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ON STABILITY OF CONFORMABLE LASOTA WAZEWSKA FRACTIONAL MODEL WITH PIECEWISE CONSTANT ARGUMENT

CEMIL BÜYÜKADALI

Abstract. We consider conformable type fractional Lasota Wazewska model with piecewise constant argument. In this model, piecewise constant argument is divided into three categories: delay, advanced and advanced delay piecewise constant argument. For each category of this model we find the existence and stability of equilibrium points separately.

1. Introduction

To model the process of production and destruction of red blood cells in animals M. Wazewska-Czyzewska and A. Lasota [22] in 1976 presented the following nonlinear differential equation with constant delay

$$N'(t) = -\mu N(t) + \beta e^{-\gamma N(t-r)}, \qquad t \geq 0.$$

Here, $\mu \in (0,\, 1)$, and $\gamma,\, \beta,\, r \in (0,\, +\infty)$. $N(t)$ is the number of red blood cells at time t, μ is the probability of death rate of red blood cells, positive constants β and γ are the production rate of red blood cells in unit time, and r is the time elapsed to produce a red blood cell [10]. The asymptotic behavior of this model has been studied by many researchers [6, 8, 19, 20] and this model is still under development recently [2, 5, 11, 17, 18, 24].

Differential equations that are using the greatest integer function $[t] = n$, $n \leq t < n+1, n \in \mathbb{Z}$ as an argument are called differential equations with piecewise constant argument. As this kind of equations has hybrid structure, that is, they include the properties of both continuous and discrete equations, they have vast amount of application areas ranging from modeling biological phenomenon to control theories [7, 23]. Lasota Wazewska model with piecewise constant argument has been studied also in literature [1, 3, 14, 16].

Differential equations of integer order are one of the fundamental tools for mathematical modeling of real world problems. The theory of fractional differential equations is the natural extension of the theory of differential equations of integer order, a branch of analysis. The subject of fractional analysis has taken attentions of many researchers for almost half century and is still being studied as it is usefulness in applications of fundamental science and engineering, and solving differential and integral equations [4, 12, 13, 15, 21].

This paper is summarized as follows. Section 2 gives the preliminary definition. In section 3, we establish the existence and asymptotic stability of equilibrium of

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conformable type fractional Lasota Wazewska model with delay piecewise constant argument. Section 4 presents the results for the existence and asymptotic stability of equilibrium of conformable type fractional Lasota Wazewska model with advanced piecewise constant argument. In section 5, we obtain the existence and instability of equilibrium of conformable type fractional Lasota Wazewska model with advanced delay piecewise constant argument.

2. Preliminaries

Let us introduce the following definition of conformal fractional derivative.

Definition 2.1 ([9]). Let $f : [t_0, +\infty) \to \mathbb{R}$ be a function. Then for $\alpha \in (0,1)$ and $t \geq t_0$ left conformable fractional derivative of function f with order α is defined by the limit

$$T_\alpha^{t_0} = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon (t - t_0)^{1-\alpha}\right) - f(t)}{\varepsilon}$$

if the limit exists. Let us denote $T_\alpha := T_\alpha^{t_0}$ for $t_0 = 0$.

If the function f is differentiable, then we have the identity

$$T_\alpha^{t_0}\left(f(t)\right) = (t - t_0)^{1-\alpha} f'(t),$$

where $f'(t)$ is first derivative of the function $f(t)$.

3. Stability of conformable type Lasota Wazewska fractional model with delay piecewise constant argument

Let us consider the following conformable type Lasota Wazewkska fractional model with delay piecewise constant argument.

$$T_\alpha^{[t]}\left(N(t)\right) = -\mu N(t) + \beta e^{-\gamma N([t])}, \qquad t \geq 0. \tag{3.1}$$

Here, $T_\alpha^{[t]}$ denotes the conformable type fractional derivative of order $\alpha \in (0,\, 1)$. $\mu \in (0,\, 1)$, and $\gamma,\, \beta,\, r \in (0,\, +\infty)$ are constants.

For $t \in [n,\, n+1)$, $n \in \mathbb{N}$ using the relation between conformable fractional derivative and integer order derivative we rewrite equation (3.1) as

$$(t-n)^{1-\alpha} \frac{dN(t)}{dt} = -\mu N(t) + \beta e^{-\gamma N(n)},$$

or

$$\frac{dN(t)}{dt} + \frac{\mu}{(t-n)^{1-\alpha}} N(t) = \frac{\beta}{(t-n)^{1-\alpha}} e^{-\gamma N(n)}.$$

By multiplying the last equation with the integration multiplier $\exp\left(\frac{\mu}{\alpha}(t-n)^\alpha\right)$ we obtain the equation

$$\frac{d}{dt}\left(N(t) e^{\frac{\mu}{\alpha}(t-n)^\alpha}\right) = \frac{\beta}{(t-n)^{1-\alpha}} e^{\frac{\mu}{\alpha}(t-n)^\alpha - \gamma N(n)}.$$

By integrating this equation over the interval $[n,\, n+1)$ we have discrete equation

$$N(n+1) e^{\mu/\alpha} - N(n) = \frac{\beta}{\mu} e^{-\gamma N(n)} \left(e^{\mu/\alpha} - 1\right),$$

or

$$N(n+1) = e^{-\mu/\alpha} N(n) + \frac{\beta}{\mu}\left(1 - e^{-\mu/\alpha}\right) e^{-\gamma N(n)} \tag{3.2}$$

for $n = 0, 1, 2, \ldots$ It is easy to see that the equilibrium point of this equation satisfies the equation

$$x^* = \frac{\beta}{\mu} e^{-\gamma x^*}. \tag{3.3}$$

Set $x_n := \gamma\left(N(n) - x^*\right)$. Then, we obtain the discrete equation

$$\begin{aligned} x_{n+1} &= \gamma\left(N(n+1) - x^*\right) = \gamma\left(e^{-\mu/\alpha} N(n) + \frac{\beta}{\mu}\left(1 - e^{-\mu/\alpha}\right) e^{-\gamma N(n)} - x^*\right) \\ &= e^{-\mu/\alpha}\left(x_n + \gamma x^*\right) + \frac{\gamma\beta}{\mu} e^{-\gamma x^*}\left(1 - e^{-\mu/\alpha}\right) e^{-x_n} - \gamma x^* \end{aligned}$$

or

$$x_{n+1} = e^{-\mu/\alpha} x_n - \gamma x^*\left(1 - e^{-\mu/\alpha}\right)\left(1 - e^{-x_n}\right), \quad n = 0, 1, 2, \ldots \tag{3.4}$$

and this equation has the equilibrium point zero.

Let us introduce asymptotic stability of the zero solution of equation (3.4).

Theorem 3.1. *If the inequality*

$$\frac{1 + e^{-\mu/\alpha}}{1 - e^{-\mu/\alpha}} > \gamma x^*$$

holds, then the zero solution of equation (3.4), *therefore the equilibrium point* x^* *of equation* (3.2), *is asymptotically stable.*

Proof. Using Taylor expansion of the function e^{-x_n} around the point zero we linearize equation (3.4) as

$$x_{n+1} = \left(e^{-\mu/\alpha - \gamma x^*\left(1 - e^{-\mu/\alpha}\right)}\right) x_n, \qquad n = 0, 1, 2, \ldots$$

In order to have the asymptotical stability of the zero solution of this equation, it is sufficient that the inequality

$$\left| e^{-\mu/\alpha} - \gamma x^*\left(1 - e^{-\mu/\alpha}\right)\right| < 1$$

is satisfied. It is easy to see that $e^{-\mu/\alpha} - \gamma x^*\left(1 - e^{-\mu/\alpha}\right) < 1$ holds for all $\mu, \alpha \in (0, 1), \gamma > 0, x^* > 0$. Moreover, by simple calculation, the inequality

$$-1 < e^{-\mu/\alpha} - \gamma x^*\left(1 - e^{-\mu/\alpha}\right)$$

is satisfied if and only if

$$\frac{1 + e^{-\mu/\alpha}}{1 - e^{-\mu/\alpha}} > \gamma x^*$$

holds. The proof is complete. □

4. Stability of conformable type Lasota Wazewska fractional model with advanced piecewise constant argument

Let us consider the following conformable type Lasota Wazewkska fractional model with advanced piecewise constant argument.

$$T_{\alpha}^{[t]}\left(N(t)\right) = -\mu N(t) + \beta e^{-\gamma N([t+1])}, \qquad t \geq 0. \tag{4.1}$$

Here, $T_{\alpha}^{[t]}$ denotes the conformable type fractional derivative of order $\alpha \in (0, 1)$. $\mu \in (0, 1)$, and γ, β, $r \in (0, +\infty)$ are constants.

For $t \in [n, n+1)$, $n \in \mathbb{N}$ using the relation between conformable fractional derivative and integer order derivative we rewrite equation (4.1) as

$$(t-n)^{1-\alpha}\frac{dN(t)}{dt} = -\mu N(t) + \beta e^{-\gamma N(n+1)},$$

or

$$\frac{dN(t)}{dt} + \frac{\mu}{(t-n)^{1-\alpha}}N(t) = \frac{\beta}{(t-n)^{1-\alpha}}e^{-\gamma N(n+1)}.$$

By multiplying the last equation with the integration multiplier $\exp\left(\frac{\mu}{\alpha}(t-n)^{\alpha}\right)$ we obtain the equation

$$\frac{d}{dt}\left(N(t)e^{\frac{\mu}{\alpha}(t-n)^{\alpha}}\right) = \frac{\beta}{(t-n)^{1-\alpha}}e^{\frac{\mu}{\alpha}(t-n)^{\alpha}-\gamma N(n+1)}.$$

By integrating this equation over the interval $[n, n+1)$ we have discrete equation

$$N(n+1)e^{\mu/\alpha} - N(n) = \frac{\beta}{\mu}e^{-\gamma N(n+1)}\left(e^{\mu/\alpha}-1\right),$$

or

$$N(n+1) = e^{-\mu/\alpha}N(n) + \frac{\beta}{\mu}\left(1-e^{-\mu/\alpha}\right)e^{-\gamma N(n+1)}, \tag{4.2}$$

in closed form for $n = 0, 1, 2, \ldots$. It is easy to see that the equilibrium point of this equation satisfies the equation

$$x^* = \frac{\beta}{\mu}e^{-\gamma x^*}. \tag{4.3}$$

Set $x_n := \gamma\left(N(n) - x^*\right)$. Then, we obtain the discrete equation

$$\begin{aligned} x_{n+1} &= \gamma\left(N(n+1)-x^*\right) = \gamma\left(e^{-\mu/\alpha}N(n) + \frac{\beta}{\mu}\left(1-e^{-\mu/\alpha}\right)e^{-\gamma N(n+1)} - x^*\right) \\ &= e^{-\mu/\alpha}\left(x_n+\gamma x^*\right) + \frac{\gamma\beta}{\mu}e^{-\gamma x^*}\left(1-e^{-\mu/\alpha}\right)e^{-x_{n+1}} - \gamma x^* \end{aligned}$$

or

$$x_{n+1} = e^{-\mu/\alpha}x_n - \gamma x^*\left(1-e^{-\mu/\alpha}\right)\left(1-e^{-x_{n+1}}\right) \tag{4.4}$$

in closed form for $n = 0, 1, 2, \ldots$ and this equation has the equilibrium point zero.

Let us introduce asymptotic stability of the zero solution of equation (4.4).

Theorem 4.1. *The zero solution of equation* (4.4), *therefore the equilibrium point* x^* *of equation* (4.2), *is asymptotically stable.*

Proof. Using Taylor expansion of the function $e^{-x_{n+1}}$ around the point zero we linearize equation (4.4) as

$$x_{n+1} = e^{-\mu/\alpha} x_n - \gamma x^* \left(1 - e^{-\mu/\alpha}\right) x_{n+1}, \qquad n = 0,\, 1,\, 2,\, \ldots$$

or

$$x_{n+1} = \frac{e^{-\mu/\alpha}}{1 + \gamma x^* \left(1 - e^{-\mu/\alpha}\right)} x_n, \qquad n = 0,\, 1,\, 2,\, \ldots$$

In order to have the asymptotical stability of the zero solution of this equation, it is sufficient that the inequality

$$\left| \frac{e^{-\mu/\alpha}}{1 + \gamma x^* \left(1 - e^{-\mu/\alpha}\right)} \right| < 1$$

is satisfied. As $\exp\left(-\mu/\alpha\right) \in (0,\, 1)$ for all $\mu,\, \alpha \in (0,\, 1)$, and $\gamma x^* > 0$, this inequality always holds. The proof is complete. ☐

5. Stability of conformable type Lasota Wazewska fractional model with advanced delay piecewise constant argument

Let us consider the following conformable type Lasota Wazewkska fractional model with advanced delay piecewise constant argument

$$T_\alpha^{[t]} \left(N(t)\right) = -\mu N(t) + \beta e^{-\gamma N\left(\left[t + \frac{1}{2}\right]\right)}, \qquad t \geq 0. \tag{5.1}$$

Here, $T_\alpha^{[t]}$ denotes the conformable type fractional derivative of order $\alpha \in (0,\, 1)$. $\mu \in (0,\, 1)$, and $\gamma,\, \beta,\, r \in (0,\, +\infty)$ are constants.

For $t \in [0,\, 1/2)$, using the relation between conformable fractional derivative and integer order derivative we rewrite equation (5.1) as

$$t^{1-\alpha} \frac{dN(t)}{dt} = -\mu N(t) + \beta e^{-\gamma N(0)},$$

or

$$\frac{dN(t)}{dt} + \frac{\mu}{t^{1-\alpha}} N(t) = \frac{\beta}{t^{1-\alpha}} e^{-\gamma N(0)}.$$

By multiplying the last equation with the integration multiplier $\exp\left(\frac{\mu}{\alpha} t^\alpha\right)$ we obtain the equation

$$\frac{d}{dt}\left(N(t) e^{\frac{\mu}{\alpha} t^\alpha}\right) = \frac{\beta}{t^{1-\alpha}} e^{\frac{\mu}{\alpha} t^\alpha - \gamma N(0)}.$$

By integrating this equation over the interval $[0,\, 1/2)$ we have discrete equation

$$N\left(\frac{1}{2}\right) \exp\left(\frac{\mu}{\alpha 2^\alpha}\right) - N(0) = \frac{\beta}{\mu} e^{-\gamma N(0)} \left(\exp\left(\frac{\mu}{\alpha 2^\alpha}\right) - 1\right),$$

or

$$N\left(\frac{1}{2}\right) = \exp\left(-\frac{\mu}{\alpha 2^\alpha}\right) N(0) + \frac{\beta}{\mu}\left(1 - \exp\left(-\frac{\mu}{\alpha 2^\alpha}\right)\right) e^{-\gamma N(0)}. \tag{5.2}$$

Similarly, for $t \in \left[n-\frac{1}{2},\, n\right)$ and $t \in \left[n,\, n+\frac{1}{2}\right)$, $n = 1,\, 2,\, \ldots$ using the relation between conformable fractional derivative and integer order derivative we rewrite equation (5.1) as

$$\begin{aligned} (t-n+1)^{1-\alpha}\frac{dN(t)}{dt} &= -\mu N(t) + \beta e^{-\gamma N(n)}, \qquad t \in \left[n-\frac{1}{2},\, n\right) \\ (t-n)^{1-\alpha}\frac{dN(t)}{dt} &= -\mu N(t) + \beta e^{-\gamma N(n)}, \qquad t \in \left[n,\, n+\frac{1}{2}\right), \end{aligned}$$

or

$$\begin{aligned} \frac{dN(t)}{dt} + \frac{\mu}{(t-n+1)^{1-\alpha}}N(t) &= \frac{\beta}{(t-n+1)^{1-\alpha}}e^{-\gamma N(n)}, \qquad t \in \left[n-\frac{1}{2},\, n\right) \\ \frac{dN(t)}{dt} + \frac{\mu}{(t-n)^{1-\alpha}}N(t) &= \frac{\beta}{(t-n)^{1-\alpha}}e^{-\gamma N(n)}, \qquad t \in \left[n,\, n+\frac{1}{2}\right). \end{aligned}$$

By multiplying the last equation correspondingly with the integration multipliers $\exp\left(\frac{\mu}{\alpha}(t-n+1)^{\alpha}\right)$ and $\exp\left(\frac{\mu}{\alpha}(t-n)^{\alpha}\right)$ we obtain the equations

$$\begin{aligned} \frac{d}{dt}\left(N(t)e^{\frac{\mu}{\alpha}(t-n+1)^{\alpha}}\right) &= \frac{\beta}{(t-n+1)^{1-\alpha}}e^{\frac{\mu}{\alpha}(t-n+1)^{\alpha}-\gamma N(n)}, \qquad t \in \left[n-\frac{1}{2},\, n\right) \\ \frac{d}{dt}\left(N(t)e^{\frac{\mu}{\alpha}(t-n)^{\alpha}}\right) &= \frac{\beta}{(t-n)^{1-\alpha}}e^{\frac{\mu}{\alpha}(t-n)^{\alpha}-\gamma N(n)}, \qquad t \in \left[n,\, n+\frac{1}{2}\right). \end{aligned}$$

By integrating these equations over the intervals $\left[n-\frac{1}{2},\, n\right)$ and $\left[n,\, n+\frac{1}{2}\right)$ correspondingly we have discrete equations

$$\begin{aligned} N(n)\exp\left(\frac{\mu}{\alpha}\right) - N\left(n-\frac{1}{2}\right)\exp\left(\frac{\mu}{\alpha 2^{\alpha}}\right) &= \frac{\beta}{\mu}e^{-\gamma N(n)}\left(\exp\left(\frac{\mu}{\alpha}\right) - \exp\left(\frac{\mu}{\alpha 2^{\alpha}}\right)\right), \\ N\left(n+\frac{1}{2}\right)\exp\left(\frac{\mu}{\alpha 2^{\alpha}}\right) - N(n) &= \frac{\beta}{\mu}e^{-\gamma N(n)}\left(\exp\left(\frac{\mu}{\alpha 2^{\alpha}}\right) - 1\right), \end{aligned}$$

or

(5.3)

$$N(n) = \exp\left(-\frac{\mu}{\alpha}\left(1-\frac{1}{2^{\alpha}}\right)\right)N\left(n-\frac{1}{2}\right) + \frac{\beta}{\mu}\left(1-\exp\left(-\frac{\mu}{\alpha}\left(1-\frac{1}{2^{\alpha}}\right)\right)\right)e^{-\gamma N(n)},$$

(5.4)

$$N\left(n+\frac{1}{2}\right) = \exp\left(-\frac{\mu}{\alpha 2^{\alpha}}\right)N(n) + \frac{\beta}{\mu}\left(1-\exp\left(-\frac{\mu}{\alpha 2^{\alpha}}\right)\right)e^{-\gamma N(n)}$$

for $n = 1,\, 2,\, \ldots$. It is easy to see that the equilibrium point of this equation satisfies the equation

(5.5)
$$x^* = \frac{\beta}{\mu}e^{-\gamma x^*}.$$

Set $z_n := \gamma\left(N(n) - x^*\right)$. Then, we obtain the discrete equations

$$z_n = \gamma\left(N(n) - x^*\right)$$

$$\begin{aligned} &= \gamma \exp\left(-\frac{\mu}{\alpha}\left(1-\frac{1}{2^\alpha}\right)\right) N\left(n-\frac{1}{2}\right) \\ &\quad +\frac{\beta\gamma}{\mu}\left(1-\exp\left(-\frac{\mu}{\alpha}\left(1-\frac{1}{2^\alpha}\right)\right)\right) e^{-\gamma N(n)} - \gamma x^* \\ &= \exp\left(-\frac{\mu}{\alpha}\left(1-\frac{1}{2^\alpha}\right)\right)\left(z_{n-1/2}+\gamma x^*\right) \\ &\quad +\frac{\beta\gamma}{\mu} e^{-\gamma x^*}\left(1-\exp\left(-\frac{\mu}{\alpha}\left(1-\frac{1}{2^\alpha}\right)\right)\right) e^{-z_n} - \gamma x^* \end{aligned}$$

$$\begin{aligned} z_{n+1/2} &= \gamma\left(N\left(n+\frac{1}{2}\right) - x^*\right) \\ &= \gamma\left(\exp\left(-\frac{\mu}{\alpha 2^\alpha}\right) N(n) + \frac{\beta}{\mu}\left(1-\exp\left(-\frac{\mu}{\alpha 2^\alpha}\right)\right) e^{-\gamma N(n)} - x^*\right) \\ &= \exp\left(-\frac{\mu}{\alpha 2^\alpha}\right)(z_n+\gamma x^*) + \frac{\beta\gamma}{\mu} e^{-\gamma x^*}\left(1-\exp\left(-\frac{\mu}{\alpha 2^\alpha}\right)\right) e^{-z_n} - \gamma x^* \end{aligned}$$

or using (5.5)

$$z_n = -\gamma x^*\left(1-\exp\left(-\frac{\mu}{\alpha}\left(1-\frac{1}{2^\alpha}\right)\right)\right)\left(1-e^{-z_n}\right) + \exp\left(-\frac{\mu}{\alpha}\left(1-\frac{1}{2^\alpha}\right)\right) z_{n-1/2},$$

$$z_{n+1/2} = -\gamma x^*\left(1-\exp\left(-\frac{\mu}{\alpha 2^\alpha}\right)\right)\left(1-e^{-z_n}\right) + \exp\left(-\frac{\mu}{\alpha 2^\alpha}\right) z_n, \qquad n = 1,\, 2,\, \ldots$$

and these equations have the equilirium point zero.

By combining these two equations for $n = 1, 2, \ldots$ we have the equation

$$\begin{aligned} z_{n+1} &= -\gamma x^*\left(1-\exp\left(-\frac{\mu}{\alpha}\left(1-\frac{1}{2^\alpha}\right)\right)\right)\left(1-e^{-z_{n+1}}\right) \\ &\quad + \exp\left(-\frac{\mu}{\alpha}\left(1-\frac{1}{2^\alpha}\right)\right) z_{n+1/2} \\ &= -\gamma x^*\left(1-\exp\left(-\frac{\mu}{\alpha}\left(1-\frac{1}{2^\alpha}\right)\right)\right)\left(1-e^{-z_{n+1}}\right) \\ &\quad + \exp\left(-\frac{\mu}{\alpha}\left(1-\frac{1}{2^\alpha}\right)\right) \\ &\qquad \left(-\gamma x^*\left(1-\exp\left(-\frac{\mu}{\alpha 2^\alpha}\right)\right)\left(1-e^{-z_n}\right) + \exp\left(-\frac{\mu}{\alpha 2^\alpha}\right) z_n\right) \end{aligned} \tag{5.6}$$

in closed form.

Let us introduce asymptotic stability of the zero solution of equation (5.6).

Theorem 5.1. *The zero solution of equation* (5.6), *therefore the equilibrium point x^* of equations* (5.3) *and* (5.4), *is unstable.*

Proof. Using Taylor expansion of function e^{-z_n} and $e^{-z_{n+1}}$ around the point zero we linearize equation (5.6) as

$$z_{n+1} = -\gamma x^*\left(1-\exp\left(-\frac{\mu}{\alpha}\left(1-\frac{1}{2^\alpha}\right)\right)\right) z_{n+1}$$

$$+\exp\left(-\frac{\mu}{\alpha}\left(1-\frac{1}{2^\alpha}\right)\right)\left(-\gamma x^*\left(1-\exp\left(-\frac{\mu}{\alpha 2^\alpha}\right)\right)+\exp\left(-\frac{\mu}{\alpha 2^\alpha}\right)\right)z_n$$

or

$$z_{n+1}=\frac{1+\gamma x^*\left(1-\exp\left(-\frac{\mu}{\alpha}\left(1-\frac{1}{2^\alpha}\right)\right)\right)}{\exp\left(-\frac{\mu}{\alpha}\left(1-\frac{1}{2^\alpha}\right)\right)\left(-\gamma x^*\left(1-\exp\left(-\frac{\mu}{\alpha 2^\alpha}\right)\right)+\exp\left(-\frac{\mu}{\alpha 2^\alpha}\right)\right)}z_n$$

In order to have the stability of the zero solution of this equation, it is sufficient that the inequality

(5.7)

$$\left|\frac{1+\gamma x^*\left(1-\exp\left(-\frac{\mu}{\alpha}\left(1-\frac{1}{2^\alpha}\right)\right)\right)}{\exp\left(-\frac{\mu}{\alpha}\left(1-\frac{1}{2^\alpha}\right)\right)\left(-\gamma x^*\left(1-\exp\left(-\frac{\mu}{\alpha 2^\alpha}\right)\right)+\exp\left(-\frac{\mu}{\alpha 2^\alpha}\right)\right)}\right|<1$$

is satisfied. Let $p=\frac{\mu}{\alpha}\left(1-\frac{1}{2^\alpha}\right)$ and $q=\frac{\mu}{\alpha 2^\alpha}$. We have two cases: either

$$h:=\left(-\gamma x^*\left(1-\exp\left(-q\right)\right)+\exp\left(-q\right)\right)$$

is positive or negative. If it is positive, by simple calculation we obtain $\gamma x^*\left(1-e^{-p-q}\right)<e^{-p-q}-1$. Then, this is a contradiction as the left hand side of this inequality is positive, while the right hand side is negative. Hence, h cannot be positive. If it is negative, inequality (5.7) can be rewritten as

$$\left(e^p+e^{-q}\right)<\frac{\gamma x^*}{1+\gamma x^*}$$

Then, we have a contradiction as the left hand side of this inequality is greater than 1, while the right hand side is less than 1. The proof is complete. ☐

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C. Büyükadali
Department of Mathematics, Faculty of Science, Van Yüzüncü Yıl University, 65090 Van, Türkiye
E-mail address: cbuyukadali@yyu.edu.tr