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ON IMPLICIT BOUNDARY VALUE PROBLEMS WITH DEFORMABLE FRACTIONAL DERIVATIVE AND DELAY IN B-METRIC SPACES

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Abstract. In this paper, we demonstrate various existence and uniqueness results for a class of deformable implicit fractional differential equations with delay in b-metric spaces with boundary conditions. We base our arguments on some suitable fixed point theorems. In the last section, we provide different examples to illustrate our obtained results.

1. Introduction

Recently, fractional differential equations have been used in engineering, mathematics, physics, and other applied disciplines. The existence of solutions to ordinary and fractional differential equations with distinct conditions has received a great deal of attention; see the monographs [1, 2, 14, 30, 31, 35] and the papers [4,5,7,11,19,23,26–29]. Recently, some results of implicit fractional differential equations have been given, see [15–18, 24], and the references therein.

Functional differential equations with delay are often utilized as equation models. Several researchers investigated differential equations with time delays [8–10,12,13]. For more details, see the papers which are concerned with finite delay [26,27], infinite delay [1, 4, 8, 11], and state-dependent delay [1, 10].

In [36], Zulfeqarr *et al.* proposed the novel notion of deformable fractional derivative, employing the limit technique as in the usual derivative. It was termed "deformable" due to its inherent ability of continuously deforming function to derivative. Deformable derivatives can be thought of as fractional order derivatives.

The author of [20] investigated further properties of the new concept of deformable derivative and used the results to study the following Cauchy problem with non-local condition:

$$\begin{aligned} \mathfrak{D}_0^\alpha x(t) &= f(t, x(t)), \quad t \in (0, T], \\ x(0) + g(x) &= x_0, \end{aligned}$$

where $\mathfrak{D}_0^\alpha$ is the deformable derivative of order $\alpha \in (0, 1)$, and $g : \mathcal{C} \to \mathbb{R}$ is a continuous function. Their arguments are based on Krasnoselskii ' s fixed point theorem.

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In [22], Meraj and Pandey studied the existence and uniqueness of mild solution for the following initial value problem:

$$\begin{aligned}\mathfrak{D}_0^\alpha x(t) &= Ax(t) + f(t, x(t)), \quad t \in J,\\ x(0) &= x_0,\end{aligned}$$

where $A : D(A) \subset X \to X$ is an infinitesimal generator of a C_0-semigroup $T(t)(t \geqslant 0)$ on a suitable space $X, x_0 \in X$, and $J = [0, b], b > 0$ is a constant. The results are obtained with the help of semigroup theory, Banach fixed point theorem, and Schauder fixed point theorem.

In this paper, first we investigate the following class of conformable fractional differential equation with finite delay:

$$\begin{cases} \left(\mathfrak{D}_0^\alpha \chi\right)(\theta) = \Psi\left(\theta, \chi_\theta, \mathfrak{D}_0^\alpha \chi(\theta)\right), & \theta \in \Theta := (0, \varpi], \\ \chi(\theta) = \zeta(\theta), & \theta \in (-\kappa, 0], \\ \imath\chi(0) + \jmath\chi(\varpi) = \varrho, \end{cases} \tag{1.1}$$

where $\mathfrak{D}_0^\alpha \chi(\theta)$ is the deformable fractional derivative of order $\alpha \in (0, 1)$, $\Psi : \Theta \times C([-\kappa, 0], \mathbb{R}) \times \mathbb{R}$ is a continuous function, $\zeta \in C((-\kappa, 0], \mathbb{R})$, $0 < \varpi < +\infty$, $\imath, \jmath, \varrho$ are real constants, and $\kappa > 0$ is the time delay. For any $\theta \in \Theta$, we give χ_θ by

$$\chi_\theta(\vartheta) = \chi(\theta + \vartheta); \text{ for } \vartheta \in [-\kappa, 0].$$

Next, we consider the following infinite delay problem:

$$\begin{cases} \left(\mathfrak{D}_0^\alpha \chi\right)(\theta) = \Psi\left(\theta, \chi_\theta, \mathfrak{D}_0^\alpha \chi(\theta)\right), & \theta \in \Theta, \\ \chi(\theta) = \zeta(\theta), & \theta \in (-\infty, 0], \\ \imath\chi(0) + \jmath\chi(\varpi) = \varrho, \end{cases} \tag{1.2}$$

where $\Psi : \Theta \times \mathcal{G} \times \mathbb{R} \to \mathbb{R}$, $\zeta : (-\infty, 0] \to \mathbb{R}$ are given continuous functions, and $\mathcal{G}$ is called a phase space that will be determined later. For any $\theta \in \Theta$, we define $\chi_\theta \in \mathcal{G}$ by

$$\chi_\theta(\vartheta) = \chi(\theta + \vartheta); \text{ for } \vartheta \in (-\infty, 0].$$

2. Preliminaries

First, we give the definitions and the notations that we will use throughout this paper. We denote by $C(\Theta, \mathbb{R})$ and $C([-\kappa, 0], \mathbb{R})$ the Banach spaces of all continuous functions from Θ and $[-\kappa, 0]$ into $\mathbb{R}$ respectively, with the following norms

$$\|\chi\|_\infty = \sup_{\theta \in \Theta}\{|\chi(\theta)|\}$$

and

$$\|\chi\|_{[-\kappa,0]} = \sup_{\theta \in [-\kappa,0]}\{|\chi(\theta)|\}.$$

Let $\mathcal{C} := C([-\kappa, \varpi])$ be a Banach space with the norm

$$\|\chi\|_C := \sup_{\theta \in [-\kappa,\varpi]} |\chi(\theta)|.$$

Consider the space $X_b^p(0,\varpi)$, $(b\in\mathbb{R},\ 1\le p\le\infty)$ of those real-valued Lebesgue measurable functions Ψ on $[0,\kappa]$ for which $\|\Psi\|_{X_b^p}<\infty$, where the norm is given by:

$$\|\Psi\|_{X_b^p}=\left(\int_0^{\varpi}|\theta^b\Psi(\theta)|^p\frac{d\theta}{\theta}\right)^{\frac{1}{p}},\quad (1\le p<\infty, b\in\mathbb{R}).$$

Definition 2.1 (The deformable fractional derivative [20,36])**.** Let $\Psi:[0,+\infty)\longrightarrow\mathbb{R}$ be a given function, the non-conformable fractional derivative of Ψ of order α is defined by

$$\left(\mathfrak{D}_0^\alpha\Psi\right)(\theta)=\lim_{\varepsilon\to 0}\frac{(1+\varepsilon\beta)\Psi\left(\theta+\varepsilon\alpha\right)-\Psi(\theta)}{\varepsilon},$$

where $\alpha+\beta=1$ and $\alpha\in(0,1]$. If the deformable fractional derivative of Ψ of order α exists, then we simply say that Ψ is α-differentiable.

Definition 2.2 (The α-fractional integral [20,21])**.** For $\alpha\in(0,1]$ and a continuous function Ψ, let

$$\left(\mathcal{J}_{0^+}^\alpha\Psi\right)(\theta)=\frac{1}{\alpha}e^{\frac{-\beta}{\alpha}\theta}\int_0^\theta e^{\frac{\beta}{\alpha}\tau}\Psi(\tau)d\tau.$$

Lemma 2.3 ([20,21])**.** *If $\alpha,\alpha_1\in(0,1]$ such that $\alpha+\beta=1$, Ψ and Φ are two α-differentiable functions at a point θ and m,n are two given numbers, then the improved conformable fractional derivative satisfies the following properties:*

- $\mathfrak{D}_0^\alpha(\lambda)=\beta\lambda$, *for any constant* λ*;*
- $\mathfrak{D}_0^\alpha(m\Psi+n\Phi)=m\mathfrak{D}_0^\alpha(\Psi)+n\mathfrak{D}_0^\alpha(\Phi)$*;*
- $\mathfrak{D}_0^\alpha(\Psi\Phi)=\Phi\mathfrak{D}_0^\alpha(\Psi)+\alpha\Psi\Phi'$*;*
- $\mathcal{J}_{0^+}^\alpha\ \mathcal{J}_{0^+}^{\alpha_1}\Psi=\mathcal{J}_{0^+}^{\alpha+\alpha_1}\Psi$.

Lemma 2.4 ([20,21])**.** *If $\alpha\in(0,1]$, f is continuous function, then we have:*

- $\left(\mathcal{J}_{0^+}^\alpha\ \mathfrak{D}_0^\alpha(\Psi)\right)(\theta)=\Psi(\theta)-e^{\frac{-\beta}{\alpha}\theta}\Psi(0)$;
- $\mathfrak{D}_0^\alpha\left(\mathcal{J}_{0^+}^\alpha\Psi\right)(\theta)=\Psi(\theta)$.

Lemma 2.5. *Let $\Phi\in L^1(\Theta)$ and $0<\alpha\le 1$. Then the boundary value problem*

$$\begin{cases}\left(\mathfrak{D}_0^\alpha\chi\right)(\theta)=\Phi(\theta);\ \theta\in\Theta:=[0,\varpi],\\ \imath\chi(0)+\jmath\chi(\varpi)=\varrho,\end{cases}\tag{2.1}$$

has a unique solution defined by

$$\chi(\theta)=\frac{\varrho}{\imath+\jmath e^{\frac{-\beta}{\alpha}\varpi}}-\frac{\jmath}{\alpha\imath e^{\frac{\beta}{\alpha}\varpi}+\alpha\jmath}\int_0^{\varpi}e^{\frac{\beta}{\alpha}\tau}\Phi(\tau)d\tau+\frac{1}{\alpha}e^{\frac{-\beta}{\alpha}\theta}\int_0^\theta e^{\frac{\beta}{\alpha}\tau}\Phi(\tau)d\tau.\tag{2.2}$$

Proof. Applying the α-fractional integral of order α to both sides the equation $\left(\mathfrak{D}_0^\alpha\chi\right)(\theta)=\Phi(\theta)$, and by using Lemma 2.3 and if $\theta\in\Theta$, we get

$$\chi(\theta)-\chi(0)e^{\frac{-\beta}{\alpha}\theta}=\frac{1}{\alpha}e^{\frac{-\beta}{\alpha}\theta}\int_0^\theta e^{\frac{\beta}{\alpha}\tau}\Phi(\tau)d\tau.\tag{2.3}$$

Hence, we get

$$\chi(\theta)=\chi(0)e^{\frac{-\beta}{\alpha}\theta}+\frac{1}{\alpha}e^{\frac{-\beta}{\alpha}\theta}\int_0^\theta e^{\frac{\beta}{\alpha}\tau}\Phi(\tau)d\tau.\tag{2.4}$$

Thus,

$$\chi(\varpi) = \chi(0)e^{\frac{-\beta}{\alpha}\varpi} + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}\varpi}\int_0^{\varpi} e^{\frac{\beta}{\alpha}\tau}\Phi(\tau)d\tau.$$

From the mixed boundary conditions $\imath\chi(0) + \jmath\chi(\varpi) = \varrho$, we get

$$\imath\chi(0) + \jmath\left(\chi(0)e^{\frac{-\beta}{\alpha}\varpi} + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}\varpi}\int_0^{\varpi} e^{\frac{\beta}{\alpha}\tau}\Phi(\tau)d\tau\right) = \varrho.$$

Thus

$$\chi(0) = \frac{\varrho - \frac{\jmath}{\alpha}e^{\frac{-\beta}{\alpha}\varpi}\int_0^{\varpi} e^{\frac{\beta}{\alpha}\tau}\Phi(\tau)d\tau}{\imath + \jmath e^{\frac{-\beta}{\alpha}\varpi}}.$$

Hence, we obtain(2.2).

Conversely, we can easily show by Lemma 2.3 that if χ verifies equation (2.2) then it satisfied the problem (2.1). □

Definition 2.6 ([3])**.** Let $\mathcal{H}$ be a set and $\varepsilon \geq 1$ be a given real number. A distance function $\delta : \mathcal{H} \times \mathcal{H} \to (0, \infty)$ is called a b-metric if the following conditions hold for all $\chi_1, \chi_2, \chi_3 \in \mathcal{H}$:

(1) $\delta(\chi_1, \chi_2) = 0 \; if \; and \; only \; if \chi_1 = \chi_2$,
(2) $\delta(\chi_1, \chi_2) = \delta(\chi_2, \chi_1)$,
(3) $\delta(\chi_1, \chi_2) \leq \varepsilon[\delta(\chi_1, \chi_3) + \delta(\chi_3, \chi_2)]$; (b-triangular inequality).

Then, the pair $(\mathcal{H}, \delta, \varepsilon)$ is called a b-metric space with parameter ε.

Let $\overline{\Phi}$ be the set of all increasing and continuous function $\psi : (0, \infty) \to (0, \infty)$ satisfying the property: $\psi(\varepsilon\chi) \leq \varepsilon\psi(\chi) \leq \varepsilon\chi$, for $\varepsilon > 1$ and $\psi(0) = 0$. We denote by Ξ the family of all nondecreasing functions $\eta : (0, \infty) \to [0, \frac{1}{\varepsilon^2})$ for some $\varepsilon \geq 1$.

Definition 2.7 ([3])**.** Let $(\mathcal{H}, \delta, \varepsilon)$ be a b-metric space, $\mathfrak{S} : \mathcal{H} \to \mathcal{H}$ is said to be a generalized ω-ψ-Geraghty mapping whenever there exists $\omega : \mathcal{H} \times \mathcal{H} \to (0, \infty)$ such that

$$\omega(\chi_1, \chi_2)\psi(\varepsilon^3 d(\mathfrak{S}(\chi_1), \mathfrak{S}(\chi_2))) \leq \eta(\psi(\delta(\chi_1, \chi_2)))\psi(\delta(\chi_1, \chi_2)),$$

for $\chi_1, \chi_2 \in \mathcal{H}$, where $\eta \in \Xi$.

Definition 2.8 ([3])**.** Let $\mathcal{H}$ be a non empty set, $\mathfrak{S} : \mathcal{H} \to \mathcal{H}$ and $\omega : \mathcal{H} \times \mathcal{H} \to (0, \infty)$ be given mappings. The operator $\mathfrak{S}$ is orbital ω-admissible if for $\chi \in \mathcal{H}$, we have

$$\omega(\chi, \mathfrak{S}(\chi)) \geq 1 \Rightarrow \omega(\mathfrak{S}(\chi), \mathfrak{S}^2(\chi)) \geq 1.$$

Theorem 2.9 ([3])**.** *Let $(\mathcal{H}, \delta)$ be a complete b-metric space and $\aleph : \mathcal{H} \to \mathcal{H}$ be a generalized ω-ψ-Geraghty mapping where*

(a)**:** *$\aleph$ is ω-admissible;*
(b)**:** *there exists $\chi_0 \in \mathcal{H}$ where $\omega(\chi_0, \aleph(\chi_0)) \geq 1$;*
(c)**:** *If $(\chi_n)_{n\in N} \subset \mathcal{H}$ with $\chi_n \to \chi$ and $\omega(\chi_n, \chi_{n+1}) \geq 1$, then $\omega(\chi_n, \chi) \geq 1$,*

Then $\aleph$ has a fixed point. Moreover, if

(d)**:** *for all fixed points $\chi, \overline{\chi}$ of $\aleph$, either*

$$\omega(\chi, \overline{\chi}) \geq 1 \quad or \quad \omega(\overline{\chi}, \chi) \geq 1,$$

Then $\aleph$ has a unique fixed point.

3. Existence Results for the First Problem

Let $(C(\Theta),\delta,2)$ be the complete b-metric space with $\varepsilon=2$, such that $\delta : C(\Theta)\times C(\Theta)\to(0,\infty)$, is given by:

$$\delta(\chi,\Im)=\|(\chi-\Im)^2\|_\infty:=\sup_{\theta\in\Theta}|\chi(\theta)-\Im(\theta)|^2.$$

Definition 3.1. A solution of problem (1.1) is a function $\chi\in C([-\kappa,\varpi],\mathbb{R})$ where

$$\chi(\theta)=\begin{cases}\dfrac{\varrho}{\imath+\jmath e^{\frac{-\beta}{\alpha}\varpi}}-\dfrac{\jmath}{\alpha\imath e^{\frac{\beta}{\alpha}\varpi}+\alpha\jmath}\displaystyle\int_0^{\varpi}e^{\frac{\beta}{\alpha}\tau}\Phi(\tau)d\tau\\ +\dfrac{1}{\alpha}e^{\frac{-\beta}{\alpha}\theta}\displaystyle\int_0^{\theta}e^{\frac{\beta}{\alpha}\tau}\Phi(\tau)d\tau,\ \ \theta\in[0,\varpi],\\ \\ \zeta(\theta),\ \ \theta\in[-\kappa,0],\end{cases}$$

such that $\Phi\in C(\Theta)$, with $\Phi(\theta)=\Psi(\theta,\chi_\theta,\Phi(\theta))$ and $\imath+\jmath e^{\frac{-\beta}{\alpha}\varpi}\neq 0$.

The hypotheses:

(H_1)**:** There exist $\overline{p}:C([-\kappa,0])\times C([-\kappa,0])\to(0,\infty)$ and $\overline{q}:\Theta\to(0,1)$ such that for each $\chi,\Im\in C([-\kappa,0])$, $\chi_1,\Im_1\in\mathbb{R}$ and $\theta\in\Theta$

$$|\Psi(\theta,\chi,\chi_1)-\Psi(\theta,\Im,\Im_1)|\leq\overline{p}(\chi,\Im)\|\chi-\Im\|_{[-\kappa,0]}+\overline{q}(\theta)|\chi_1-\Im_1|$$

with

$$\left\|-\frac{\jmath\displaystyle\int_0^{\varpi}e^{\frac{\beta}{\alpha}\tau}\frac{\overline{p}(\chi,\Im)}{1-\overline{q}*}d\tau}{\alpha\imath e^{\frac{\beta}{\alpha}\varpi}+\alpha\jmath}+\frac{1}{\alpha}e^{\frac{-\beta}{\alpha}\theta}\int_0^{\theta}e^{\frac{\beta}{\alpha}\tau}\frac{\overline{p}(\chi,\Im)}{1-\overline{q}*}d\tau\right\|_C^2\leq\psi(\|(\chi-\Im)^2\|_C).$$

(H_2)**:** There exist $\psi\in\overline{\Phi}$ and $\overline{\lambda}_0\in\mathcal{C}$ and a function $\overline{\theta}:\mathcal{C}\times\mathcal{C}\to\mathbb{R}$, such that

$$\overline{\theta}\left(\overline{\lambda}_0(\theta),\zeta(\theta)\right)\geq 0\quad if\ \theta\in[-\kappa,0],$$

$$\overline{\theta}\left(\overline{\lambda}_0(\theta),-\frac{\jmath}{\alpha\imath e^{\frac{\beta}{\alpha}\varpi}+\alpha\jmath}\int_0^{\varpi}e^{\frac{\beta}{\alpha}\tau}\Phi(\tau)d\tau+\frac{1}{\alpha}e^{\frac{-\beta}{\alpha}\theta}\int_0^{\theta}e^{\frac{\beta}{\alpha}\tau}\Phi(\tau)d\tau\right)$$
$$\geq 0,\qquad if\ \theta\in\Theta,$$

where $\Phi\in C(\Theta)$ such that $\Phi(\theta)=\Psi(\theta,\overline{\lambda}_0(\theta),\Phi(\theta))$.

(H_3)**:** For each $\theta\in\Theta$, and $\chi,\Im\in\mathcal{C}$, we have

$$\overline{\theta}(\chi(\theta),\Im(\theta))\geq 0$$

implies

$$\overline{\theta}\left(\zeta(\theta),\zeta(\theta)\right)\geq 0\quad if\ \theta\in[-\kappa,0],$$

$$\overline{\theta}\left(\frac{\varrho}{\imath+\jmath e^{\frac{-\beta}{\alpha}\varpi}}-\frac{\jmath\displaystyle\int_0^{\varpi}e^{\frac{\beta}{\alpha}\tau}\Phi(\tau)d\tau}{\alpha\imath e^{\frac{\beta}{\alpha}\varpi}+\alpha\jmath}+\frac{1}{\alpha}e^{\frac{-\beta}{\alpha}\theta}\int_0^{\theta}e^{\frac{\beta}{\alpha}\tau}\Phi(\tau)d\tau,\frac{\varrho}{\imath+\jmath e^{\frac{-\beta}{\alpha}\varpi}}\right.$$

$$-\frac{\jmath\int_0^{\varpi} e^{\frac{\beta}{\alpha}\tau}\Phi'(\tau)d\tau}{\alpha\imath e^{\frac{\beta}{\alpha}\varpi}+\alpha\jmath}+\frac{1}{\alpha}e^{\frac{-\beta}{\alpha}\theta}\int_0^{\theta} e^{\frac{\beta}{\alpha}\tau}\Phi'(\tau)d\tau\Bigg)\geq 0 \quad if\ \theta\in\Theta,$$

where $\Phi,\overline{\Phi}\in C(\Theta)$ such that

$$\Phi(\theta)=\Psi(\theta,\chi(\theta),\Phi(\theta))$$

and

$$\overline{\Phi}(\theta)=\Psi(\theta,\Im(\theta),\overline{\Phi}(\theta)).$$

(H_4): If $(\chi_n)_{n\in N}\subset\mathcal{C}$ with $\chi_n\to\chi$ and $\overline{\theta}(\chi_n,\chi_{n+1})\geq 1$,then

$$\overline{\theta}(\chi_n,\chi)\geq 1.$$

(H_5): For all fixed solutions $\chi,\overline{\chi}$ of problem (1.1), either

$$\overline{\theta}(\chi(\theta),\overline{\chi}(\theta))\geq 0,$$

or

$$\overline{\theta}(\overline{\chi}(\theta),\chi(\theta))\geq 0.$$

Theorem 3.2. *Assume that the hypotheses (H_1)-(H_4) hold. Then the problem (1.1) has a least one solution defined on $[-\kappa,\varpi]$. Moreover, if (H_5) holds, then we get a unique solution.*

Proof. Consider the operator $\mathcal{K}:\ \mathcal{C}\to\mathcal{C}$ defined by:

$$(\mathcal{K}\chi)(\theta)=\begin{cases}\dfrac{\varrho}{\imath+\jmath e^{\frac{-\beta}{\alpha}\varpi}}-\dfrac{\jmath}{\alpha\imath e^{\frac{\beta}{\alpha}\varpi}+\alpha\jmath}\displaystyle\int_0^{\varpi}e^{\frac{\beta}{\alpha}\tau}\Phi(\tau)d\tau\\ +\dfrac{1}{\alpha}e^{\frac{-\beta}{\alpha}\theta}\displaystyle\int_0^{\theta}e^{\frac{\beta}{\alpha}\tau}\Phi(\tau)d\tau,\ \theta\in[0,\varpi],\\ \\ \zeta(\theta),\ \theta\in[-\kappa,0],\end{cases}\tag{3.1}$$

where $\Phi\in C(\Theta)$ such that $\Phi(\theta)=\Psi(\theta,\chi_\theta,\Phi(\theta))$.
The function $\omega:\ \mathcal{C}\times\mathcal{C}\to(0,\infty)$ is given by:

$$\begin{cases}\omega(\chi,\overline{\chi})=1; & if\ \overline{\theta}(\chi(\theta),\overline{\chi}(\theta))\geq 0,\ \theta\in\Theta,\\ \omega(\chi,\overline{\chi})=0; & eles.\end{cases}$$

First, we prove that $\mathcal{K}$ is a generalized ω-ψ-Geraghty operator:
For any $\chi,\overline{\chi}\in\mathcal{C}$.Then, for each $\theta\in[-\kappa,0]$ we have

$$|(\mathcal{K}\chi)(\theta)-(\mathcal{K}\overline{\chi})(\theta)|=0,$$

where $\Phi,\ \overline{\Phi}\in C(\Theta)$ such that

$$\Phi(\theta)=\Psi(\theta,\chi(\theta),\Phi(\theta))\ and\ \Phi'(\theta)=\Psi(\theta,\overline{\chi}(\theta),\overline{\Phi}(\theta)).$$

Thus

$$\omega(\chi,\overline{\chi})|(\mathcal{K}\chi)(\theta)-(\mathcal{K}\overline{\chi})(\theta)|^2\leq\|(\chi-\overline{\chi})^2\|_C\psi(\|(\chi-\overline{\chi})^2\|_C),$$

where $\eta \in \Xi$, $\psi \in \overline{\Phi}$, with $\eta(\theta) = \frac{1}{8}\theta$, and $\psi(\theta) = \theta$. So, $\mathcal{K}$ is generalized ω-ψ-Geraghty operator on $[-\kappa, 0]$. And, for each $\theta \in \Theta$, we have

$$|(\mathcal{K}\chi)(\theta) - (\mathcal{K}\overline{\chi})(\theta)| \leq \frac{\jmath}{\alpha\imath e^{\frac{\beta}{\alpha}\varpi} + \alpha\jmath} \int_0^{\varpi} e^{\frac{\beta}{\alpha}\tau} |\Phi(\tau) - \overline{\Phi}(\tau)| d\tau + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}\theta} \int_0^{\theta} e^{\frac{\beta}{\alpha}\tau} |\Phi(\tau) - \overline{\Phi}(\tau)| d\tau.$$

From (H_1) we have

$$\|\Phi - \Phi'\|_\infty \leq \frac{\overline{p}(\chi, \overline{\chi})}{1 - \overline{q}*} \|(\chi - \overline{\chi})^2\|_C^{\frac{1}{2}},$$

where $\overline{q}* = \sup_{\theta\in\Theta} |\overline{q}(\theta)|$.
Next, we have

$$|(\mathcal{K}\chi)(\theta) - (\mathcal{K}\overline{\chi})(\theta)| \leq \frac{\jmath}{\alpha\imath e^{\frac{\beta}{\alpha}\varpi} + \alpha\jmath} \int_0^{\varpi} e^{\frac{\beta}{\alpha}\tau} \frac{\overline{p}(\chi, \overline{\chi})}{1 - \overline{q}*} \|(\chi - \overline{\chi})^2\|_C^{\frac{1}{2}} d\tau + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}\theta} \int_0^{\theta} e^{\frac{\beta}{\alpha}\tau} \frac{\overline{p}(\chi, \overline{\chi})}{1 - \overline{q}*} \|(\chi - \overline{\chi})^2\|_C^{\frac{1}{2}} d\tau.$$

Thus

$$\begin{aligned}
&\omega(\chi, \overline{\chi})|(\mathcal{K}\chi)(\theta) - (\mathcal{K}\overline{\chi})(\theta)|^2 \\
&\leq \|(\chi - \overline{\chi})^2\|_C \omega(\chi, \overline{\chi}) \left\| \frac{\jmath}{\alpha\imath e^{\frac{\beta}{\alpha}\varpi} + \alpha\jmath} \int_0^{\varpi} e^{\frac{\beta}{\alpha}\tau} \frac{\overline{p}(\chi, \overline{\chi})}{1 - \overline{q}*} d\tau \right. \\
&\left. + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}\theta} \int_0^{\theta} e^{\frac{\beta}{\alpha}\tau} \frac{\overline{p}(\chi, \overline{\chi})}{1 - \overline{q}*} d\tau \right\|_C^2 \\
&\leq \|(\chi - \overline{\chi})^2\|_C \psi(\|(\chi - \overline{\chi})^2\|_C).
\end{aligned}$$

Hence

$$\omega(\chi, \overline{\chi})\psi(2^3 d(\mathcal{K}(\chi), \mathcal{K}(\overline{\chi})) \leq \eta(\psi(\delta(\chi, \overline{\chi}))\psi(\delta(\chi, \overline{\chi})),$$

where $\eta \in \Xi$, $\psi \in \overline{\Phi}$, with $\eta(\theta) = \frac{1}{8}\theta$, and $\psi(\theta) = \theta$. So, $\mathcal{K}$ is generalized ω-ψ-Geraghty operator on Θ.

Let $\chi, \overline{\chi} \in \mathcal{C}$ such that

$$\omega(\chi, \overline{\chi}) \geq 1.$$

Thus, for each $\theta \in \Theta$, we have

$$\overline{\theta}(\chi(\theta), \overline{\chi}(\theta)) \geq 0.$$

This implies from (H_3) that

$$\overline{\theta}(\mathcal{K}\chi(\theta), \mathcal{K}\overline{\chi}(\theta)) \geq 0,$$

which gives

$$\omega(\mathcal{K}(\chi), \mathcal{K}(\overline{\chi})) \geq 1.$$

Hence, $\mathcal{K}$ is a ω-admissible.

Now, by (H_2), there exist $\overline{\lambda}_0 \in \mathcal{C}$ such that

$$\omega(\overline{\lambda}_0, \aleph(\overline{\lambda}_0)) \geq 1.$$

Thus, by (H_4), if $\left(\overline{\lambda}_n\right)_{n\in N} \subset \mathcal{H}$ with $\overline{\lambda}_n \to \overline{\lambda}$ and $\omega(\overline{\lambda}_n, \overline{\lambda}_{n+1}) \geq 1$, then

$$\omega(\overline{\lambda}_n, \overline{\lambda}) \geq 1.$$

From an application of Theorem 2.9, we conclude that $\mathcal{K}$ has a fixed point χ which is a solution of problem (1.1).

Moreover, (H_5) implies that if χ and $\overline{\chi}$ are fixed points of $\mathcal{K}$, then either

$$\overline{\theta}(\chi, \overline{\chi}) \geq 0 \quad or \quad \overline{\theta}(\overline{\chi}, \chi) \geq 0.$$

This implies that either

$$\omega(\chi, \overline{\chi}) \geq 1 \quad or \quad \omega(\overline{\chi}, \chi) \geq 1.$$

Hence, problem (1.1) has a unique solution. □

4. Existence Results for the Second Problem

In this section, we are concerned with the existence results of (1.2). Let the space $(\mathcal{G}, \|\cdot\|_{\mathcal{G}})$ is a seminormed linear space of functions mapping $(-\infty, 0]$ into $\mathbb{R}$, and verifying the following axioms which were derived from Hale and Kato's originals [8]:

- **(Ax_1):** If $y : (-\infty, 0] \to \mathbb{R}$, and $\chi_0 \in \mathcal{G}$, then there exist constants $\xi_1, \xi_2, \xi_3 > 0$, such that for each $\theta \in \Theta$; we have:
  - **(i):** *χ_θ is in $\mathcal{G}$,*
  - **(ii):** $\|\chi_\theta\|_{\mathcal{G}} \leq \xi_1 \|\chi_1\|_{\mathcal{G}} + \xi_2 \sup_{\vartheta \in [0,\theta]} |\chi(\vartheta)|$,
  - **(iii):** $|\chi(\theta)| \leq \xi_3 \|\chi_\theta\|_{\mathcal{G}}$.
- **(Ax_2):** For the function $\chi(\cdot)$ *in* (Ax_1), y_θ is a $\mathcal{G}-$ valued continuous function on Θ.
- **(Ax_3):** The space $\mathcal{G}$ is complete.

Consider the space

$$\Xi = \{\chi : (-\infty, \varpi] \to \mathbb{R},\ \chi|_{(-\infty,0]} \in \mathcal{G},\ \chi|_{\Theta} \in C([0, \varpi], \mathbb{R})\}.$$

Definition 4.1. By a solution of problem (1.2), we mean a function $\chi \in \Xi$ such that

$$\chi(\theta) = \begin{cases} \dfrac{\varrho}{\imath + \jmath e^{\frac{-\beta}{\alpha}\varpi}} - \dfrac{\jmath}{\alpha \imath e^{\frac{\beta}{\alpha}\varpi} + \alpha\jmath} \displaystyle\int_0^{\varpi} e^{\frac{\beta}{\alpha}\tau} \Phi(\tau) d\tau \\ + \dfrac{1}{\alpha} e^{\frac{-\beta}{\alpha}\theta} \displaystyle\int_0^{\theta} e^{\frac{\beta}{\alpha}\tau} \Phi(\tau) d\tau, \ \theta \in [0, \varpi], \\ \\ \zeta(\theta), \ \theta \in (-\infty, 0], \end{cases}$$

where $\Phi \in C(\Theta)$, with $\Phi(\theta) = \Psi(\theta, \chi_\theta, \Phi(\theta))$ and $\imath + \jmath e^{\frac{-\beta}{\alpha}\varpi} \neq 0$.

The hypothesis:

- **(H_{01}):** There exist $\overline{M} : \mathcal{G} \times \mathcal{G} \to (0, \infty)$ and $\overline{N} : \Theta \to (0, 1)$ such that for each $\chi, \Im \in \mathcal{G}$, $\chi_1, \Im_1 \in \mathbb{R}$ and $\theta \in \Theta$

$$|\Psi(\theta, \chi, \chi_1) - \Psi(\theta, \Im, \Im_1)| \leq \overline{M}(\chi, \Im) \|\chi - \Im\|_{\mathcal{G}} + \overline{N}(\theta) |\chi_1 - \Im_1|$$

with

$$\left\| -\frac{\jmath\int_0^{\varpi} e^{\frac{\beta}{\alpha}\tau}\frac{\overline{M}(\chi,\Im)}{1-\overline{N}*}d\tau}{\alpha\imath e^{\frac{\beta}{\alpha}\varpi}+\alpha\jmath} + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}\theta}\int_0^{\theta} e^{\frac{\beta}{\alpha}\tau}\frac{\overline{M}(\chi,\Im)}{1-\overline{N}*}d\tau \right\|_{\mathcal{G}}^2 \leq \psi(\|(\chi-\Im)^2\|_{\mathcal{G}}).$$

(H_{02}): There exist $\psi \in \overline{\Phi}$ and $\overline{\lambda}_0 \in \mathcal{G}$ and a function $\overline{\theta} : \mathcal{G}\times\mathcal{G} \to \mathbb{R}$, such that

$$\overline{\theta}\left(\overline{\lambda}_0(\theta), -\frac{\jmath}{\alpha\imath e^{\frac{\beta}{\alpha}\varpi}+\alpha\jmath}\int_0^{\varpi} e^{\frac{\beta}{\alpha}\tau}\Phi(\tau)d\tau + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}\theta}\int_0^{\theta} e^{\frac{\beta}{\alpha}\tau}\Phi(\tau)d\tau\right) \geq 0,$$

where $\Phi \in C(\Theta)$ such that $\Phi(\theta) = \Psi(\theta, \overline{\lambda}_0(\theta), \Phi(\theta))$.

(H_{03}): For each $\theta \in \Theta$, and $\chi, \Im \in \mathcal{G}$, we have:

$$\overline{\theta}(\chi(\theta), \Im(\theta)) \geq 0$$

implies

$$\overline{\theta}\Bigg(-\frac{\jmath}{\alpha\imath e^{\frac{\beta}{\alpha}\varpi}+\alpha\jmath}\int_0^{\varpi} e^{\frac{\beta}{\alpha}\tau}\Phi(\tau)d\tau + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}\theta}\int_0^{\theta} e^{\frac{\beta}{\alpha}\tau}\Phi(\tau)d\tau,$$
$$-\frac{\jmath}{\alpha\imath e^{\frac{\beta}{\alpha}\varpi}+\alpha\jmath}\int_0^{\varpi} e^{\frac{\beta}{\alpha}\tau}\Phi'(\tau)d\tau + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}\theta}\int_0^{\theta} e^{\frac{\beta}{\alpha}\tau}\Phi'(\tau)d\tau\Bigg) \geq 0,$$

where $\Phi, \overline{\Phi} \in C(\Theta)$ such that

$$\Phi(\theta) = \Psi(\theta, \chi(\theta), \Phi(\theta))$$

and

$$\overline{\Phi}(\theta) = \Psi(\theta, \Im(\theta), \overline{\Phi}(\theta)).$$

(H_{04}): If $(\chi_n)_{n\in N} \subset \mathcal{G}$ with $\chi_n \to \chi$ and $\overline{\theta}(\chi_n, \chi_{n+1}) \geq 1$,then

$$\overline{\theta}(\chi_n, \chi) \geq 1.$$

(H_{05}): For all fixed solutions $\chi, \overline{\chi}$ of problem (1.2), either

$$\overline{\theta}(\chi(\theta), \overline{\chi}(\theta)) \geq 0,$$

or

$$\overline{\theta}(\overline{\chi}(\theta), \chi(\theta)) \geq 0.$$

Theorem 4.2. *Assume that the hypotheses (H_{01})-(H_{04}) hold. Then the problem (1.2) has a least one solution defined on $(-\infty, \varpi]$. Moreover, if (H_{05}) holds, then we get a unique solution.*

Proof. Consider the operator $\aleph : \Xi \to \Xi$ defined by:

$$(\aleph\chi)(\theta) = \begin{cases} \dfrac{\varrho}{\imath+\jmath e^{\frac{-\beta}{\alpha}\varpi}} - \dfrac{\jmath}{\alpha\imath e^{\frac{\beta}{\alpha}\varpi}+\alpha\jmath}\displaystyle\int_0^{\varpi} e^{\frac{\beta}{\alpha}\tau}\Phi(\tau)d\tau \\ +\dfrac{1}{\alpha}e^{\frac{-\beta}{\alpha}\theta}\displaystyle\int_0^{\theta} e^{\frac{\beta}{\alpha}\tau}\Phi(\tau)d\tau,\ \theta\in[0,\varpi], \\ \zeta(\theta),\ \theta\in(-\infty,0], \end{cases} \tag{4.1}$$

where $\Phi \in C(\Theta)$ such that $\Phi(\theta) = \Psi(\theta, \chi_\theta, \Phi(\theta))$.
Let $\varkappa_1 : (-\infty, \varpi] \to \mathbb{R}$ be a function given by

$$\varkappa_1(\theta) = \begin{cases} \zeta(\theta); & \theta \in (-\infty, 0], \\ \chi(\varpi) & \theta \in \Theta. \end{cases}$$

Then $\varkappa_{10} = \zeta$. For each $\varkappa_2 \in C(\Theta)$, with $\varkappa_2(0) = 0$, we denote by $\overline{\varkappa_2}$ the function defined by

$$\overline{\varkappa_2} = \begin{cases} 0, & \theta \in \theta \in (-\infty, 0], \\ \varkappa_2(\theta), & \theta \in \Theta. \end{cases}$$

If $\chi(\cdot)$ satisfies the integral equation

$$\chi(\theta) \quad = \frac{\varrho}{\imath + \jmath e^{\frac{-\beta}{\alpha}\varpi}} - \frac{\jmath}{\alpha \imath e^{\frac{\beta}{\alpha}\varpi} + \alpha \jmath} \int_0^\varpi e^{\frac{\beta}{\alpha}\tau} \Phi(\tau) d\tau + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}\theta} \int_0^\theta e^{\frac{\beta}{\alpha}\tau} \Phi(\tau) d\tau,$$

we can decompose $\chi(\cdot)$ as $\chi(\theta) = \overline{\varkappa_2}(\theta) + \varkappa_1(\theta)$; for $\theta \in \Theta$, which implies that $\chi_\theta = \overline{\varkappa_2}_\theta + \varkappa_{1\theta}$ for every $\theta \in \Theta$, and the function $\varkappa_2(\cdot)$ satisfies

$$\varkappa_2(\theta) = -\frac{\jmath}{\alpha \imath e^{\frac{\beta}{\alpha}\varpi} + \alpha \jmath} \int_0^\varpi e^{\frac{\beta}{\alpha}\tau} \Phi(\tau) d\tau + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}\theta} \int_0^\theta e^{\frac{\beta}{\alpha}\tau} \Phi(\tau) d\tau,$$

where

$$\Phi(\theta) = \Psi(\theta, \overline{\varkappa_2}_\theta + \varkappa_{1\theta}, \Phi(\theta)); \ \theta \in \Theta.$$

Set

$$\mathcal{D}_0 = \{\varkappa_2 \in C(\Theta); \ \varkappa_{20} = 0\},$$

and let $\|\cdot\|_\varpi$ be the norm in $\mathcal{D}_0$ defined by

$$\|\varkappa_2\|_\varpi = \|\varkappa_{20}\|_{\mathcal{G}} + \sup_{\theta \in \Theta} |\varkappa_2(\theta)| = \sup_{\theta \in \Theta} |\varkappa_2(\theta)|; \ \varkappa_2 \in \mathcal{D}_0,$$

where $\mathcal{D}_0$ is a Banach space with norm $\|\cdot\|_\varpi$. Define the operator $\mathcal{K} : \mathcal{D}_0 \to \mathcal{D}_0$ by

$$(\mathcal{K}\varkappa_2)(\theta) = -\frac{\jmath}{\alpha \imath e^{\frac{\beta}{\alpha}\varpi} + \alpha \jmath} \int_0^\varpi e^{\frac{\beta}{\alpha}\tau} \Phi(\tau) d\tau + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}\theta} \int_0^\theta e^{\frac{\beta}{\alpha}\tau} \Phi(\tau) d\tau, \tag{4.2}$$

where

$$\Phi(\theta) = \Psi(\theta, \overline{\varkappa_2}_\theta + \varkappa_{1\theta}, \Phi(\theta)); \ \theta \in \Theta.$$

The function $\omega : \ \mathcal{D}_0 \times \mathcal{D}_0 \to (0, \infty)$ is given by:

$$\begin{cases} \omega(\varkappa_2, \varkappa_2') = 1; & if \ \overline{\theta}(\varkappa_2(\theta), \varkappa_2'(\theta)) \geq 0, \ \theta \in \Theta, \\ \omega(\varkappa_2, \varkappa_2') = 0; & eles. \end{cases}$$

First, we prove that $\mathcal{K}$ is a generalized ω-ψ-Geraghty operator:
For any $\varkappa_2, \varkappa_2' \in \mathcal{D}_0$.Then, for each $\theta \in \Theta$, we have

$$\begin{aligned} \|(\mathcal{K}\varkappa_2)(\theta) - (\mathcal{K}\varkappa_2')(\theta)\| \leq & \frac{\jmath}{\alpha \imath e^{\frac{\beta}{\alpha}\varpi} + \alpha \jmath} \int_0^\varpi e^{\frac{\beta}{\alpha}\tau} \|\Phi(\tau) - \overline{\Phi}(\tau)\| d\tau \\ & + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}\theta} \int_0^\theta e^{\frac{\beta}{\alpha}\tau} \|\Phi(\tau) - \overline{\Phi}(\tau)\| d\tau, \end{aligned}$$

where $\Phi, \ \overline{\Phi} \in C(\Theta)$ such that

$$\Phi(\theta) = \Psi(\theta, \overline{\varkappa_2}_{n\theta} + \varkappa_{1\theta}, \Phi(\theta)) \ and \ \Phi'(\theta) = \Psi(\theta, \overline{\varkappa'_2}_{n\theta} + \varkappa_{1\theta}, \overline{\Phi}(\theta)).$$

From (H_{01}) we have

$$\|\Phi - \Phi'\|_\infty \leq \frac{\overline{M}(\varkappa_2, {\varkappa_2}')}{1 - \overline{N}*} \|(\varkappa_2 - {\varkappa_2}')^2\|_\varpi^{\frac{1}{2}},$$

where $\overline{N}* = \sup_{\theta \in \Theta} |\overline{N}(\theta)|$. Next, we have

$$\begin{aligned}|(\mathcal{K}\varkappa_2)(\theta) - (\mathcal{K}{\varkappa_2}')(\theta)| \leq & \frac{\jmath}{\alpha\imath e^{\frac{\beta}{\alpha}\varpi} + \alpha\jmath} \int_0^\varpi e^{\frac{\beta}{\alpha}\tau} \frac{\overline{M}(\varkappa_2, {\varkappa_2}')}{1 - \overline{N}*} \|(\varkappa_2 - {\varkappa_2}')^2\|_\varpi^{\frac{1}{2}} d\tau \\ & + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}\theta} \int_0^\theta e^{\frac{\beta}{\alpha}\tau} \frac{\overline{M}(\varkappa_2, {\varkappa_2}')}{1 - \overline{N}*} \|(\varkappa_2 - {\varkappa_2}')^2\|_\varpi^{\frac{1}{2}} d\tau.\end{aligned}$$

Thus

$$\begin{aligned}&\omega(\varkappa_2, {\varkappa_2}')|(\mathcal{K}\varkappa_2)(\theta) - (\mathcal{K}{\varkappa_2}')(\theta)|^2 \\ &\leq \|(\varkappa_2 - {\varkappa_2}')^2\|_\varpi \omega(\varkappa_2, {\varkappa_2}') \left\| \frac{\jmath}{\alpha\imath e^{\frac{\beta}{\alpha}\varpi} + \alpha\jmath} \int_0^\varpi e^{\frac{\beta}{\alpha}\tau} \frac{\overline{M}(\varkappa_2, {\varkappa_2}')}{1 - \overline{N}*} d\tau \right. \\ &\left. + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}\theta} \int_0^\theta e^{\frac{\beta}{\alpha}\tau} \frac{\overline{M}(\varkappa_2, {\varkappa_2}')}{1 - \overline{N}*} d\tau \right\|_\varpi^2 \\ &\leq \|(\varkappa_2 - {\varkappa_2}')^2\|_\varpi \psi(\|(\varkappa_2 - {\varkappa_2}')^2\|_\varpi).\end{aligned}$$

Hence

$$\omega(\varkappa_2, {\varkappa_2}')\psi(2^3 d(\mathcal{K}(\varkappa_2), \mathcal{K}({\varkappa_2}')) \leq \eta(\psi(\delta(\varkappa_2, {\varkappa_2}'))\psi(\delta(\varkappa_2, {\varkappa_2}')),$$

where $\eta \in \Xi$, $\psi \in \overline{\Phi}$, with $\eta(\theta) = \frac{1}{8}\theta$, and $\psi(\theta) = \theta$. So, $\mathcal{K}$ is generalized ω-ψ-Geraghty operator.

Let $\varkappa_2, {\varkappa_2}' \in \mathcal{D}_0$ such that

$$\omega(\varkappa_2, {\varkappa_2}') \geq 1.$$

Thus, for each $\theta \in \Theta$, we have

$$\overline{\theta}(\varkappa_2(\theta), {\varkappa_2}'(\theta)) \geq 0.$$

This implies from (H_{03}) that

$$\overline{\theta}(\mathcal{K}\varkappa_2(\theta), \mathcal{K}{\varkappa_2}'(\theta)) \geq 0,$$

which gives

$$\omega(\mathcal{K}(\varkappa_2), \mathcal{K}({\varkappa_2}')) \geq 1.$$

Hence, $\mathcal{K}$ is a ω-admissible.

Now, by (H_{02}), there exist $\overline{\lambda}_0 \in \mathcal{C}$ such that

$$\omega(\overline{\lambda}_0, \mathcal{K}(\overline{\lambda}_0)) \geq 1.$$

Thus, by (H_{04}), if $\left(\overline{\lambda}_n\right)_{n\in N} \subset \mathcal{H}$ with $\overline{\lambda}_n \to \overline{\lambda}$ and $\omega(\overline{\lambda}_n, \overline{\lambda}_{n+1}) \geq 1$, then

$$\omega(\overline{\lambda}_n, \overline{\lambda}) \geq 1.$$

From an application of Theorem 2.9, we conclude that $\mathcal{K}$ has a fixed point. Consequently, $\aleph$ has a fixed point which is the unique solution of problem (1.2).

Moreover, (H_{05}) implies that if $\varkappa_2$ and $\varkappa_2'$ are fixed points of $\mathcal{K}$, then either

$$\overline{\theta}(\varkappa_2, \varkappa_2') \geq 0 \quad or \quad \overline{\theta}(\varkappa_2', \varkappa_2) \geq 0.$$

This implies that either

$$\omega(\varkappa_2, \varkappa_2') \geq 1 \quad or \quad \omega(\varkappa_2', \varkappa_2) \geq 1,$$

Hence, problem (1.2) has a unique solution. □

5. Some Examples

We give now some examples that illustrate our obtained results throughout the paper.

Example 5.1. Consider the following problem:

$$(5.1) \qquad \begin{cases} (\mathfrak{D}_0^{\frac{1}{2}}\chi)(\theta) = \dfrac{1+\sin(\|\chi_\theta\|_{[-1,0]})}{4(1+\|\chi_\theta\|_{[-1,0]})} + \dfrac{1}{4(1+|(\mathcal{D}_0^{\frac{1}{2}}\chi)(\theta)|)}; \ \theta \in [0,1], \\ \chi(\theta) = \theta + 1; \ \theta \in (-1,0], \\ \chi(0) + \chi(1) = 0. \end{cases}$$

Set

$$\Psi(\theta, \chi, \Im) = \frac{1+\sin(\|\chi\|_{[-1,0]})}{4(1+\|\chi\|_{[-1,0]})} + \frac{1}{4(1+|\Im|)},$$

where $\theta \in [0,1], \ \chi \in C([-1,0]), \Im \in \mathbb{R}$.
For any $\chi, \Im \in C([-1,0]), \ \overline{\chi}, \overline{\Im} \in \mathbb{R}$ and $\theta \in [0,1]$. If $\|\chi\|_{[-1,0]} \leq \|\Im\|_{[-1,0]}$, then

$$\begin{aligned}
&|\Psi(\theta,\chi,\overline{\chi}) - \Psi(\theta,\Im,\overline{\Im})| \\
&\leq \left| \frac{1+\sin(\|\chi\|_{[-1,0]})}{4(1+\|\chi\|_{[-1,0]}+|\overline{\chi}|)} - \frac{1+\sin(\|\Im\|_{[-1,0]})}{4(1+\|\Im\|_{[-1,0]}+|\overline{\Im}|)} \right| + \frac{|\overline{\chi}-\overline{\Im}|}{4} \\
&\leq \frac{1}{4}|\|\chi\|_{[-1,0]} - \|\Im\|_{[-1,0]}| + \frac{1}{4}|\sin(\|\chi\|_{[-1,0]}) - \sin(\|\Im\|_{[-1,0]})| \\
&\quad + \left|\|\chi\|_{[-1,0]}\sin(\|\Im\|_{[-1,0]}) - \|\Im\|_{[-1,0]}\sin(\|\chi\|_{[-1,0]})\right| + \frac{|\overline{\chi}-\overline{\Im}|}{4} \\
&\leq |\chi - \Im| + \frac{1}{4}|\sin(\|\chi\|_{[-1,0]}) - \sin(\|\Im\|_{[-1,0]})| + \frac{|\overline{\chi}-\overline{\Im}|}{4} \\
&\quad + \left|\|\Im\|_{[-1,0]}\sin(\|\Im\|_{[-1,0]}) - \|\Im\|_{[-1,0]}\sin(\|\chi\|_{[-1,0]})\right| \\
&= |\chi) - \Im| + (1+\|\Im\|_{[-1,0]})\left|\sin(\|\chi\|_{[-1,0]}) - \sin(\|\Im\|_{[-1,0]})\right| + \frac{|\overline{\chi}-\overline{\Im}|}{4} \\
&\leq |\chi - \Im| + \frac{1}{2}(1+\|\Im\|_{[-1,0]}) \\
&\quad \times \left|\sin\left(\tfrac{\|\chi\|_{[-1,0]}-\|\Im\|_{[-1,0]}}{2}\right)\right| \left|\cos\left(\tfrac{\|\chi\|_{[-1,0]}+\|\Im\|_{[-1,0]}}{2}\right)\right| \\
&\leq (2+\|\Im\|_{[-1,0]})\|\chi - \Im\|_{[-1,0]} + \frac{\|\overline{\chi}-\overline{\Im}\|_\infty}{4}.
\end{aligned}$$

The case when $\|\Im\|_{[-1,0]} \leq \|\chi\|_{[-1,0]}$, we get

$$|\Psi(\theta,\chi,\overline{\chi}) - \Psi(\theta,\Im,\overline{\Im})| \leq (2+\|\chi\|_{[-1,0]})\|\chi-\Im\|_{[-1,0]} + \frac{\|\overline{\chi}-\overline{\Im}\|_\infty}{4}.$$

Hence

$$|\Psi(\theta,\chi,\overline{\chi}) - \Psi(\theta,\Im,\overline{\Im})| \leq \min\{2+\|\chi\|_{[-1,0]}, 2+\|\Im\|_{[-1,0]}\|\chi-\Im\|_{[-1,0]} + \frac{\|\overline{\chi}-\overline{\Im}\|_\infty}{4}.$$

Thus, hypothesis (H_2) is satisfied with

$$\overline{p}(\chi,\Im) = \min\{2+\|\chi\|_{[-1,0]}, 2+\|\Im\|_\infty\}, \text{ and } \overline{q}(\theta) = \frac{1}{4}.$$

Define the functions $\eta(\theta) = \frac{1}{8}\theta$, $\psi(\theta) = \theta$, $\omega : C(\Theta)\times C(\Theta) \to \mathbb{R}_+^*$ with

$$\begin{cases} \omega(\chi,\Im) = 1; \ if \ \mathcal{Y}(\chi(\theta),\Im(\theta)) \geq 0, \ \theta\in\Theta, \\ \omega(\chi,\Im) = 0; \ else, \end{cases}$$

and $\mathcal{Y} : C([0,1])\times C([0,1]) \to \mathbb{R}$ with $\mathcal{Y}(\chi,\Im) = \|\chi-\Im\|_\infty$.
Hypothesis (H_2) is satisfied with $\overline{\lambda}_0(\theta) = \chi_0$. Also, (H_3) holds from the definition of the function $\mathcal{Y}$.

Simple computations show that all conditions of Theorem 3.2 are satisfied. It follows that problem (5.1) has at least one solution defined on $(-1,1]$.

Example 5.2. Consider the following example:

$$\begin{cases} (\mathfrak{D}_0^{\frac{1}{2}}\chi)(\theta) = \dfrac{1+\sin(\|\chi\|_{B_\gamma})}{8(1+\|\chi\|_{B_\gamma})} + \dfrac{e^{-\theta}}{2\left(1+\left|\left(\mathfrak{D}_0^{\frac{1}{2}}\chi\right)(\theta)\right|\right)}; \ \theta\in[0,1], \\ \chi(\theta) = \theta+1; \ \theta\in(-\infty,0], \\ \chi(0)+\chi(1) = 0. \end{cases} \tag{5.2}$$

Let γ be a positive real constant and

$$B_\gamma = \{\chi\in C((-\infty,1],\mathbb{R},) : \lim_{\tau\to-\infty} e^{\gamma\tau}\chi(\tau) \ exists \ in \ \mathbb{R}\}. \tag{5.3}$$

The norm of B_γ is given by

$$\|\chi\|_\gamma = \sup_{\tau\in(-\infty,1]} e^{\gamma\tau}|\chi(\tau)|.$$

Let $\chi : (-\infty,0] \to \mathbb{R}$ be such that $\chi_0 \in B_\gamma$. Then

$$\begin{aligned} \lim_{\tau\to-\infty} e^{\gamma\tau}\chi_\theta(\tau) &= \lim_{\tau\to-\infty} e^{\gamma\tau}\chi(\theta+\tau-1) = \lim_{\tau\to-\infty} e^{\gamma(\tau-\theta+1)}\chi(\tau) \\ &= e^{\gamma(-\theta+1)}\lim_{\tau\to-\infty} e^{\gamma(\tau)}\chi_1(\tau) < \infty. \end{aligned}$$

Hence $\chi_\theta \in B_\gamma$. Finally we prove that

$$\|\chi_\theta\|_\gamma \leq \xi_1\|\chi_1\|_\gamma + \xi_2 \sup_{\vartheta\in[0,\theta]} |\chi(\vartheta)|,$$

where $\xi_1 = \xi_2 = 1$ and $\xi_3 = 1$. We have

$$|\chi_\theta(\tau)| = |\chi(\theta+\tau)|.$$

If $\theta + \tau \le 1$, we get

$$|\chi_\theta(\xi)| \le \sup_{\vartheta\in(-\infty,0]} |\chi(\vartheta)|.$$

For $\theta + \tau \ge 0$, then we have

$$|\chi_\theta(\xi)| \le \sup_{\vartheta\in[0,\theta]} |\chi(\vartheta)|.$$

Thus for all $\theta + \tau \in \Theta$, we get

$$|\chi_\theta(\xi)| \le \sup_{\vartheta\in(-\infty,0]} |\chi(\vartheta)| + \sup_{\vartheta\in[0,\theta]} |\chi(\vartheta)|.$$

Then

$$\|\chi_\theta\|_\gamma \le \|\chi_0\|_\gamma + \sup_{\vartheta\in[0,\theta]} |\chi(\vartheta)|.$$

It is clear that $(B_\gamma, \|\cdot\|)$ is a Banach space. We can conclude that B_γ a phase space. Set

$$\Psi(\theta,\chi,\Im) = \frac{1+\sin(\|\chi\|_{B_\gamma})}{8(1+\|\chi\|_{B_\gamma})} + \frac{e^{-\theta}}{2(1+|\Im|)},$$

where $\theta\in[0,1],\ \chi\in B_\gamma, \Im\in\mathbb{R}$.

Let $(C([0,1]),\delta,2)$ be the complete b-metric space with $\varepsilon = 2$, such that $\delta : C([0,1])\times C([0,1]) \to (0,\infty)$, is given by:

$$\delta(\chi,\Im) = \|(\chi-\Im)^2\|_\infty := \sup_{\theta\in[0,1]} |\chi(\theta)-\Im(\theta)|^2.$$

For any $\chi,\Im\in B_\gamma,\ \overline{\chi},\overline{\Im}\in C(\Theta)$ and $\theta\in[0,1]$. If $\|\chi\|_{B_\gamma}\le\|\Im\|_{B_\gamma}$, then

$$|\Psi(\theta,\chi,\overline{\chi}) - \Psi(\theta,\Im,\overline{\Im})| \le \min\{2+\|\chi\|_{B_\gamma}, 2+\|\Im\|_{B_\gamma}\}\|\chi-\Im\|_{B_\gamma} + \frac{\|\overline{\chi}-\overline{\Im}\|_\infty}{4}.$$

Then, hypothesis (H_{03}) is satisfied with

$$\overline{M}(\chi,\Im) = \frac{1}{8}\min\{2+\|\chi\|_{B_\gamma}, 2+\|\Im\|_{B_\gamma}\}$$

and

$$\overline{N}(\theta) = \frac{e^{-\theta}}{2}.$$

Define the functions $\lambda(\theta) = \frac{1}{8}\theta$, $\phi(\theta)=\theta$, $\alpha : C([0,1])\times C([0,1]) \to \mathbb{R}_+^*$ with

$$\begin{cases} \alpha(\chi,\Im) = 1;\ if\ \delta(\chi(\theta),\Im(\theta)) \ge 0,\ \theta\in\Theta, \\ \alpha(\chi,\Im) = 0;\ else, \end{cases}$$

and $\delta : C([0,1])\times C([0,1]) \to \mathbb{R}$ with $\delta(\chi,\Im) = \|\chi-\Im\|_\infty$.
Hypothesis (H_{02}) is satisfied with $\overline{\mu}_0(\theta) = \chi_0$. Also, (H_{03}) holds from the definition of the function δ.

Since all requirements of Theorem 4.2 are verified. Then, we conclude the existence the uniqueness of solutions and for problem (5.2).

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