F^{\nu}(t), q, D^{\nu} q). \tag{2.6}$$

Theorem 2.5. *Let us consider a fractal functional as follows*

$$J^{\nu}[q] = \int_a^b \ln \left[L^{\nu}(S_F^{\nu}(t), q, D^{\nu} q) \right] d_F^{\nu} t, \tag{2.7}$$

where $q \in C^2(F, R^n)$ and $L^{\nu}(.,.,.) \in C^2(F, R^n \times R^n)$. If $q(.)$ is an extremal, then we have

$$\left(1 + \frac{\alpha\beta}{S_F^{\nu}(t)} \right) \frac{\partial L^{\nu}(S_F^{\nu}(t), q, D^{\nu} q)}{\partial q}$$

$$+ (1 + \alpha) \left[\frac{D_{F,t}^{\nu} L^{\nu}(S_F^{\nu}(t), q, D^{\nu} q)}{L^{\nu}(S_F^{\nu}(t), q, D^{\nu} q)} - D_F^{\nu} \left(\frac{\partial L^{\nu}(S_F^{\nu}(t), q, D^{\nu} q)}{\partial D_F^{\nu} q}, \right) \right] = 0. \tag{2.8}$$

which is called the fractal Euler-Lagrange equation.

Definition 2.6. Fractal functional (2.2) is called to be fractal s-invariant under of group of diffeomorphism $\Psi_i = \{\psi_i(s, .)\}_{s \in R}, i = 1, 2$ if it satisfies

$$L^{\nu}(S_F^{\nu}(t), q(t), (1 + \alpha) D_F^{\nu} q(t) + \frac{\alpha\beta}{S_F^{\nu}(t)} q(t))$$

$$= L^{\nu} \left(\psi_1(s, t), \psi_2(s, q(t)), (1 + \alpha) \frac{D_{F,t}^{\nu} \psi_2(s, q(t))}{D_{F,t}^{\nu} \psi_1(s, t)} + \frac{\alpha\beta}{S_F^{\nu}(t)} \frac{\psi_2(s, q(t))}{\psi_1(s, t)} \right)$$

$$\times D_{F,t}^{\nu} \psi_1(s, t), \tag{2.9}$$

for any $q(.) \in \tilde{W}^{1,p}$ where $\tilde{W}^{1,p}$ is the fractal Sobolev space [9].

Theorem 2.7. *If fractal functional* (2.2) *is invariant under Eq.*(2.9) *,then we have*

$$\begin{aligned}
&\left(1-\frac{\alpha\beta}{S_F^\nu(t)^2}q(t)\right)D_{F,t}^\nu L^\nu(S_F^\nu(t),q(t),D^\nu q)D_{F,t}^\nu\psi_1(0,t)\\
&+\left(1+\frac{\alpha\beta}{S_F^\nu(t)}\right)\frac{\partial L^\nu(S_F^\nu(t),q(t),D^\nu q)}{\partial q}\cdot\frac{\partial\psi_2}{\partial s}(0,q(t))\\
&+(1+\alpha)\frac{\partial L^\nu(S_F^\nu(t),q(t),D^\nu q)}{\partial D_F^\nu q}\left(D_{F,t}^\nu\frac{\partial\psi_2}{\partial s}(0,q(t))-D_{F,t}^\nu q(t)D_F^\nu\frac{\partial\psi_1}{\partial s}(0,t)\right)\\
&+L^\nu(S_F^\nu(t),q(t),D^\nu q)\left(D_{F,t}^\nu\frac{\partial\psi_1}{\partial s}(0,t)\right)+F\cdot\left(\frac{\partial\psi_2}{\partial s}(0,t)-D_F^\nu q\frac{\partial\psi_1}{\partial s}(0,t)\right).
\end{aligned}\tag{2.10}$$

Theorem 2.8. *If the fractal Lagrange equation is as*

$$D_{F,t}^\nu\left(\frac{\partial L^\nu(S_F^\nu(t),q(t),D^\nu q)}{\partial D_F^\nu q}\right)=F\cdot\left(\frac{\partial\psi_2}{\partial s}(0,t)-D_F^\nu q\frac{\partial\psi_1}{\partial s}(0,t)\right),\tag{2.11}$$

then the quantity $C^\nu(S_F^\nu(t),q(t),D^\nu q)$ is defined by:

$$\begin{aligned}
&(1+\alpha)\frac{\partial\psi_2}{\partial s}(0,q(t))\cdot\frac{\partial L^\nu(S_F^\nu(t),q(t),D^\nu q)}{\partial D_F^\nu q}\\
&+\frac{\partial\psi_1}{\partial s}(0,t)\left(L^\nu(S_F^\nu(t),q(t),D^\nu q)-(1+\alpha)D_F^\nu q\cdot\frac{\partial L^\nu(S_F^\nu(t),q(t),D^\nu q)}{\partial D_F^\nu q}\right)\\
&+f^\nu(S_F^\nu(t),q(t),D^\nu q),
\end{aligned}\tag{2.12}$$

is a constant of motion.

Example 2.9. Consider the fractal Lagrangian as

$$L^\nu(S_F^\nu(t),q(t),D^\nu q)\equiv L^\nu(S_F^\nu(t),(1+\alpha)D_F^\nu q),\tag{2.13}$$

then using Eq.(2.12), and $\partial\psi_1/\partial s=1,\ \partial\psi_2/\partial s=0$ we have

$$L^\nu(S_F^\nu(t),(1+\alpha)D_F^\nu q)-(1+\alpha)\,D_F^\nu q\cdot\frac{\partial L^\nu(S_F^\nu(t),(1+\alpha)D_F^\nu q)}{\partial D_F^\nu q}\equiv\text{constant}.\tag{2.14}$$

If $\partial\psi_1/\partial s=0,\ \partial\psi_2/\partial s=1$. Then

$$\frac{\partial L^\nu(S_F^\nu(t),(1+\alpha)D_F^\nu q(t))}{\partial D_F^\nu q(t)}\equiv\text{constant}.\tag{2.15}$$

Example 2.10. Consider the fractal functional in the following form

$$J^\alpha[q(\cdot)]=\frac{1}{2}\int_a^b\left(\|D^\nu q(t)\|^2+\|q(t)\|^2\right)d_F^\nu t,\tag{2.16}$$

By minimizing Eq.(2.16), we arrive at

$$\begin{aligned}
&(1+\alpha)^3D_F^{2\nu}q(t)+(1+\alpha)\left(\frac{\alpha^2\beta}{S_F^\nu(t)}-\frac{\alpha^2\beta^2}{S_F^\nu(t)^2}\right)D_F^\nu q(t)\\
&+\left(\alpha\beta\frac{-(1+\alpha)^2-\alpha\beta}{S_F^\nu(t)^2}-\frac{\alpha^3\beta^3}{S_F^\nu(t)^3}-\frac{\alpha\beta}{S_F^\nu(t)}-1\right)q(t)=0.
\end{aligned}\tag{2.17}$$

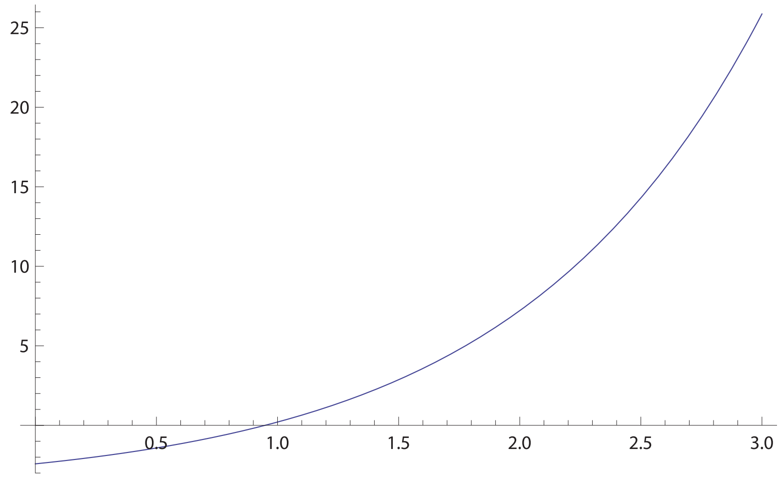

FIGURE 1. Graph of Eq.(2.20) for case of $\nu = \frac{1}{2}$

Simplifying Eq.(2.17) and setting $\alpha = \beta = 0.1$ and neglecting $\alpha^2\beta$ and $\alpha^2\beta^2$ terms we arrive at

$$1.030301 D_F^{2\nu} q(t) + \left(-\frac{0.01}{S_F^\nu(t)^2} - \frac{0.01}{S_F^\nu(t)} + 1 \right) q(t) = 0, \tag{2.18}$$

The solution of Eq.(2.18) is

$$q(t) = e^{0.981585 S_F^\nu(t)} S_F^\nu(t)^{0.99} \times \Big(c_5 U(0.9895 - 0.00492593i, 1.96117, 1.97037 S_F^\nu(t)) \tag{2.19}$$
$$+ c_6 L^{0.9895}_{-0.99-0.00492583i}(1.97037 S_F^\nu(t)) \Big)$$

By using $S_F^\nu(t) \leq t^\nu$ we have

$$q(t) \propto e^{0.981585 t^\nu} t^{0.99\nu} \times \Big(c_5 U(0.9895 - 0.00492593i, 1.96117, 1.97037 t^\nu)$$
$$+ c_6 L^{0.9895}_{-0.99-0.00492583i}(1.97037 t^\nu) \Big) \tag{2.20}$$

where $U(.,.,.)$ is the confluent hypergeometric function of the $2^n d$ kind and $L_a^n(x)$ is the associated Laguerre polynomial (see Figure 1). For the case of $\alpha = \beta = 0$, we have $q(t) = c_3 \exp(S_F^\nu(t)) + c_4 \exp(-S_F^\nu(t))$.

In the table 1, two methods, i.e. fractional and fractal space models, are compared.

3. Conclusion

In this paper, the fractal Frechet derivatives have been defined. The fractal generalized Euler-Lagrange equation and fractal Dubois-Reymond optimality condition have been given and compared fractal space approach by fractional space model to present the effect of space with fractional dimension on dynamics.

TABLE 1. Comparison of generalized derivative operator [5] and generalized fractal derivative operator

Generalized Derivative Operator [23]	Generalized Fractal Derivative
$J_\chi[q] = \int_a^b L(t, q, Dq) t^{\chi-1} dt$	$J^\nu[q] = \int_a^b L^\nu(S_F^\nu(t), q, D^\nu q) d_F^\nu t$
$DJ_\chi[q](h) = \int_a^b \left[\left(1+\frac{\alpha\beta}{t}\right)\frac{\partial L(t,q,Dq)}{\partial q}\cdot h + (1+\alpha)\frac{\partial L(t,q,Dq)}{\partial \dot{q}}\cdot \dot{h}\right] t^{\chi-1} dt$	$D^\nu J^\nu[q](h) = \int_a^b \left[\left(1+\frac{\alpha\beta}{S_F^\nu(t)}\right)\frac{\partial L^\nu}{\partial q} h + (1+\alpha)\frac{\partial L^\nu}{\partial D^\nu q} h\right] d_F^\nu t$
$\left(1+\frac{\alpha\beta}{t}\right)\frac{\partial L}{\partial q} - (1+\alpha)\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{(1+\alpha)(\chi-1)}{t}\frac{\partial L}{\partial \dot{q}}$	$\left(1+\frac{\alpha\beta}{S_F^\nu(t)}\right)\frac{\partial L^\nu}{\partial q} - (1+\alpha)D_F^\nu\left(\frac{\partial L^\nu}{\partial D_F^\nu q}\right) = 0$
$\frac{d}{dt}\left[L - (1+\alpha)\dot{q}\cdot\frac{\partial L}{\partial \dot{q}}\right] = \left(1-\frac{\alpha\beta}{t^2}q\right)\frac{\partial L}{\partial t} + q\cdot\frac{(1+\alpha)(\chi-1)}{t}\frac{\partial L}{\partial \dot{q}}$	$D_F^\nu\left[L^\nu - (1+\alpha)D_F^\nu q\cdot\frac{\partial L^\nu}{\partial D_F^\nu q}\right] = \left(1-\frac{\alpha\beta}{S_F^\nu(t)^2}q\right) D_{F,t}^\nu L^\nu$
$J_\chi[q] = \int_a^b \ln[L(t,q,Dq)t^{\chi-1}]dt$	$J^\nu[q] = \int_a^b \ln\left[L^\nu(S_F^\nu(t), q, D^\nu q)\right] d_F^\nu t$
$\left(1+\frac{\alpha\beta}{t}\right)\frac{\partial L}{\partial q} + (1+\alpha)\left[\frac{\dot{L}}{L} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right] = \frac{(1+\alpha)(\chi-1)}{t}\frac{\partial L}{\partial \dot{q}}$	$\left(1+\frac{\alpha\beta}{S_F^\nu(t)}\right)\frac{\partial L^\nu}{\partial q} + (1+\alpha)\left[\frac{D_{F,t}^\nu L^\nu}{L^\nu} - D_F^\nu\left(\frac{\partial L^\nu}{\partial D_F^\nu q},\right)\right] = 0$
$\frac{\partial L}{\partial \dot{q}}(t,(1+\alpha)\dot{q}) + \int_a^b \frac{\chi-1}{t}\frac{\partial L}{\partial \dot{q}}(t,(1+\alpha)\dot{q})dt \equiv const.$	$L^\nu(S_F^\nu(t),(1+\alpha)D_F^\nu q) - (1+\alpha)\,D_F^\nu q\cdot\frac{\partial L^\nu(S_F^\nu(t),(1+\alpha)D_F^\nu q)}{\partial D_F^\nu q} \equiv const.$

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