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FORWARD-BACKWARD ALGORITHM FOR SOLUTIONS OF VARIATIONAL INEQUALITY PROBLEM INVOLVING QUASI-MONOTONE OPERATORS

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Abstract. In this paper, a forward-backward iterative algorithm induced by a certain dynamical system for solutions of variational inequality problem involving quasi-monotone operator is introduced and studied. Weak convergence of the sequence generated by the said algorithm is proved in the setting of real Hilbert space. Numerical examples are given to demonstrate the efficiency and workability of the algorithm. The theorem obtained augments, generalizes, improves and unifies several results announced recently.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|.\|$. Let C be a closed convex nonempty subset of H, and $A : H \to H$ is a continuous operator. The *Variational Inequality Problem* (VIP) involving A is to;

$$\text{find } x^* \in C \text{ such that } \langle Ax^*, y - x^* \rangle \geq 0 \qquad \forall\, y \in C. \tag{1.1}$$

The set of solutions of VIP(1.1) shall be denoted by S; that is, $S = \{x \in C \,:\, \langle Ax, y - x\rangle \geq 0 \;\forall\, y \in C\}$.

Minty formulation of VIP (see Minty [18], see also Crespi and Rocca [5]), is to;

$$\text{find } x^* \in C \text{ such that } \langle Ay, y - x^* \rangle \geq 0 \qquad \forall\, y \in C. \tag{1.2}$$

The set of solutions of VIP(1.2) shall be denoted by S_M. Thus, $S_M = \{x \in C \,:\, \langle Ay, y - x\rangle \geq 0 \;\forall\, y \in C\}$.

It can be shown that S_M is a closed and convex subset of C; and using the continuity of the operator A, it can further be shown that $S_M \subseteq S$.

The study of variational inequality problem has proved useful as several transportation, programming, engineering, biological, and optimization problems can be modeled by VIP(1.1) (see for example, [1,2,7,12,13,17]). One of the popular method for solving VIP(1.1) is the extragradient method introduced by Korpelevich [13]. The extragradient method studied by Korpelevich [13] is given by

$$\begin{cases} x_1 \in C \\ y_n = P_C(x_n - \gamma_n A x_n), \\ x_{n+1} = P_C(x_n - \gamma_n A y_n),\ n \in \mathbb{N}, \end{cases} \tag{1.3}$$

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where A is monotone (pseudo-monotone) Lipschitz continuous operator, $\gamma_n \in (0, \frac{1}{L})$ and $L > 0$ is the Lipschitz constant of the operator A. The method was shown to converge weakly to the solution of VIP(1.1) (see for example, [8,24]). However, the extragradient method has two attributes which makes it difficult and computationally expensive. At each iteration, it involves two projections onto the feasible set C, and two evaluation of the operator A.

In trying to improve on the extragradient method, several variants were introduced which addressed one or both drawbacks. Popov [20] introduced his variant of extragradient method which required only one evaluation of the operator A. Several authors have established weak convergence of Popov's subgradient extragradient methods when A is monotone (or pseudo-monotone) and Lipschitz continuous (see for example, [4,6,15,22,26]). The subgradient extragradient method introduced by Censor *et al.* [3] involves two projections at each iteration, where one projection is on a certain half space. Tseng [23] introduced another variant of extragradient method known as the forward-backward-forward method which involves only one projection onto the feasible set C at each iteration. Tseng method is given by

$$(1.4) \qquad \begin{cases} x_1 \in C \\ y_n = P_C(x_n - \gamma_n A x_n), \\ x_{n+1} = y_n + \gamma_n(Ax_n - Ay_n), \ n \in \mathbb{N}, \end{cases}$$

where $\gamma_n \in (0, \frac{1}{L})$ and $L > 0$ is the Lipschitz constant of the operator A. Tseng [23] showed that (1.4) converges weakly to the solution of VIP(1.1).

The results mentioned above were obtained with A being a monotone (or pseudo-monotone) operator, but are very difficult to use to approximate the zeros of A with a weaker assumption that A is quasi-monotone. This is true in a sense as the convergence analysis used for monotone operator fails when A is quasi-monotone. Given thas H is an infinite dimensional Hilbert space, Lin and Yang [14] and Salahuddin [21] independently proved that their forward-backward-forward and extragradient methods respectively converges weakly to a solution of VIP(1.1) when A is quasi-monotone, Lipschitz continuous and sequentially weakly continuous. Using inertial projection and contraction method, Wang *et al.* [25] obtained a weak solution of VIP(1.1) when A is quasi-monotone and Lipschitz continuous. The iterative scheme proposed in [25] requires computation of two projections onto the feasible set C and two evaluation of A at each iteration.

Izuchukwu *et al.* [10] proposed an inertial forward-backward type method with self-adaptive step sizes for solving VIP(1.1) which involves only one projection onto feasible set C and one evaluation of A at each iteration. In fact, the algorithm of Izuchukwu *et al.* [10] is given by

$$(1.5) \qquad \begin{cases} x_0, x_1 \in H \\ w_n = x_n + \theta(x_n - x_{n-1}) \\ x_{n+1} = P_C(w_n - \gamma_n A x_n - \alpha_n(Ax_n - Ax_{n-1})), \ n \in \mathbb{N}. \end{cases}$$

They proved that (1.5) converges to a weak solution of VIP(1.1) when A is quasi-monotone and Lipschitz continuous. A similar algorithm was also introduced and studied by Izuchukwu *et al.* [11]. The results obtained in [10] and [11] improved and augumented the results of Lin and Yang [14], Salahuddin [21], and Wang *et al.* [25].

Motivated by the works of Izuchukwu *et al.* [10, 11] and recent research trend on obtaining solution of VIP(1.1) with a quasi-monotone and Lipschitz continuous operator, we examine the possibility of developing a method with a simpler and more encompassing step size that has mild constraint unlike the step size used in [10]. This led to the following question:

Question 1. Can a more general inertial forward-backward-type method with a simpler and more encompassing step size which yields the conclusions of Izuchukwu *et al.* [10] and Izuchukwu *et al.* [11] be constructed?

Question 2. Can a more general inertial forward-backward-type method that naturally avoids the assumption $\alpha_n = \gamma_{n-1}\ \forall\, n \in \mathbb{N}$ as observed in [10] be constructed?

Question 3. Can the works of Izuchukwu *et al.* [10] and Izuchukwu *et al.* [11] be improved upon?

It is our purpose in this paper to give an affirmative answers to Questions 1, 2 and 3 above in the setting of real Hilbert space. Our result will complement, generalize, improve and unify corresponding results of the authors cited above.

2. Preliminaries

All through this paper, the weak convergence of the sequence $\{u_n\}$ to a point u^*, shall be denoted by $u_n \rightharpoonup u^*$ as $n \to \infty$; and in what follows, the following definitions and lemmas shall play crucial and important roles in the sequel:

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. Let $A : D(A) \subset H \to R(A) \subset H$ be an operator, then A is

(a) $L-$*Lipschitz continuous* if there exists an $L > 0$ such that for all $x, y \in D(A)$,
$$\|Ax - Ay\| \leq L\|x - y\|,$$

(b) $\eta-$*strongly monotone* if there exists an $\eta > 0$ such that for all $x, y \in D(A)$,
$$\langle Ax - Ay, x - y \rangle \geq \eta\|x - y\|^2,$$

(c) *monotone* if for all $x, y \in H$,
$$\langle Ax - Ay, x - y \rangle \geq 0,$$

(d) η-*strongly pseudo-monotone* if there exists $\eta > 0$ such that for all $x, y \in D(A)$,
$$\langle Ay, x - y \rangle \geq 0 \Rightarrow \langle Ax, x - y \rangle \geq \eta\|x - y\|^2,$$

(e) *pseudo-monotone* if for all $x, y \in D(A)$,
$$\langle Ay, x - y \rangle \geq 0 \Rightarrow \langle Ax, x - y \rangle \geq 0,$$

(f) *quasi-monotone* if for all $x, y \in D(A)$,
$$\langle Ay, x - y \rangle > 0 \Rightarrow \langle Ax, x - y \rangle \geq 0,$$

(g) *sequentially weakly-strongly continuous*, if for every sequence $\{x_n\}$ that converges weakly to a point y, the sequence $\{Ax_n\}$ converges strongly to Ay,

(h) *sequentially weakly continuous*, if for every sequence $\{x_n\}$ that converges weakly to a point y, the sequence $\{Ax_n\}$ converges weakly to Ay.

Clearly, $(b) \Rightarrow (c)$, $(d) \Rightarrow (e)$, and $(c) \Rightarrow (e) \Rightarrow (f)$. The converse is not always true (see [24] and references therein for examples).

Let C be a nonempty closed convex subset of a real Hilbert space, H. The mapping $P_C : H \to C$ is called projection mapping if and only if for all $x \in H$,

$$\|P_C x - x\| = \inf_{z \in C} \|x - z\|.$$

The following statements are equivalent (see, for example, [19] for details):

$$\begin{cases} (i.) \quad P_C : H \to C \text{ is a projection of } H \text{ onto } C, \\ (ii.) \quad \text{for all } x \in H, \quad \langle x - P_C x, z - P_C x \rangle \le 0, \ \forall\, z \in C, \\ (iii.) \quad \text{for all } x \in H, \ \|P_C x - z\|^2 \le \|x - z\|^2 - \|P_C x - x\|^2, \ \forall\, z \in C. \end{cases} \tag{2.1}$$

Lemma 2.1. Let H be a real Hilbert space, then for any $x, y \in H$, and for $\lambda \in [0, 1]$, the following inequalities hold;

$$2\langle x, y \rangle = \|x\|^2 + \|y\|^2 - \|x - y\|^2 = \|x + y\|^2 - \|x\|^2 - \|y\|^2,$$

and

$$\|\lambda x + (1 - \lambda) y\|^2 = \lambda \|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2.$$

Lemma 2.2. [27] Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Then, S_D is nonempty if either of the following holds;

i.) A is pseudomonotone on C and $S \neq \emptyset$,

ii.) A is the gradient of G, where G is a differential quasiconvex function on an open set $K \supset C$ and attains its global minimum on C,

iii.) A is quasi-monotone on C, $A \neq 0$ on C and C is bounded,

iv.) A is quasi-monotone on C, $A \neq 0$ on C and there exists $r > 0$ such that, for every $y \in C$ with $\|y\| \ge r$, there exists $x \in C$ such that $\|x\| \le r$ and $\langle Ay, x - y \rangle \ge 0$,

v.) A is quasi-monotone on C, *int* C is nonempty and there exists $y^* \in S$ such that $Ay^* \neq 0$.

3. Method and derived algorithm

The algorithm proposed in this work is derived from the following implicit first-order dynamical system associated with the VIP(1.1):

$$\begin{cases} \dot{x}(t) + x(t) = P_C\Big(\theta(t)\dot{x}(t) + x(t) - z(t) - \gamma(t-1)\dot{y}(t)\Big), \\ y(t) = Ax(t), \\ z(t) = \gamma(t) A x(t), \end{cases} \tag{3.1}$$

where $\theta, \gamma : \mathbb{R} \to [0, \infty)$ are Lebesgue measurable functions. If $\theta(s) = 0$ and $\gamma(s) = \gamma > 0$ for all $s \in \mathbb{R}$, then (3.1) the continuous dynamical system associated with VIP(1.1) whose discrete version is the forward-backward splitting algorithm studied in [16].

If we perform a forward discretization of $\dot{x}(t)$ on the left hand side of (3.1) (that is, $\dot{x}(t) \approx x_{n+1} - x_n, n \in \mathbb{N}$) and backward discretization of $\dot{x}(t)$ and $\dot{y}(t)$ on the right hand side of (3.1) (that is, for $n \in \mathbb{N}$, $\dot{x}(t) \approx x_n - x_{n-1}$ and $\dot{y}(t) \approx y_n - y_{n-1}$), then we obtain the following iterative algorithm:

$$x_{n+1} = P_C\Big(x_n + \theta_n(x_n - x_{n-1}) - \gamma_n Ax_n - \gamma_{n-1}(Ax_n - Ax_{n-1})\Big). \tag{3.2}$$

With some mild conditions on the iterative parameters θ_n, γ_n, $n \in \mathbb{N}$, (3.2) is the inertial-type forward-backward method studied in this paper for approximation of solution of the VIP(1.1). We now present the proposed algorithm for approximate solution of VIP(1.1) in details as follows:

Algorithm 1.

1. Choose $\theta_1, \gamma_0, \gamma_1 > 0$. Let $x_0, x_1 \in C$ be fixed and set $n := 1$.

2. Compute

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = P_C\Big(w_n - \big((\gamma_n + \gamma_{n-1})Ax_n - \gamma_{n-1}Ax_{n-1}\big)\Big), \end{cases} \tag{3.3}$$

where $\{\theta_n\}_{n=1}^{\infty}, \{\gamma_n\}_{n=1}^{\infty}$ are sequences in $[0, \infty)$.

3. Set $n \leftarrow n + 1$ and go to 2.

We make the following assumptions for weak convergence of Algorithm 1;

(a.) $S \neq \emptyset$,
(b.) $A : C \to H$ is $L-$Lipschitz continuous,
(c.) The mapping $\|A(\cdot)\| : C \to \mathbb{R}$ is weakly lower semicontinuous; in the sense that for any sequence $\{u_n\}_{n=1}^{\infty}$ in C such that $u_n \rightharpoonup u^*$ as $n \to \infty$, then $\|A(u^*)\| \leq \liminf_{n\to\infty} \|Au_n\|$,
(d.) A is a quasi-monotone mapping.

4. Main results

In this section, the convergence theorems obtained in this paper is presented and proved. Let us proceed as follows:

4.1. **Weak convergence result.**

Lemma 4.1. Let $\{x_n\}_{n=0}^{\infty}$ be generated by Algorithm 1 such that assumptions (a.) and (b.) hold. Suppose that $\{\theta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are such that there exists $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$, and for some $\delta \in (0, 1)$, θ_n is monotone non-decreasing, $2\theta_n < \frac{\delta}{2}$, and $\gamma_{n-1} < \frac{1-\delta}{L}$, then $\{x_n\}_{n=0}^{\infty}$ is bounded.

Proof. Fix $u_0 \in S_M \subset S$, then using (2.1*ii.*), (3.3), and lemma 2.1, we obtain for all $n \in \mathbb{N}$ that

$$\begin{aligned} 0 &\leq 2\langle x_{n+1} - w_n + (\gamma_n + \gamma_{n-1})Ax_n - \gamma_{n-1}Ax_{n-1}, u_0 - x_{n+1}\rangle \\ &= 2\langle x_{n+1} - w_n, u_0 - x_{n+1}\rangle \\ &\quad + 2\gamma_n\langle Ax_n, u_0 - x_{n+1}\rangle + 2\gamma_{n-1}\langle Ax_n - Ax_{n-1}, u_0 - x_{n+1}\rangle \end{aligned}$$

$$= \|w_n - u_0\|^2 - \|x_{n+1} - w_n\|^2 - \|x_{n+1} - u_0\|^2 + 2\gamma_n\langle Ax_n, u_0 - x_{n+1}\rangle$$
$$+ 2\gamma_{n-1}\langle Ax_n - Ax_{n-1}, u_0 - x_{n+1}\rangle \tag{4.1}$$

Now, since $x_{n+1} \in C$ and $u_0 \in S_M \subset S \subset C$, we obtain from (1.2) that $\langle Ax_{n+1}, x_{n+1} - u_0\rangle \geq 0$ for all $n \in \mathbb{N}$. This implies that for all $n \geq 1$,

$$\langle Ax_n, u_0 - x_{n+1}\rangle \leq \langle Ax_n - Ax_{n+1}, u_0 - x_{n+1}\rangle.$$

Thus, (4.1) gives

$$\|x_{n+1} - u_0\|^2 \leq \|w_n - u_0\|^2 - \|x_{n+1} - w_n\|^2$$
$$+ 2\gamma_n\langle Ax_n - Ax_{n+1}, u_0 - x_{n+1}\rangle$$
$$+ 2\gamma_{n-1}\langle Ax_n - Ax_{n-1}, u_0 - x_n\rangle + 2\gamma_{n-1}\langle Ax_n - Ax_{n-1}, x_n - x_{n+1}\rangle \tag{4.2}$$

Using the fact that A is $L-$Lipschitz continuous, we obtain that,

$$2\gamma_{n-1}\langle Ax_n - Ax_{n-1}, x_n - x_{n+1}\rangle \leq 2\gamma_{n-1}\|Ax_n - Ax_{n-1}\| \ \|x_{n+1} - x_n\|$$
$$\leq 2\gamma_{n-1}L\|x_n - x_{n-1}\| \ \|x_{n+1} - x_n\|$$
$$\leq \gamma_{n-1}L\Big(\|x_n - x_{n-1}\|^2 + \|x_{n+1} - x_n\|^2\Big) \tag{4.3}$$

Thus, for all $n \geq N_0$, we obtain from (4.3) that,

$$2\gamma_{n-1}\langle Ax_n - Ax_{n-1}, x_n - x_{n+1}\rangle \leq (1-\delta)\big(\|x_n - x_{n-1}\|^2 + \|x_{n+1} - x_n\|^2\big) \tag{4.4}$$

So, for all $n \geq N_0$, we obtain using (4.4) in (4.2) that,

$$\|x_{n+1} - u_0\|^2 \leq \|w_n - u_0\|^2 - \|x_{n+1} - w_n\|^2$$
$$+ 2\gamma_n\langle Ax_n - Ax_{n+1}, u_0 - x_{n+1}\rangle + 2\gamma_{n-1}\langle Ax_n - Ax_{n-1}, u_0 - x_n\rangle$$
$$+ (1-\delta)\|x_n - x_{n-1}\|^2 + (1-\delta)\|x_{n+1} - x_n\|^2 \tag{4.5}$$

By (3.3) and lemma 2.1($ii.$), we obtain that,

$$\|w_n - u_0\|^2 = \|(1+\theta_n)(x_n - u_0) - \theta_n(x_{n-1} - u_0)\|^2$$
$$= (1+\theta_n)\|x_n - u_0\|^2 - \theta_n\|x_{n-1} - u_0\|^2 + \theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2 \tag{4.6}$$

Also,

$$\|x_{n+1} - w_n\|^2 = \|x_{n+1} - x_n - \theta_n(x_n - x_{n-1})\|^2$$
$$= \|x_{n+1} - x_n\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 - 2\theta_n\langle x_{n+1} - x_n, x_n - x_{n-1}\rangle$$
$$\geq \|x_{n+1} - x_n\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 - 2\theta_n\|x_{n+1} - x_n\| \ \|x_n - x_{n-1}\|$$
$$\geq (1-\theta_n)\|x_{n+1} - x_n\|^2 + (\theta_n^2 - \theta_n)\|x_n - x_{n-1}\|^2 \tag{4.7}$$

Using (4.6) and (4.7) in (4.5) gives that for all $n \geq N_0$, we have that,

$$\|x_{n+1} - u_0\|^2 \leq (1+\theta_n)\|x_n - u_0\|^2 - \theta_n\|x_{n-1} - u_0\|^2 + \theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2$$
$$- (1-\theta_n)\|x_{n+1} - x_n\|^2 - (\theta_n^2 - \theta_n)\|x_n - x_{n-1}\|^2$$
$$+ 2\gamma_n\langle Ax_n - Ax_{n+1}, u_0 - x_{n+1}\rangle + 2\gamma_{n-1}\langle Ax_n - Ax_{n-1}, u_0 - x_n\rangle$$
$$+ (1-\delta)\|x_n - x_{n-1}\|^2 + (1-\delta)\|x_{n+1} - x_n\|^2,$$

so that

$$\|x_{n+1} - u_0\|^2 - \theta_n\|x_n - u_0\|^2 + \|x_{n+1} - x_n\|^2$$

$$
\begin{aligned}
&\quad + (\delta - \theta_n)\|x_{n+1} - x_n\|^2 + 2\gamma_n\langle Ax_{n+1} - Ax_n, u_0 - x_{n+1}\rangle \\
&\le \|x_n - u_0\|^2 - \theta_{n-1}\|x_{n-1} - u_0\|^2 + \|x_n - x_{n-1}\|^2 \\
&\quad - (\delta - 2\theta_n)\|x_n - x_{n-1}\|^2 + 2\gamma_{n-1}\langle Ax_n - Ax_{n-1}, u_0 - x_n\rangle \\
&\quad + (\theta_{n-1} - \theta_n)\|x_{n-1} - u_0\|^2 \qquad (4.8)
\end{aligned}
$$

Since for all $n \ge N_0$, $2\theta_n < \frac{\delta}{2}$ and $\theta_{n-1} \le \theta_n$, we obtain from (4.8) that,

$$
\begin{aligned}
&\|x_{n+1} - u_0\|^2 - \theta_n\|x_n - u_0\|^2 + \|x_{n+1} - x_n\|^2 \\
&\quad + (\delta - \theta_n)\|x_{n+1} - x_n\|^2 + 2\gamma_n\langle Ax_{n+1} - Ax_n, u_0 - x_{n+1}\rangle \\
&\le \|x_n - u_0\|^2 - \theta_{n-1}\|x_{n-1} - u_0\|^2 + \|x_n - x_{n-1}\|^2 \\
&\quad + 2\gamma_{n-1}\langle Ax_n - Ax_{n-1}, u_0 - x_n\rangle. \qquad (4.9)
\end{aligned}
$$

Now, for all $n \ge N_0$, define

$a_n := \|x_n - u_0\|^2 - \theta_{n-1}\|x_{n-1} - u_0\|^2 + \|x_n - x_{n-1}\|^2 + 2\gamma_{n-1}\langle Ax_n - Ax_{n-1}, u_0 - x_n\rangle.$

We show that $\{a_n\}_{n=1}^{\infty}$ is a non-negative sequence of real numbers and that $\lim_{n\to\infty} a_n$ exists. Observe that,

$$
\begin{aligned}
a_n &= \|x_n - u_0\|^2 - \theta_{n-1}\|x_{n-1} - u_0\|^2 + \|x_n - x_{n-1}\|^2 \\
&\quad + 2\gamma_{n-1}\langle Ax_n - Ax_{n-1}, u_0 - x_n\rangle \\
&\ge \|x_n - u_0\|^2 - \theta_{n-1}\|x_{n-1} - u_0\|^2 + \|x_n - x_{n-1}\|^2 \\
&\quad - 2\gamma_{n-1}L\|x_n - x_{n-1}\|\ \|u_0 - x_n\| \\
&\ge \|x_n - u_0\|^2 - \gamma_{n-1}L\big(\|x_n - x_{n-1}\|^2 + \|u_0 - x_n\|^2\big) \\
&\quad - \theta_{n-1}\|x_{n-1} - u_0\|^2 + \|x_n - x_{n-1}\|^2 \\
&= (1 - \gamma_{n-1}L)\Big[\|x_n - u_0\|^2 + \|x_n - x_{n-1}\|^2\Big] - \theta_{n-1}\|x_{n-1} - u_0\|^2 \qquad (4.10)
\end{aligned}
$$

Observe that by applying lemma 2.1($i.$), we have that,

$$
\begin{aligned}
\|x_{n-1} - u_0\|^2 &= \|(x_{n-1} - x_n) + (x_n - u_0)\|^2 \\
&= \|x_n - x_{n-1}\|^2 + \|x_n - u_0\|^2 + 2\langle x_{n-1} - x_n, x_n - u_0\rangle \\
&\le 2\|x_n - x_{n-1}\|^2 + 2\|x_n - u_0\|^2 \qquad (4.11)
\end{aligned}
$$

If we use (4.11) in (4.10), and the fact that for all $n \ge N_0$, $\theta_{n-1} \le \theta_n$, we obtain that,

$$
\begin{aligned}
a_n &\ge (1 - \gamma_{n-1}L)\Big[\|x_n - u_0\|^2 + \|x_n - x_{n-1}\|^2\Big] \\
&\quad - 2\theta_{n-1}\Big[\|x_n - x_{n-1}\|^2 + \|x_n - u_0\|^2\Big] \\
&= (1 - \gamma_{n-1}L - 2\theta_{n-1})\Big[\|x_n - u_0\|^2 + \|x_n - x_{n-1}\|^2\Big] \\
&\ge (1 - \gamma_{n-1}L - 2\theta_n)\Big[\|x_n - u_0\|^2 + \|x_n - x_{n-1}\|^2\Big] \qquad (4.12)
\end{aligned}
$$

It is easy to see (using the fact that for all $n \ge N_0$, $2\theta_n < \frac{\delta}{2}$ and $\gamma_{n-1} < \frac{1-\delta}{L}$) that

$$1 - \gamma_{n-1}L - 2\theta_n > 0.$$

Thus, we obtain from (4.12) that for all $n \geq N_0$,

$$a_n \geq 0.$$

Also, we obtain from (4.9) that for all $n \geq N_0$,

$$a_{n+1} \leq a_n - (\delta - \theta_n)\|x_{n+1} - x_n\|^2. \tag{4.13}$$

Since for all $n \geq N_0$, $2\theta_n < \frac{\delta}{2}$, if follows that for all $n \geq N_0$, $\delta - \theta_n > 0$. Thus, we obtain from (4.13) that for all $n \geq N_0$,

$$a_{n+1} \leq a_n.$$

Therefore, the sequence $\{a_n\}_{n=N_0}^{\infty}$ is a monotone decreasing sequence of real numbers bounded below by 0. Thus, $\lim\limits_{n\to\infty} a_n$ exists. As a result, we obtain from (4.13) (using the fact that $\lim\limits_{n\to\infty} \theta_n$ exists) that,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{4.14}$$

Moreover, since $w_n = x_n + \theta_n(x_n - x_{n-1})$ and $\lim\limits_{n\to\infty} \theta_n$ exists, we obtain that,

$$\lim_{n\to\infty} \|w_n - x_n\| = 0. \tag{4.15}$$

Using (4.14) and (4.15) gives that,

$$\lim_{n\to\infty} \|x_{n+1} - w_n\| = 0. \tag{4.16}$$

Finally, observe from (4.12) that for all $n \geq N_0$,

$$\begin{aligned} a_n &\geq (1 - \gamma_{n-1}L - 2\theta_n)\Big[\|x_n - u_0\|^2 + \|x_n - x_{n-1}\|^2\Big] \\ &\geq \frac{\delta}{2}\Big[\|x_n - u_0\|^2 + \|x_n - x_{n-1}\|^2\Big] \end{aligned}$$

This implies that for all $n \geq N_0$,

$$\begin{aligned} \|x_n - u_0\|^2 &\leq \frac{2}{\delta}a_n - \|x_n - x_{n-1}\|^2 \\ &\leq \frac{2}{\delta}a_n \end{aligned} \tag{4.17}$$

Since $\lim\limits_{n\to\infty} a_n$ exists, we obtain from (4.17) that $\{x_n\}_{n=1}^{\infty}$ is bounded. □

Lemma 4.2. Let $\{x_n\}_{n=1}^{\infty}$ be generated by Algorithm 1 such that $\{\theta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are as lemma 4.1. Suppose assumptions $(a)-(d)$ hold, and that $\liminf\limits_{n\to\infty} \gamma_n > 0$. Let W_C be the set of weak cluster points of $\{x_n\}_{n=1}^{\infty}$, then $W_C \subseteq S_M \subseteq S$.

Proof. By lemma 4.1, $\{x_n\}_{n=1}^{\infty}$ is bounded. Thus, for $w^* \in W_C$, there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $x_{n_k} \rightharpoonup w^*$ as $k \to \infty$. Since assumption (c) holds, we obtain that,

$$\|A(w^*)\| \leq \liminf_{k\to\infty} \|A(x_{n_k})\|.$$

If $\liminf\limits_{k\to\infty} \|A(x_{n_k})\| = 0$, then, $A(w^*) = 0$. Thus, for all $y \in C$, $\langle Aw^*, y - w^*\rangle = 0$; and since A is quasi-monotone (see assumption (d)), we obtain that for all $y \in C$, $\langle Ay, y - w^*\rangle \geq 0$. Thus, $w^* \in S$.

Suppose $\liminf_{k\to\infty} \|A(x_{n_k})\| \neq 0$. Using (2.1$ii.$), we obtain that for $y \in C$,

$$
\begin{aligned}
0 &\leq \Big\langle x_{n_k+1} - w_{n_k} + \big(\gamma_{n_k} + \gamma_{n_k+1}\big)Ax_{n_k} - \gamma_{n_k-1}Ax_{n_k-1}, y - x_{n_k+1}\Big\rangle \\
&\leq \|x_{n_k+1} - w_{n_k}\| \; \|y - x_{n_k+1}\| + \gamma_{n_k}\langle Ax_{n_k}, y - x_{n_k}\rangle \\
&\quad + \gamma_{n_k}\|Ax_{n_k}\| \; \|x_{n_k} - x_{n_k+1}\| + \gamma_{n_k-1}L\|x_{n_k} - x_{n_k-1}\| \; \|y - x_{n_k+1}\| \qquad (4.18)
\end{aligned}
$$

Using (4.14), (4.16), and the fact that $\liminf_{n\to\infty} \gamma_n > 0$, we obtain from (4.18) that,

$$\liminf_{k\to\infty} \gamma_{n_k}\langle Ax_{n_k}, y - x_{n_k}\rangle \geq 0.$$

This leads to two possible cases:

Case 1: Suppose $\liminf_{k\to\infty} \gamma_{n_k}\langle Ax_{n_k}, y - x_{n_k}\rangle > 0$. Let $\liminf_{k\to\infty} \gamma_{n_k}\langle Ax_{n_k}, y - x_{n_k}\rangle = \beta_0 > 0$, then there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$,

$$\gamma_{n_k}\langle Ax_{n_k}, y - x_{n_k}\rangle > \frac{\beta_0}{2}.$$

This implies that for all $k \geq k_0$, for all $y \in C$,

$$\langle Ax_{n_k}, y - x_{n_k}\rangle > \frac{\beta_0}{2\gamma_{n_k}} > 0. \qquad (4.19)$$

Since A is quasi-monotone on C, (4.19) implies that for all $y \in C$, for all $k \geq k_0$, $\langle Ax_{n_k}, y - x_{n_k}\rangle \geq 0$. But,

$$\langle Ay, y - w^*\rangle = \langle Ay, y - x_{n_k}\rangle + \langle Ay, x_{n_k} - w^*\rangle.$$

So, for all $k \geq k_0$, for all $y \in C$,

$$
\begin{aligned}
\langle Ay, y - w^*\rangle &= \langle Ay, y - x_{n_k}\rangle + \langle Ay, x_{n_k} - w^*\rangle \\
&\geq \langle Ay, x_{n_k} - w^*\rangle \qquad (4.20)
\end{aligned}
$$

Since $x_{n_k} \rightharpoonup w^*$ as $k \to \infty$, we obtain from (4.20) that for all $y \in C$, $\langle Ay, y - w^*\rangle \geq 0$, which implies that $w^* \in S_M$.

Case 2: Suppose that for all $y \in C$, $\liminf_{k\to\infty} \gamma_{n_k}\langle Ax_{n_k}, y - x_{n_k}\rangle = 0$, then there exists a subsequence $\{x_{n_{k_j}}\}_{j=1}^{\infty}$ of $\{x_{n_k}\}_{k=1}^{\infty}$ such that $\lim_{j\to\infty} \gamma_{n_{k_j}}\langle Ax_{n_{k_j}}, y - x_{n_{k_j}}\rangle = 0$. It is easy to see that for all $y \in C$,

$$\gamma_{n_{k_j}}\langle Ax_{n_{k_j}}, y - x_{n_{k_j}}\rangle + \gamma_{n_{k_j}}|\langle Ax_{n_{k_j}}, y - x_{n_{k_j}}\rangle| + \frac{\gamma_{n_{k_j}}}{2^j} > 0 \qquad (4.21)$$

Since $\liminf_{k\to\infty} \|Ax_{n_k}\| > 0$, then $\liminf_{j\to\infty} \|Ax_{n_{k_j}}\| > 0$. Let $\liminf_{j\to\infty} \|Ax_{n_{k_j}}\| = \beta_1$, for some $\beta_1 > 0$. Then, there exists $j_1 \in \mathbb{N}$ such that for all $j \geq j_1$,

$$\|Ax_{n_{k_j}}\| > \frac{\beta_1}{2}.$$

So, for all $j \geq j_1$, we obtain that $\langle Ax_{n_{k_j}}, \xi_{n_{k_j}}\rangle = 1$, where $\xi_{n_{k_j}} := \frac{1}{\|Ax_{n_{k_j}}\|^2} \cdot Ax_{n_{k_j}}$. Thus, (4.21) becomes

$$\gamma_{n_{k_j}}\langle Ax_{n_{k_j}}, y - x_{n_{k_j}}\rangle + \gamma_{n_{k_j}}|\langle Ax_{n_{k_j}}, y - x_{n_{k_j}}\rangle| \; \langle Ax_{n_{k_j}}, \xi_{n_{k_j}}\rangle + \frac{\gamma_{n_{k_j}}}{2^j}\langle Ax_{n_{k_j}}, \xi_{n_{k_j}}\rangle > 0$$

This implies that,

$$\gamma_{n_{k_j}}\Big\langle Ax_{n_{k_j}}, y+\Big(|\langle Ax_{n_{k_j}}, y-x_{n_{k_j}}\rangle|+\frac{1}{2^j}\Big)\xi_{n_{k_j}}-x_{n_{k_j}}\Big\rangle>0. \tag{4.22}$$

Since A is quasi-monotone, we obtain from (4.22) that

$$\gamma_{n_{k_j}}\Big\langle A\Big(y+\Big(|\langle Ax_{n_{k_j}}, y-x_{n_{k_j}}\rangle|+\frac{1}{2^j}\Big)\xi_{n_{k_j}}\Big), \\ y+\Big(|\langle Ax_{n_{k_j}}, y-x_{n_{k_j}}\rangle|+\frac{1}{2^j}\Big)\xi_{n_{k_j}}-x_{n_{k_j}}\Big\rangle\geq 0.$$

This implies that

$$\begin{aligned}
&\gamma_{n_{k_j}}\Big\langle Ay, y+\Big(|\langle Ax_{n_{k_j}}, y-x_{n_{k_j}}\rangle|+\frac{1}{2^j}\Big)\xi_{n_{k_j}}-x_{n_{k_j}}\Big\rangle\\
&\quad\geq \gamma_{n_{k_j}}\Big\langle Ay-A\Big(y+|\langle Ax_{n_{k_j}}, y-x_{n_{k_j}}\rangle|+\frac{1}{2^j}\Big)\xi_{n_{k_j}},\\
&\qquad\qquad y+\Big(|\langle Ax_{n_{k_j}}, y-x_{n_{k_j}}\rangle|+\frac{1}{2^j}\Big)\xi_{n_{k_j}}-x_{n_{k_j}}\Big\rangle\\
&\quad\geq -\gamma_{n_{k_j}}\Big\|Ay-A\Big(y+|\langle Ax_{n_{k_j}}, y-x_{n_{k_j}}\rangle|+\frac{1}{2^j}\Big)\xi_{n_{k_j}}\Big\|\\
&\qquad\qquad \Big\|y+\Big(|\langle Ax_{n_{k_j}}, y-x_{n_{k_j}}\rangle|+\frac{1}{2^j}\Big)\xi_{n_{k_j}}-x_{n_{k_j}}\Big\|\\
&\quad\geq -\gamma_{n_{k_j}}L\Big(|\langle Ax_{n_{k_j}}, y-x_{n_{k_j}}\rangle|+\frac{1}{2^j}\Big)\|\xi_{n_{k_j}}\|\\
&\qquad\qquad \Big\|y+\Big(|\langle Ax_{n_{k_j}}, y-x_{n_{k_j}}\rangle|+\frac{1}{2^j}\Big)\xi_{n_{k_j}}-x_{n_{k_j}}\Big\|\\
&\quad= -\frac{\gamma_{n_{k_j}}L}{\|Ax_{n_{k_j}}\|}\Big(|\langle Ax_{n_{k_j}}, y-x_{n_{k_j}}\rangle|+\frac{1}{2^j}\Big)\\
&\qquad\qquad \Big\|y+\Big(|\langle Ax_{n_{k_j}}, y-x_{n_{k_j}}\rangle|+\frac{1}{2^j}\Big)\xi_{n_{k_j}}-x_{n_{k_j}}\Big\|\\
&\quad\geq -\frac{2\gamma_{n_{k_j}}L}{\beta_1}\Big(|\langle Ax_{n_{k_j}}, y-x_{n_{k_j}}\rangle|+\frac{1}{2^j}\Big)M^*,
\end{aligned}$$

for some $M^*>0$.

Thus,

$$\begin{aligned}
&\gamma_{n_{k_j}}\Big\langle Ay, y+\Big(|\langle Ax_{n_{k_j}}, y-x_{n_{k_j}}\rangle|+\frac{1}{2^j}\Big)\xi_{n_{k_j}}-x_{n_{k_j}}\Big\rangle\\
&\qquad\geq -\frac{2L\gamma_{n_{k_j}}}{\beta_1}\Big(|\langle Ax_{n_{k_j}}, y-x_{n_{k_j}}\rangle|+\frac{1}{2^j}\Big)M^*\\
&\qquad\geq -\frac{2LM^*}{\beta_1}|\gamma_{n_{k_j}}\langle Ax_{n_{k_j}}, y-x_{n_{k_j}}\rangle|-\frac{2LM^*\gamma_0}{\beta_1}\frac{1}{2^j}
\end{aligned} \tag{4.23}$$

for some $\gamma_0>0$. But,

$$\begin{aligned}
&\gamma_{n_{k_j}}\Big\langle Ay, y+\Big(|\langle Ax_{n_{k_j}}, y-x_{n_{k_j}}\rangle|+\frac{1}{2^j}\Big)\xi_{n_{k_j}}-x_{n_{k_j}}\Big\rangle\\
&\quad=\gamma_{n_{k_j}}\Big\langle Ay, y-w^*+w^*+\Big(|\langle Ax_{n_{k_j}}, y-x_{n_{k_j}}\rangle|+\frac{1}{2^j}\Big)\xi_{n_{k_j}}-x_{n_{k_j}}\Big\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \gamma_{n_{k_j}} \langle Ay, y - w^* \rangle + \gamma_{n_{k_j}} \langle Ay, w^* - x_{n_{k_j}} \rangle \\
&\quad + \gamma_{n_{k_j}} \Big\langle Ay, \Big(|\langle Ax_{n_{k_j}}, y - x_{n_{k_j}} \rangle| + \frac{1}{2^j} \Big) \xi_{n_{k_j}} \Big\rangle \\
&= \gamma_{n_{k_j}} \langle Ay, y - w^* \rangle + \gamma_{n_{k_j}} \langle Ay, w^* - x_{n_{k_j}} \rangle \\
&\quad + \Big(|\gamma_{n_{k_j}} \langle Ax_{n_{k_j}}, y - x_{n_{k_j}} \rangle| + \frac{\gamma_{n_{k_j}}}{2^j} \Big) \langle Ay, \xi_{n_{k_j}} \rangle
\end{aligned} \tag{4.24}
$$

So, using (4.24) in (4.23), we obtain that

$$
\begin{aligned}
\gamma_{n_{k_j}} \langle Ay, y - w^* \rangle &\geq \gamma_{n_{k_j}} \langle Ay, x_{n_{k_j}} - w^* \rangle - \Big(|\gamma_{n_{k_j}} \langle Ax_{n_{k_j}}, y - x_{n_{k_j}} \rangle| + \frac{\gamma_{n_{k_j}}}{2^j} \Big) \langle Ay, \xi_{n_{k_j}} \rangle \\
&\quad - \frac{2LM^*}{\beta_1} |\gamma_{n_{k_j}} \langle Ax_{n_{k_j}}, y - x_{n_{k_j}} \rangle| - \frac{2LM^* \gamma_0}{\beta_1} \frac{1}{2^j} \\
&\geq -M_0 |\langle Ay, x_{n_{k_j}} - w^* \rangle| - |\gamma_{n_{k_j}} \langle Ax_{n_{k_j}}, y - x_{n_{k_j}} \rangle| M_0 - \frac{M_0}{2^j}
\end{aligned} \tag{4.25}
$$

for some $M_0 > 0$. So, taking $\liminf_{j\to\infty}$ in (4.25), we obtain using the fact that

$$
0 = \lim_{j\to\infty} \langle Ay, x_{n_{k_j}} - w^* \rangle = \lim_{j\to\infty} \gamma_{n_{k_j}} \langle Ax_{n_{k_j}}, y - x_{n_{k_j}} \rangle = \lim_{j\to\infty} \frac{1}{2^j} = 0
$$

that,

$$
\liminf_{j\to\infty} \gamma_{n_{k_j}} \langle Ay, y - w^* \rangle \geq 0. \tag{4.26}
$$

Since $\liminf_{j\to\infty} \gamma_{n_{k_j}} > 0$, we obtain from (4.26) that,

$$
\langle Ay, y - w^* \rangle \geq 0.
$$

Thus, $w^* \in S_M$. Hence, $W_C \subseteq S_M \subseteq S$. □

Theorem 4.3. *Let $\{x_n\}_{n=1}^\infty$ be generated by Algorithm 1 such that assumptions $(a) - (d)$ hold, and such that for all $x \in C$, $Ax \neq 0$. Suppose that conditions of Lemma 4.2 hold, then $\{x_n\}_{n=1}^\infty$ converges weakly to some element of $S_M \subseteq S$.*

Proof. Recall that from lemma 4.1, $\lim_{n\to\infty} a_n$ exists, where for any $u_0 \in W_C$;

$$
a_n = \|x_n - u_0\|^2 - \theta_{n-1}\|x_{n-1} - u_0\|^2 + \|x_n - x_{n-1}\|^2 - 2\gamma_{n-1}\langle Ax_n - Ax_{n-1}, u_0 - x_n \rangle. \tag{4.27}
$$

Using (4.14) and the fact that A is $L-$Lipschitz continuous, we obtain from (4.27) that

$$
\lim_{n\to\infty} \left[|x_n - u_0\|^2 - \theta_{n-1}\|x_{n-1} - u_0\|^2 \right] \tag{4.28}
$$

exists for any $u_0 \in W_C$. Now, since $\{x_n\}_{n=1}^\infty$ is bounded, then reflexivity of H gives that there exists a subsequences; $\{x_{n_p}\}_{p=1}^\infty$ of $\{x_n\}_{n=1}^\infty$ such that $x_{n_p} \rightharpoonup w^*$ as $p \to \infty$, for some $w^* \in H$. Suppose there is another subsequence $\{x_{n_q}\}_{n=1}^\infty$ of $\{x_n\}_{n=1}^\infty$ such that $x_{n_q} \rightharpoonup y^*$ as $q \to \infty$, for some $y^* \in H$, then observe that

$$
2\langle x_n, y^* - w^* \rangle = \|x_n - w^*\|^2 - \|x_n - y^*\|^2 - \|w^*\|^2 + \|y^*\|^2 \tag{4.29}
$$

and

$$
2\langle -\theta_{n-1} x_{n-1}, y^* - w^* \rangle = -\theta_{n-1}\|x_{n-1} - w^*\|^2 + \theta_{n-1}\|x_{n-1} - y^*\|^2
$$

$$+ \theta_{n-1}\|w^*\|^2 - \theta_{n-1}\|y^*\|^2. \tag{4.30}$$

Thus, we obtain the following by adding (4.29) and (4.30):

$$\begin{aligned} 2\langle x_n - \theta_{n-1}x_{n-1}, y^* - w^*\rangle &= \left[\|x_n - w^*\|^2 - \theta_{n-1}\|x_{n-1} - w^*\|^2\right] \\ &\quad - \left[\|x_n - y^*\|^2 - \theta_{n-1}\|x_{n-1} - y^*\|^2\right] \\ &\quad - (1 - \theta_{n-1}\left[\|y^*\|^2 - \|w^*\|^2\right] \end{aligned} \tag{4.31}$$

Using (4.28), we obtain from (4.31) that $\lim\limits_{n\to\infty} \langle x_n - \theta_{n-1}x_{n-1}, y^* - w^*\rangle$ exists. Suppose $\lim\limits_{n\to\infty} \theta_n = \theta$, then we have that

$$\begin{aligned} \langle w^* - \theta w^*, y^* - w^*\rangle &= \lim_{p\to\infty} \langle x_{n_p} - \theta_{n_p-1}x_{n_p}, y^* - w^*\rangle \\ &= \lim_{n\to\infty} \langle x_n - \theta_{n-1}x_n, y^* - w^*\rangle \\ &= \lim_{q\to\infty} \langle x_{n_q} - \theta_{n_q-1}x_{n_q}, y^* - w^*\rangle \\ &= \langle y^* - \theta y^*, y^* - w^*\rangle. \end{aligned} \tag{4.32}$$

Thus,

$$(1 - \theta)\|y^* - w^*\|^2 = 0.$$

Since $0 < \theta_n < \frac{\delta}{4} < 1$ for all $n \in \mathbb{N}$, then $0 \leq \theta < 1$. Thus, we have that $y^* = w^*$. Therefore, $\{x_n\}_{n=1}^{\infty}$ converges weakly to some element of $S_M \subseteq S$ □

5. Numerical examples and illustrations

Example 5.1. [14] Let $C = [-1, 1]$ and

$$A(x) = \begin{cases} 2x - 1, & x > 1, \\ x^2, & x \in C, \\ -2x - 1, & x < -1. \end{cases}$$

A is quasi-monotone and $L-$Lipschitz continuous on C with $L = 2$. Also, $S_M = \{-1\}$ and $S = \{-1, 0\}$.

Example 5.2. [24] Let $H = \left\{x = (x_1, x_2, ..., x_n, ...) \mid \sum\limits_{i\geq 1} |x_i|^2 < \infty\right\}$. Let $\alpha, \beta \in \mathbb{R}$ be such that $0 < \frac{\beta}{2} < \alpha < \beta$, and take $C = C_\alpha = \{x \in H \mid \|x\| \leq \alpha\}$. We define

$$A(x) = A_\beta(x) = \begin{cases} (\beta - \|x\|)x, & \text{if } x \in C, \\ x, & \text{otherwise.} \end{cases}$$

A is pseudo-monotone (and hence quasi-monotone) and $L-$Lipschitz continuous on C with $L = 3\beta$. Also, $S_M = S = \{\bar{0}\}$.

5.1. **Experiment 1.**

We examine the convergence of our Algorithm (3.3) with varying parameters θ_n and γ_n where A is a defined in Example 5.1. Here, $x_0 = 0.1$, $x_1 = 0.2$, and the stopping criterion is $\max\left\{\|x_{n+1} - x_n\|, \|x_n - x_{n-1}\|\right\} < 10^{-8}$.

5.2. **Experiments 2.**
For this experiment, we consider A as described in Example 5.2, with $\beta = 4$, and $\alpha = 3$. Using the same varying sequences as defined in Experiment 1, we use a different stopping criterion; $\max\left\{\|x_{n+1} - x_n\|^2, \|x_n - x_{n-1}\|^2\right\} < 10^{-7}$. Also, we let x_0 and x_1 as follows: $x_0 = (1, \frac{1}{2}, \frac{1}{4}, \dots)$, $x_1 = (\frac{4}{5}, \frac{16}{25}, \frac{125}{64}, \dots)$.

All the computations are performed using Spyder (Python 3.8) which is running on a personal computer with an Intel(R) Core(TM) i5-4300 CPU at 2.50GHz and 8.00 Gb-RAM. In Table 1, "Iter" and "CPU" refers to the number of iterations and CPU time in seconds for computation respectively.

Cases	θ_n	γ_n	Experiment 1		Experiment 2	
			iter (n)	CPU	iter (n)	CPU
1	$\frac{n}{1.45(n+1)}$	$\frac{1}{32}(3n+3)^{\frac{1}{n+1}}$	266	0.1549	11	0.0180
2	$\frac{n^2-4}{7.5n^2}$	$\frac{n+1}{4(n+2)}$	171	0.0460	46	0.0560
3	$\frac{n}{1.45(n+1)}$	$\frac{n+1}{4(n+2)}$	02	0.0010	10	0.01701
4	$\frac{n^2-4}{7.5n^2}$	$\frac{1}{32}(3n+3)^{\frac{1}{n+1}}$	333	0.1729	39	0.06397

TABLE 1. Computation of Example 5.1 and Example 5.2 using Algorithm (3.3) while varying θ_n and γ_n

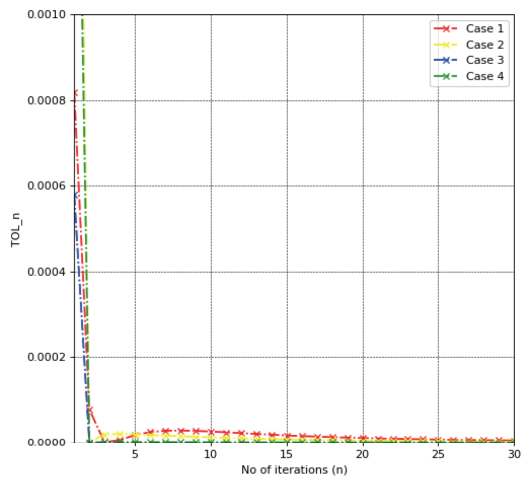

FIGURE 1. The Graphs of TOL_n against No of Iteration for Experiment 1

Remark 5.3. In our experiments, θ_n and γ_n are chosen to satisfy the assumptions of our algorithm. It is easy to see that faster convergence is obtained if γ_n is closer to $\frac{1}{4}$. The parameter θ_n aided the best and fastest approximation when δ is chosen to be as large as possible. Our scheme and analysis did not require

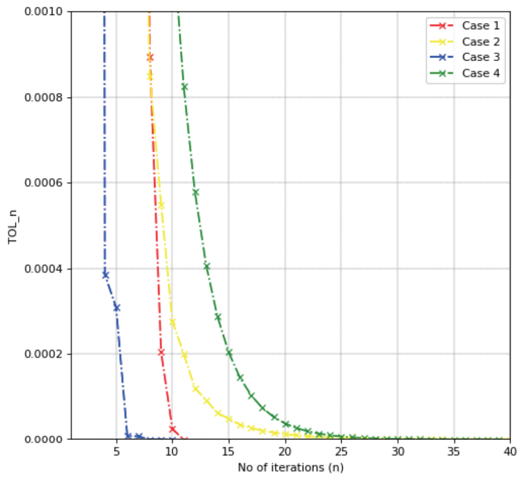

FIGURE 2. The Graphs of TOL_n against No of Iteration for Experiment 2

the stringent choice of $\mu \in [0, 1)$ which constituted certain level of drawback in the schemes used in [9] and [14]. Our scheme which required only one projection yielded convergence result comparable with other results obtained in [9, 14, 16, 25] in CPU and number of iterations. Finally, it is worthy to note that considering our setting in this paper, linear convergence results (see [10]) are easily obtainable when A is strongly pseudo-monotone without further assumptions either on the iterative parameters (step sizes) or the inertia factors, but with slight modification of lines of argument in the corresponding results of Izuchukwu *et al.* [10]. Theorem 4.3 extends, generalizes, improves and unifies the corresponding results of Izuchukwu *et al.* [10], Izuchukwu *et al.* [11], and that of a host of other authors. The numerical experiment presented in this paper is of independent interest.

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