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AN INERTIAL SELF-ADAPTIVE ITERATIVE PROCEDURE FOR MINIMUM NORM SOLUTIONS OF SPLIT GENERALIZED MIXED EQUILIBRIUM AND FIXED POINT PROBLEMS

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Abstract. In this paper, we introduce an inertial self-adaftive iterative algorithm for approximating minimum solutions of split generalized mixed equilibrium problem and fixed point of demimetric mapping in real Hilbert spaces. Strong convergence theorem of the propose scheme is established. Our results generalize and improve many recent results in the literature.

1. Introduction

Let H be a real Hilbert space and K be a nonempty, closed convex subset of H. A mapping $S : K \to K$ is called nonexpansive if $\|Sx - Sy\| \leq \|x - y\| \ \forall x, y \in K$. A point $x \in K$ is a fixed point of S if $x = Sx$. We denote by $F(S)$ the set of all fixed points of S i.e. $F(S) = \{x \in K : x = Sx\}$. S is quasi nonexpansive if $F(S) \neq \emptyset$ and $\|Sx - z\| \leq \|x - z\| \ \forall x \in K$ and $z \in F(S)$. S is $(\vartheta, \zeta)-$generalized hybrid, [27] if there exists real numbers ϑ, ζ such that for all $x, y \in K$

$$\begin{aligned} \vartheta\|Sx - Sy\|^2 &+ (1-\vartheta)\|x - Sy\|^2 \\ &\leq \zeta\|Sx - y\|^2 + (1-\zeta)\|x - y\|^2. \end{aligned} \tag{1.1}$$

The mapping S is called $\tau-$demicontractive, see [17] if $F(S) \neq \emptyset$ and for some $\tau \in (0, 1)$, we have

$$\|Sx - z\|^2 \leq \|x - z\|^2 + \tau\|x - Sx\|^2, \ \forall x \in K, \ z \in F(S).$$

S is called $\tau-$demimetric, see [42] if $F(S) \neq \emptyset$ and for some $\tau \in (-\infty, 1)$, we have

$$\langle x - z, x - Sx\rangle \geq \frac{1-\tau}{2}\|x - Sx\|^2, \ \forall x \in K, \ z \in F(S).$$

Remark 1.1. It is clear from (1.1) that if $\vartheta = 1$ and $\zeta = 0$, then S is nonexpansive. Hence the class of nonexpansive mappings is contained in the class of generalized hybrid mappings. Moreover, the class of $\tau-$demicontractive mappings contains the class of nonexpansive and quasi nonexpansive mappings. Furtheremore, every $\tau-$demecontractive mapping is $\tau-$demimetric mapping.

Numerous studies have been conducted and are ongoing in fixed point theory of various classes of nonlinear mappings due to its applications such as in theory of differential equations, image recovery and signal processing, game theory and market economy and so on, (see for example Byrne [4], Chidume *et al.* [6], Nash [32, 33], Suantai et al. [41]).

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Let $f, g : K \times K \to \mathbb{R}$ be two bifunctions, $B : K \to H$ be a nonlinear operator and $\psi : K \to \mathbb{R}$ be real valued function. The Generalized Mixed Equilibrium Problem (GMEP), see [8,16] is to find $z \in K$ such that

$$f(z,y) + g(z,y) - g(z,z) + \langle Bz, y - z\rangle + \psi(y) - \psi(z) \geq 0 \ \forall y \in K. \tag{1.2}$$

We denote by $GMEP(f,g,B)$ the set of solutions of generalized mixed equilibrium problem, $GMEP$.

Remark 1.2. We observe that

(1) If $\psi = 0$, then $GMEP$ (1.2) reduces to the problem studied by Harbau et al. [16], i.e., find $z \in K$ such that

$$f(z,y) + g(z,y) - g(z,z) + \langle Bz, y - z\rangle \geq 0 \ \forall y \in K. \tag{1.3}$$

(2) If $\psi = 0, B = 0$, then $GMEP$ problem (1.2) reduces to the following generalized equilibrium problem see for example [24], i.e., find $z \in K$ such that

$$f(z,y) + g(z,y) - g(z,z) \geq 0 \ \forall y \in K. \tag{1.4}$$

(3) If $g = 0$, then then $GMEP$ problem (1.2) reduces to mixed equilibrium problem as follows: find $z \in K$ such that

$$f(z,y) + \langle Bz, y - z\rangle + \psi(y) - \psi(z) \geq 0 \ \forall y \in K. \tag{1.5}$$

(4) If $\psi = 0, B = 0$ and $g = 0$, then $GMEP$ problem (1.2) reduces to the following classical equilibrium problem introduced by Blum and Oettli [3], i.e., find $z \in K$ such that

$$f(z,y) \geq 0 \ \forall y \in K. \tag{1.6}$$

Equilibrium problems and their generalizations have been studied by numerous mathematicians due to its importance in solving problems arising from linear and nonlinear programming, optimization problems, variational inequalities and also problems in physics, economics, engineering and so on, see for example [18,19,28,36,44] and the references contained therein.

In 2012, He [13] introduced the following split equilibrium problems in Hilbert spaces:

Let $K_1,\ K_2$ be nonempty closed convex subsets of real Hilbert spaces $H_1,\ H_2$ respectively. Let $f_1 : K_1 \times K_1 \to \mathbb{R}$, $f_2 : K_2 \times K_2 \to \mathbb{R}$ be two bifunctions and $A : H_1 \to H_2$ be a bounded linear map, then the split equilibrium problem (SEP) is

$$\begin{aligned} &\text{find } z \in K_1 \text{ such that } f_1(z,y) \geq 0 \ \forall\ y \in K_1,\\ &\text{and } v' = Az \in K_2 \text{ solves } f_2(v',v) \geq 0 \ \forall\ v \in K_2. \end{aligned} \tag{1.7}$$

Kazmi and Rizvi [25] considered the following split generalized equilibirium problem which is a generalization of split equilibrium problem studied by He [13] in (1.7):

$$\begin{aligned} &\text{find } z \in K_1 \text{ such that } f_1(z,y) + g_1(z,y) \geq 0 \ \forall\ y \in K_1,\\ &\text{and } v' = Az \in K_2 \text{ solves } f_2(v',v) + g_2(v',v) \geq 0 \ \forall\ v \in K_2, \end{aligned} \tag{1.8}$$

where $g_1 : K_1 \times K_1 \to \mathbb{R}$, and $g_2 : K_2 \times K_2 \to \mathbb{R}$ are bifunctions.

Split equilibrium problem contains two equilibrium problems in two different subsets of spaces, under which solutions are splitted such that the image of one equilibrium problem under a given bounded linear map is a solution of another equilibirm problem, see for example [13] and the references contained therein. Many authors have introduced iterative algorithms for finding solutions of split equilibrium problems and their generalizations in real Hilbert spaces, see [9, 10, 21, 23, 26, 34, 43] and the references continued therein.

Numerous authors have proposed modifcations of Picard [38], Mann [30] and Ishikawa [14] iterative procedures to approximate fixed points of various classes of nonlinear mappings, see for example [7, 11, 31, 40]. Moreover, inertial extrapolation method introduced by Polyak [37] to speed up the convergence rate of iteration procedures has attracted attention of researchers, see for example [2, 5, 15, 22] and the references contained therein.

Recently, Husain and Asad [20] proposed the following algorithm for solving split generalized equilibrium problem (1.8) in a real Hilbert space H:

$$\begin{cases} u_0,\ u_1 \in H_1 \textit{ chosen arbitrarily}, \\ w_k = u_k + \theta_k(u_k - u_{k-1}), \\ v_k = (1-\xi_k)w_k + \xi_k \mathcal{F} w_k, \\ z_k = (1-\zeta_k)v_k + \zeta_k \mathcal{F} v_k, \\ u_{n+1} = \mathcal{F} z_k, \quad \forall k \geq 1. \end{cases} \tag{1.9}$$

Where $\mathcal{F}_k = T_{r_k}^{(f_1,g_1)}(I - \gamma A^*(I - T_{r_k}^{(f_2,g_2)})A),\ \gamma \in (0, \frac{1}{\mu})$, μ is the spectral radius of A^*A, the authors proved weak convergence of (1.9) under the following conditions imposed on the control sequences $\{\theta_k\},\ \{\xi_k\},\ \{\zeta_k\}$;

(i) $\sum_{k=1}^{\infty} \theta_k \|u_k - u_{k-1}\| < \infty$;
(ii) $0 < \liminf_{k\to\infty} \xi_k \leq \limsup_{k\to\infty} \xi_k < 1$;
(iii) $0 < \liminf_{k\to\infty} \zeta_k \leq \limsup_{k\to\infty} \zeta_k < 1$.

We observe that apart from weak convergence of algorithm (1.9) established, the step size γ depends on the spectral radius of A^*A which is difficult compute. Moreover, condition (i), i.e summability condition makes implementation of algorithm (1.9) difficult.

Motivated and inspired by the above mentioned works, we study and analyze an inertial self-adaftive iterative algorithm for approximating minimum solutions of split generalized mixed equilibrium problem and fixed point of demimetric mapping in real Hilbert spaces. To be specified, we consider the following Split Generalized Mixed Equilibrium Problem (SGMEP): Find $z \in K_1$ such that

$$f_1(z,y) + g_1(z,y) - g_1(z,z) + \langle B_1 z, y - z\rangle + \psi_1(y) - \psi_1(z) \geq 0 \ \forall\ y \in K_1$$

and $v' = Az \in K_2$ solves

$$f_2(v',v) + g_2(v',v) - g_2(v',v') + \langle B_2 v', v - v'\rangle + \psi_2(v) - \psi_2(v') \geq 0 \ \forall\ v \in K_2. \tag{1.10}$$

Where $B_1 : K_1 \to H_1,\ B_2 : K_2 \to H_2$ are nonlinear operators and $\psi_1 : K_1 \to \mathbb{R},\ \psi_2 : K_2 \to \mathbb{R}$ are real valued functions.

We denote by Γ the set of solutions of SGMEP, i.e.

$$\Gamma = \{z \in GMEP(f_1, g_1, B_1) : Az \in GMEP(f_2, g_2, B_2)\}.$$

The algorithm constructed in this paper has the following properties:

(1) The step size η_n in the propose method is chosen self-adaptively and does not requires computation of spectral radius of A^*A;
(2) In the propose algorithm, the summability condition, i.e. $\sum_{k=1}^{\infty} \theta_k \|u_k - u_{k-1}\| < \infty$ of algorithm (1.9) of [20] is dispense with and this make the propose method simple to implement;
(3) The propose algorithm solves split generalized mixed equilibrium problem and fixed point of demimetric mapping as against (1.9) that solves split generalized equilibrium problem;
(4) The convergence analysis of the propose method does not follow the usual two cases approach that has been used by many authors in obtaining strong convergence .

2. Preliminaries

Let H be a real Hilbert space. Then following identities are well known:

$$||\lambda x+(1-\lambda)y||^2 = \lambda||x||^2+(1-\lambda)||y||^2-\lambda(1-\lambda)||x-y||^2,\ \forall x, y \in H,\ \lambda \in \mathbb{R}. \tag{2.1}$$

$$||x - y||^2 = ||x||^2 - ||y||^2 - 2\langle x - y, y\rangle,\ \forall x, y \in H. \tag{2.2}$$

$$||x + y||^2 \leq ||x||^2 + 2\langle y, x + y\rangle\ \forall x, y \in H. \tag{2.3}$$

It is also known that for any $x \in H$, there exists a unique element denoted by $P_C x$ in C, such that

$$||x - P_C x|| \leq ||x - y||, \quad \forall y \in C. \tag{2.4}$$

The mapping P_C is called the metric projection from H onto C. In addition, P_C has the following characteristics, (see, for example Goebel and Reich [12]):

(i) $\langle x - y, P_C x - P_C y\rangle \geq ||P_C x - P_C y||^2,\ \forall x, y \in H$;
(ii) for $x \in H$, and $x^* \in C$,

$$x^* = P_C x,\ \Leftrightarrow \langle x - x^*, x^* - y\rangle \geq 0,\ \forall y \in C; \tag{2.5}$$

(iii) for $x \in H$ and $y \in C$,

$$||x - P_C x||^2 + ||y - P_C x||^2 \leq ||x - y||^2. \tag{2.6}$$

To solve the generalized equilibrium problem, see [29], we have the following assumptions:

Let $f, g : K_1 \times K_1 \to \mathbb{R}$, $B : K \to H_1$ and $\psi : K_1 \to \mathbb{R}$ satisfiying the following conditions:

(C1) $f(x, x) = 0$ for all $x, \in K_1$;
(C2) f is monotone; that is $f(x, y) + f(y, x) \leq 0$ for all $x, y \in K_1$;
(C3) for all $x, y, z \in K_1$, $\limsup_{t\to 0} f(tz + (1 - t)x, y) \leq f(x, y)$;
(C4) for all $x \in K_1$, $f(x, .)$ is convex and lower semicotinuous.

(C5) g is skew symmetric, i.e.,

$$g(x,x) - g(x,y) - g(y,x) + g(y,y) \geq 0, \ \forall x,y \in K_1;$$

(C6) for all $x \in K_1$, $g(x,.)$ is convex;
(C7) g is continuous.
(C8) B is continuous monotone
(C9) ψ is convex and lower semicontinuous

The following Lemmas will be needed in the proof of the main results.

Lemma 2.1 ([1]). *Let H be a real Hilbert space and K be a nonempty closed convex subset of H. Let $\tau \in (-\infty, 1)$ and $S: K \to H$ be $\tau-$demimetric mapping such that $F(S) \neq \emptyset$. Then $F(S)$ is closed and convex.*

Lemma 2.2 ([29]). *Let $f, g : K \times K \to \mathbb{R}$ satisfiy conditions (C1)-(C9). Let $r > 0$ and $x \in H$, then there exists $z \in K$ such that*

$$\begin{aligned} f(z,y) &+ g(z,y) - g(z,z) + \langle Bz, y-z \rangle \\ &+ \psi(y) - \psi(z) + \frac{1}{r}\langle y-z, z-x \rangle \geq 0, \ \forall y \in K. \end{aligned}$$

Lemma 2.3. *Assume that $f, g : K \times K \to \mathbb{R}$ satisfiy conditions (C1)-(C9). For $r > 0$ and $x \in H$, define a mapping $T_r^{(f,g,B)} : H \to K$ as follows:*

$$\begin{aligned} T_r^{(f,g,B)}(x) = \Big\{ z \in K \quad : \quad & f(z,y) + g(z,y) - g(z,z) + \langle Bz, y-z \rangle \\ &+ \psi(y) - \psi(z) + \frac{1}{r}\langle y-z, z-x \rangle \geq 0, \ \forall y \in K \Big\}. \end{aligned}$$

Then, the following hold:

(i) $T_r^{(f,g,B)}$ *is single-valued,*
(ii) $T_r^{(f,g,B)}$ *is firmly nonexpansive, i.e., for $x, y \in H$,*

$$||T_r^{(f,g,B)}x - T_r^{(f,g,B)}y||^2 \leq \left\langle T_r^{(f,g,B)}x - T_r^{(f,g,B)}y, x-y \right\rangle,$$

(iii) $F(T_r^{(f,g,B)}) = GMEP(f,g,B)$,

(iv) $GMEP(f,g,B)$ *is closed and convex.*

Lemma 2.4 ([39]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers, $\{b_n\}$ be a sequence of real numbers and $\{\delta_n\}$ be sequence of real numbers in $(0,1)$ such that $\sum_{n=1}^{\infty} \delta_n = \infty$. Suppose that*

$$a_{n+1} \leq (1-\delta_n)a_n + \delta_n b_n, \ \forall n \geq 1.$$

If $\limsup_{k\to\infty} b_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying $\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \geq 0$. Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

In this section we propose an inertial algorithm with self-adaptive step size for approximating minimum norm solutions of fixed point of demimetric mapping and split generalized mixed equilibrium problems in real Hilbert spaces;

Assumption 3.1. (D1) $K_1,\ K_2$ are nonempty closed convex subsets of the real Hilbert spaces $H_1,\ H_2$ respectively;

(D2) $f_1, g_1 : K_1 \times K_1 \to \mathbb{R},\ f_2, g_2 : K_2 \times K_2 \to \mathbb{R}$ are equilibrium bifunctions satisfying assumptions $(C1) - (C9)$, $A : H_1 \to H_2$ is a bounded linear operator with adjoint $A^* : H_2 \to H_1$ and $S : K_1 \to H_1$ be $\tau-$demimetric mapping and $I - S$ demiclosed at 0 such that $\Omega = F(S) \cap \Gamma \neq \emptyset$, where $\Gamma = \{x^* \in GMEP(f_1, g_1, B_1) : Ax^* \in GMEP(f_2, g_2, B_2)\}$;

(D3) $\{\alpha_n\},\ \{\beta_n\},\ \{\gamma_n\},\ \{\delta_n\}$ are sequences in $(0,1)$ $\{\mu_n\}$ is a positive sequence such that $\mu_n = \circ(\delta_n)$, $\lim\limits_{n\to\infty} \delta_n = 0$, $\sum_{n=1}^{\infty} \delta_n = +\infty$, $\gamma_n + \delta_n < 1$, $\alpha_n,\ \beta_n,\ \gamma_n \in (a, 1-a)$ for some $a \in (0,1)$ and $\inf\limits_{n\geq 1}(1 - \gamma_n - \delta_n) > 0$.

Algorithm 3.2. Choose $x_0, x_1 \in H_1$. Given the iterates x_{n-1} and x_n for every $n \geq 1, \theta > 0$, select θ_n such that $0 \leq \theta_n \leq \bar{\theta_n}$ and

$$\bar{\theta_n} = \begin{cases} min\{\frac{\mu_n}{||x_n - x_{n-1}||}, \theta\}, & if\ x_n \neq x_{n-1} \\ \theta, \quad Otherwise, \end{cases} \tag{3.1}$$

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = (1-\alpha_n)w_n + \alpha_n \mathcal{J}_n w_n, \\ z_n = (1-\beta_n)w_n + \beta_n \mathcal{J}_n y_n, \\ x_{n+1} = (1-\gamma_n - \delta_n)w_n + \gamma_n \mathcal{J}_n z_n, \quad \forall n \geq 0, \end{cases} \tag{3.2}$$

where $\mathcal{J}_n = S_{\lambda_n}\Big(T_{r_n}^{(f_1,g_1,B_1)}(I - \eta_n A^*(I - T_{r_n}^{(f_2,g_2,B_2)})A)\Big)$, $r_n \in [\epsilon, \infty),\ \epsilon > 0$, $S_{\lambda_n} = (1-\lambda_n)I + \lambda_n S$, $\lambda_n \in (0,1)$ such that $0 < b \leq \lambda_n < c < 1 - \tau$ and for $\eta,\ \xi > 0$, the step size η_n is chosen as follows:

$$0 < \xi \leq \eta_n = \begin{cases} min\{\frac{||(I - T_{r_n}^{(f_2,g_2,B_2)})Az_n||^2}{||A^*(I - T_{r_n}^{(f_2,g_2,B_2)})Az_n||^2}, \eta\}, & if\ T_{r_n}^{(f_2,g_2,B_2)}Az_n \neq Az_n \\ \eta, \quad Otherwise. \end{cases} \tag{3.3}$$

Remark 3.3. We note from $(D3)$ that $\lim\limits_{n\to\infty} \frac{\mu_n}{\delta_n} = 0$. Therefore, from (3.1) in Algorithm 3.2, for each $n \geq 1$ with $x_n \neq x_{n-1}$ we obtain $\theta_n \leq \frac{\mu_n}{||x_n - x_{n-1}||}$, so that $0 < \lim\limits_{n\to\infty} \frac{\theta_n}{\delta_n}||x_n - x_{n-1}|| \leq \lim\limits_{n\to\infty} \frac{\mu_n}{\delta_n} = 0$.

Remark 3.4. The step size η_n is well defined. To show this we proceed as follows: Let $x^* \in \Omega$, then as $Ax^* = T_{r_n}^{(f_2,g_2,B_2)})Ax^*$, we have

$$-\langle z_n - x^*, A^*(I - T_{r_n}^{(f_2,g_2,B_2)})Az_n\rangle = \langle x^* - z_n, A^*(I - T_{r_n}^{(f_2,g_2,B_2)})Az_n\rangle.$$

But

$$\begin{aligned} \langle x^* - z_n, A^*(I - T_{r_n}^{(f_2,g_2,B_2)})Az_n\rangle &= \langle A^*(x^* - z_n), (I - T_{r_n}^{(f_2,g_2,B_2)})Az_n\rangle \\ = \quad & \langle A^*(x^* - z_n) + (I - T_{r_n}^{(f_2,g_2,B_2)})Az_n, (I - T_{r_n}^{(f_2,g_2,B_2)})Az_n\rangle \end{aligned}$$

$$
\begin{aligned}
& \quad - \|(I - T_{r_n}^{(f_2,g_2,B_2)})Az_n\|^2) \\
&= \Big(\frac{1}{2}\Big(\|Ax^* - T_{r_n}^{(f_2,g_2,B_2)})Az_n\|^2 + \|(I - T_{r_n}^{(f_2,g_2,B_2)})Az_n\|^2 - \|Ax^* - Az_n\|^2\Big)\Big) \\
& \quad - \|(I - T_{r_n}^{(f_2,g_2,B_2)})Az_n\|^2 \\
&\le -\frac{1}{2}\|(I - T_{r_n}^{(f_2,g_2,B_2)})Az_n\|^2.
\end{aligned}
$$

Therefore,

$$
-\langle z_n - x^*, A^*(I - T_{r_n}^{(f_2,g_2,B_2)})Az_n\rangle \le -\frac{1}{2}\|(I - T_{r_n}^{(f_2,g_2,B_2)})Az_n\|^2. \tag{3.4}
$$

Observe that from (3.4), we have

$$
\begin{aligned}
\|z_n - x^*\|\|A^*(I - T_{r_n}^{(f_2,g_2,B_2)})Az_n\| &\ge \langle z_n - x^*, A^*(I - T_{r_n}^{(f_2,g_2,B_2)})Az_n\rangle \\
&\ge \frac{1}{2}\|(I - T_{r_n}^{(f_2,g_2,B_2)})Az_n\|^2. \qquad (3.5)
\end{aligned}
$$

It is clear that if $T_{r_n}^{(f_2,g_2,B_2)}Az_n \neq Az_n$, then $\|Az_n - T_{r_n}^{(f_2,g_2,B_2)}Az_n\| > 0$. Therefore, from (3.5), we get $\|z_n - x^*\|\|A^*(I - T_{r_n}^{(f_2,g_2,B_2)})Az_n\| > 0$, which shows that $\|A^*(I - T_{r_n}^{(f_2,g_2,B_2)})Az_n\| \neq 0$.

Remark 3.5. From the Definition of S_{λ_n} in Algorithm 3.2, it is easy to see that $x \in F(S_{\lambda_n})$ if and only if $x \in F(S)$.

Lemma 3.6. *Let S be as in $(D2)$ of Assumption 3.1 and S_{λ_n} as in Algorithm 3.2. Then for $x \in K_1$, $x^* \in \Omega$, we have*

$$
\|S_{\lambda_n}x - x^*\| \le \|x - x^*\|^2 - \lambda_n(1 - \tau - \lambda_n)\|x - Sx\|^2.
$$

Proof. Let $x \in K_1$ and $x^* \in \Omega$. Then

$$
\begin{aligned}
\|S_{\lambda_n}x - x^*\|^2 &= \|(1-\lambda_n)x + \lambda_n Sx - x^*\|^2 \\
&= \|x - x^* + \lambda_n(Sx - x)\|^2 \\
&= \|x - x^*\|^2 + 2\lambda_n\langle x - x^*, Sx - x\rangle + \lambda_n^2\|Sx - x\|^2 \\
&= \|x - x^*\|^2 - 2\lambda_n\langle x - x^*, x - Sx\rangle + \lambda_n^2\|Sx - x\|^2 \\
&\le \|x - x^*\|^2 - (1-\tau)\lambda_n\|x - Sx\|^2 + \lambda_n^2\|Sx - x\|^2 \\
&= \|x - x^*\|^2 - \lambda_n(1 - \tau - \lambda_n)\|x - Sx\|^2.
\end{aligned}
$$

□

Remark 3.7. (1) Since $0 < \lambda_n < 1 - \tau$, It follows from Lemma 3.6 that

$$
\|S_{\lambda_n}x - x^*\|^2 \le \|x - x^*\|^2. \tag{3.6}
$$

(2) Since by Lemma 2.3 $T_{r_n}^{(f_1,g_1,B_1)}$ is firmly nonexpansive and $I - \eta_n A^*(I - T_{r_n}^{(f_2,g_2,B_2)})A$ is nonexpansive, see [20], then for $x^* \in \Omega$, it follows from (3.6) $\mathcal{J}_n$ is quasi nonexpansive.

Lemma 3.8. *Assume conditions $(C1) - (C9)$ hold. Let $\{x_n\}$ be as in Algorithm 3.2 such that Assumption 3.1 holds. Then $\{x_n\}$ is bounded.*

Proof. Let $x^* \in \Omega$. By Remark 3.3, $\{\frac{\theta_n}{\delta_n}||x_n - x_{n-1}||\}$ is bounded. Therefore there exists $M_1 > 0$ such that $\frac{\theta_n}{\delta_n}||x_n - x_{n-1}|| \leq M_1$ for all $n \geq 1$. From (3.2), we have

$$\begin{aligned} \|w_n - x^*\| &= \|x_n + \theta_n(x_n - x_{n-1}) - x^*\| \\ &= \|x_n - x^* + \theta_n(x_n - x_{n-1})\| \\ &\leq \|x_n - x^*\| + \delta_n\Big(\frac{\theta_n}{\delta_n}||x_n - x_{n-1}||\Big) \\ &\leq \|x_n - x^*\| + \delta_n M_1. \end{aligned} \tag{3.7}$$

Furthermore, from (3.2) we have the following estimates;

$$\begin{aligned} \|y_n - x^*\| &= \|(1-\alpha_n)(w_n - x^*) + \alpha_n(\mathcal{J}_n w_n - x^*)\| \\ &\leq (1-\alpha_n)\|w_n - x^*\| + \alpha_n\|\mathcal{J}_n w_n - x^*\| \\ &\leq (1-\alpha_n)\|w_n - x^*\| + \alpha_n\|w_n - x^*\| \\ &= \|w_n - x^*\|. \end{aligned} \tag{3.8}$$

Using (3.8), we have

$$\begin{aligned} \|z_n - x^*\| &= \|(1-\beta_n)(w_n - x^*) + \beta_n(\mathcal{J}_n y_n - x^*)\| \\ &\leq (1-\beta_n)\|w_n - x^*\| + \beta_n\|\mathcal{J}_n y_n - x^*\| \\ &\leq (1-\beta_n)\|w_n - x^*\| + \beta_n\|y_n - x^*\| \\ &\leq (1-\beta_n)\|w_n - x^*\| + \beta_n\|w_n - x^*\| \\ &= \|w_n - x^*\|. \end{aligned} \tag{3.9}$$

Observe that

$$\begin{aligned} \|(1-\gamma_n &- \delta_n)(w_n - x^*) + \gamma_n(\mathcal{J}_n z_n - x^*)\|^2 \\ &= (1-\gamma_n-\delta_n)^2\|w_n - x^*\|^2 + \gamma_n^2\|\mathcal{J}_n z_n - x^*\|^2 \\ &\quad + 2(1-\gamma_n-\delta_n)\gamma_n\langle w_n - x^*, \mathcal{J}_n z_n - x^*\rangle \\ &\leq (1-\gamma_n-\delta_n)^2\|w_n - x^*\|^2 + \gamma_n^2\|z_n - x^*\|^2 \\ &\quad + 2(1-\gamma_n-\delta_n)\gamma_n\|w_n - x^*\|\|z_n - x^*\| \\ &\leq (1-\delta_n)^2\|w_n - x^*\|^2. \end{aligned}$$

Thus,

$$\|(1-\gamma_n-\delta_n)(w_n - x^*) + \gamma_n(\mathcal{J}_n z_n - x^*)\| \leq (1-\delta_n)\|w_n - x^*\|. \tag{3.10}$$

Therefore, from (3.2), (3.7), (3.9) and (3.10) we obtain

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1-\gamma_n-\delta_n)(w_n - x^*) + \gamma_n(\mathcal{J}_n z_n - x^*) - \delta_n x^*\| \\ &\leq \|(1-\gamma_n-\delta_n)(w_n - x^*) + \gamma_n(\mathcal{J}_n z_n - x^*)\| + \delta_n\|x^*\| \\ &\leq (1-\delta_n)\|w_n - x^*\| + \delta_n\|x^*\| \\ &\leq (1-\delta_n)[\|x_n - x^*\| + \delta_n M_1] + \delta_n\|x^*\| \\ &\leq (1-\delta_n)\|x_n - x^*\| + \delta_n(M_1 + \|x^*\|) \\ &\leq \max\{\|x_n - x^*\|, M_1 + \|x^*\|\ \}. \end{aligned}$$

By induction, we have $\{x_n\}$ is bounded. Consequently, $\{w_n\}$, $\{y_n\}$ and $\{z_n\}$ are all bounded. □

Lemma 3.9. *Under the conditions $(C1)-(C9)$, let $\{x_n\}$ be a sequence generated by Algorithm 3.2 satisfying Assumption 3.1. Then*

$$\|x_{n+1}-x^*\|^2 \le (1-\delta_n)\|x_n-x^*\|^2+\delta_n\Big[\frac{\theta_n}{\delta_n}||x_n-x_{n-1}||M_2+\|x^*\|^2\Big] - (1-\alpha_n)\alpha_n\beta_n\gamma_n\|w_n-\mathcal{J}_nw_n\|^2-(1-\beta_n)\beta_n\gamma_n\|w_n-\mathcal{J}_ny_n\|^2 - (1-\gamma_n-\delta_n)\gamma_n\|w_n-\mathcal{J}_nz_n\|^2-\eta_n\gamma_n\|A^*(I-T_{r_n}^{(f_2,g_2,B_2)})Az_n\|^2.$$

Proof. Let $x^*\in\Omega$. Then Algorithm 3.2, using (2.1), Remark 3.7(2) and 3.8, we have

$$\begin{aligned}\|y_n-x^*\|^2 &= \|(1-\alpha_n)(w_n-x^*)+\alpha_n(\mathcal{J}_nw_n-x^*)\|^2\\ &= (1-\alpha_n)\|w_n-x^*\|^2+\alpha_n\|\mathcal{J}_nw_n-x^*\|^2-(1-\alpha_n)\alpha_n\|w_n-\mathcal{J}_nw_n\|^2\\ &\le \|w_n-x^*\|^2-(1-\alpha_n)\alpha_n\|w_n-\mathcal{J}_nw_n\|^2. \qquad (3.11)\end{aligned}$$

And

$$\begin{aligned}\|z_n-x^*\|^2 &= \|(1-\beta_n)(w_n-x^*)+\beta_n(\mathcal{J}_ny_n-x^*)\|^2\\ &= (1-\beta_n)\|w_n-x^*\|^2+\beta_n\|\mathcal{J}_ny_n-x^*\|^2-(1-\beta_n)\beta_n\|w_n-\mathcal{J}_ny_n\|^2\\ &\le (1-\beta_n)\|w_n-x^*\|^2+\beta_n\|y_n-x^*\|^2-(1-\beta_n)\beta_n\|w_n-\mathcal{J}_ny_n\|^2\\ &\le (1-\beta_n)\|w_n-x^*\|^2+\beta_n(\|w_n-x^*\|^2-(1-\alpha_n)\alpha_n\|w_n-\mathcal{J}_nw_n\|^2)\\ &\quad -(1-\beta_n)\beta_n\|w_n-\mathcal{J}_ny_n\|^2\\ &= \|w_n-x^*\|^2-(1-\alpha_n)\alpha_n\beta_n\|w_n-\mathcal{J}_nw_n\|^2\\ &\quad -(1-\beta_n)\beta_n\|w_n-\mathcal{J}_ny_n\|^2. \qquad (3.12)\end{aligned}$$

Observe from, (3.4)

$$\begin{aligned}\|z_n-x^*-\eta_nA^*(I-T_{r_n}^{(f_2,g_2,B_2)})Az_n\|^2 &= \|z_n-x^*\|^2\\ &\quad +\eta_n^2\|A^*(I-T_{r_n}^{(f_2,g_2,B_2)})Az_n\|^2-2\eta_n\langle z_n-x^*,A^*(I-T_{r_n}^{(f_2,g_2,B_2)})Az_n\rangle\\ &\le \|z_n-x^*\|^2+\eta_n^2\|A^*(I-T_{r_n}^{(f_2,g_2,B_2)})Az_n\|^2\\ &\quad -\eta_n\|(I-T_{r_n}^{(f_2,g_2,B_2)})Az_n\|^2\\ &= \|z_n-x^*\|^2-\eta_n(\|(I-T_{r_n}^{(f_2,g_2,B_2)})Az_n\|^2-\eta_n\|A^*(I-T_{r_n}^{(f_2,g_2,B_2)})Az_n\|^2).\end{aligned}$$

Therefore,

$$\begin{aligned}\|z_n-x^*-\eta_nA^*(I-T_{r_n}^{(f_2,g_2,B_2)})Az_n\|^2 &\le \|z_n-x^*\|^2\\ &\quad -\eta_n(\|(I-T_{r_n}^{(f_2,g_2,B_2)})Az_n\|^2-\eta_n\|A^*(I-T_{r_n}^{(f_2,g_2,B_2)})Az_n\|^2). \qquad (3.13)\end{aligned}$$

From the Definition of Step size η_n we have

$$\eta_n\|A^*(I-T_{r_n}^{(f_2,g_2,B_2)})Az_n\|^2 \le \|(I-T_{r_n}^{(f_2,g_2,B_2)})Az_n\|^2 \text{ and}$$
$$0<\|A^*(I-T_{r_n}^{(f_2,g_2,B_2)})Az_n\|^2.$$

Thus,

$$\eta_n\|A^*(I-T_{r_n}^{(f_2,g_2,B_2)})Az_n\|^2 \le \|(I-T_{r_n}^{(f_2,g_2,B_2)})Az_n\|^2-\|A^*(I-T_{r_n}^{(f_2,g_2,B_2)})Az_n\|^2,$$

so that

$$\eta_n\|A^*(I-T_{r_n}^{(f_2,g_2,B_2)})Az_n\|^2 \le \eta_n(\|(I-T_{r_n}^{(f_2,g_2,B_2)})Az_n\|^2-\eta_n\|A^*(I-T_{r_n}^{(f_2,g_2,B_2)})Az_n\|^2).$$

Hence, from (3.13) we obtain

$$\begin{aligned} &\|z_n - x^* - \eta_n A^*(I - T_{r_n}^{(f_2,g_2,B_2)})Az_n\|^2 \leq \|z_n - x^*\|^2 \\ &- \eta_n\|A^*(I - T_{r_n}^{(f_2,g_2,B_2)})Az_n\|^2. \end{aligned} \tag{3.14}$$

Also from the Definition of $\mathcal{J}_n$ in algorithm 3.1, (3.6) and (3.14), we have

$$\begin{aligned} \|\mathcal{J}_n z_n - x^*\|^2 &= \|S_{\lambda_n}\Big(T_{r_n}^{(f_1,g_1,B_1)}(z_n - \eta_n A^*(I - T_{r_n}^{(f_2,g_2,B_2)})Az_n)\Big) - x^*\|^2 \\ &\leq \|T_{r_n}^{(f_1,g_1,B_1)}(z_n - \eta_n A^*(I - T_{r_n}^{(f_2,g_2,B_2)})Az_n) - x^*\|^2 \\ &\leq \|z_n - x^* - \eta_n A^*(I - T_{r_n}^{(f_2,g_2,B_2)})Az_n\|^2 \\ &\leq \|z_n - x^*\|^2 - \eta_n\|A^*(I - T_{r_n}^{(f_2,g_2,B_2)})Az_n\|^2. \end{aligned} \tag{3.15}$$

Now from Algorithm 3.2, (2.1), (3.15) and (3.12)

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= (1 - \gamma_n - \delta_n)w_n + \gamma_n\mathcal{J}_n z_n - x^*\|^2 \\ &= \|(1 - \gamma_n - \delta_n)(w_n - x^*) + \gamma_n(\mathcal{J}_n z_n - x^*) - \delta_n x^*\|^2 \\ &= (1 - \gamma_n - \delta_n)\|w_n - x^*\|^2 + \gamma_n\|\mathcal{J}_n z_n - x^*\|^2 \\ &\quad + \delta_n\|x^*\|^2 - (1 - \gamma_n - \delta_n)\gamma_n\|w_n - \mathcal{J}_n z_n\|^2 \\ &\leq (1 - \gamma_n - \delta_n)\|w_n - x^*\|^2 + \gamma_n\|z_n - x^*\|^2 \\ &\quad - \eta_n\gamma_n\|A^*(I - T_{r_n}^{(f_2,g_2,B_2)})Az_n\|^2 + \delta_n\|x^*\|^2 \\ &\quad - (1 - \gamma_n - \delta_n)\gamma_n\|w_n - \mathcal{J}_n z_n\|^2 \\ &\leq (1 - \gamma_n - \delta_n)\|w_n - x^*\|^2 + \gamma_n\|w_n - x^*\|^2 \\ &\quad - (1 - \alpha_n)\alpha_n\beta_n\gamma_n\|w_n - \mathcal{J}_n w_n\|^2 \\ &\quad - (1 - \beta_n)\beta_n\gamma_n\|w_n - \mathcal{J}_n y_n\|^2 \\ &\quad - \eta_n\gamma_n\|A^*(I - T_{r_n}^{(f_2,g_2,B_2)})Az_n\|^2 + \delta_n\|x^*\|^2 \\ &\quad - (1 - \gamma_n - \delta_n)\gamma_n\|w_n - \mathcal{J}_n z_n\|^2 \end{aligned}$$

Thus,

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq (1 - \delta_n)\|w_n - x^*\|^2 \\ &\quad - (1 - \alpha_n)\alpha_n\beta_n\gamma_n\|w_n - \mathcal{J}_n w_n\|^2 \\ &\quad - (1 - \beta_n)\beta_n\gamma_n\|w_n - \mathcal{J}_n y_n\|^2 \\ &\quad - \eta_n\gamma_n\|A^*(I - T_{r_n}^{(f_2,g_2,B_2)})Az_n\|^2 + \delta_n\|x^*\|^2 \\ &\quad - (1 - \gamma_n - \delta_n)\gamma_n\|w_n - \mathcal{J}_n z_n\|^2. \end{aligned} \tag{3.16}$$

But,

$$\begin{aligned} \|w_n - x^*\|^2 &= \|x_n - x^* + \theta_n(x_n - x_{n-1})\|^2 \\ &\leq (\|x_n - x^*\| + \theta_n\|x_n - x_{n-1}\|)^2 \\ &= \|x_n - x^*\|^2 + 2\theta_n\|x_n - x^*\|\|x_n - x_{n-1}\| + \theta_n^2\|x_n - x_{n-1}\|^2 \\ &= \|x_n - x^*\|^2 + \theta_n\|x_n - x_{n-1}\|(2\|x_n - x^*\| + \theta_n\|x_n - x_{n-1}\|) \\ &\leq \|x_n - x^*\|^2 + \theta_n\|x_n - x_{n-1}\|M_2, \end{aligned} \tag{3.17}$$

for some $M_2 > 0$. Putting (3.17) in (3.16), we get

$$\|x_{n+1} - x^*\|^2 \leq (1 - \delta_n)\|x_n - x^*\|^2 + \delta_n\Big[\frac{\theta_n}{\delta_n}||x_n - x_{n-1}||M_2 + \|x^*\|^2\Big]$$
$$- (1 - \alpha_n)\alpha_n\beta_n\gamma_n\|w_n - \mathcal{J}_n w_n\|^2 - (1 - \beta_n)\beta_n\gamma_n\|w_n - \mathcal{J}_n y_n\|^2$$
$$- (1 - \gamma_n - \delta_n)\gamma_n\|w_n - \mathcal{J}_n z_n\|^2 - \eta_n\gamma_n\|A^*(I - T_{r_n}^{(f_2,g_2,B_2)})Az_n\|^2.$$

□

Lemma 3.10. *Let $\{x_n\}$ be defined as in Algorithm 3.2 satisfying Assumption 3.1. Then for $x^* \in \Omega$, we have*

$$\|x_{n+1} - x^*\|^2 \leq (1 - \delta_n)\|x_n - x^*\|^2 + \delta_n\Big[\frac{\theta_n}{\delta_n}||x_n - x_{n-1}||M_2 + 2\langle x^*, x^* - x_{n+1}\rangle\Big].$$

Proof. Let $x^* \in \Omega$, then from (2.3), (3.9) and Remark 3.7(2), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(1 - \gamma_n - \delta_n)(w_n - x^*) + \gamma_n(\mathcal{J}_n z_n - x^*) - \delta_n x^*\|^2 \\
&\leq \|(1 - \gamma_n - \delta_n)(w_n - x^*) + \gamma_n(\mathcal{J}_n z_n - x^*)\|^2 \\
&\quad - 2\delta_n\langle x^*, x_{n+1} - x^*\rangle \\
&\leq \Big(\|(1 - \gamma_n - \delta_n)(w_n - x^*)\| + \|\gamma_n(\mathcal{J}_n z_n - x^*)\|\Big)^2 \\
&\quad + 2\delta_n\langle x^*, x^* - x_{n+1}\rangle \\
&\leq \Big((1 - \gamma_n - \delta_n)\|w_n - x^*\| + \gamma_n\|z_n - x^*\|\Big)^2 \\
&\quad + 2\delta_n\langle x^*, x^* - x_{n+1}\rangle \\
&\leq \Big((1 - \gamma_n - \delta_n)\|w_n - x^*\| + \gamma_n\|w_n - x^*\|\Big)^2 \\
&\quad + 2\delta_n\langle x^*, x^* - x_{n+1}\rangle \\
&\leq (1 - \delta_n)\|w_n - x^*\|^2 + 2\delta_n\langle x^*, x^* - x_{n+1}\rangle. \qquad (3.18)
\end{aligned}$$

Combining (3.17) and (3.18) we obtain

$$\|x_{n+1} - x^*\|^2 \leq (1 - \delta_n)\|x_n - x^*\|^2 + \delta_n\Big[\frac{\theta_n}{\delta_n}||x_n - x_{n-1}||M_2 + 2\langle x^*, x^* - x_{n+1}\rangle\Big]$$

as required. □

Theorem 3.11. *Let $\{x_n\}$ be defined as in Algorithm 3.2 satisfying Assumption 3.1. Then $\{x_n\}$ converges strongly to an element $x' \in \Omega$ such that $\|x'\| = \{\min \|d\| : d \in \Omega\}$.*

Proof. Let $x^* \in \Omega$. Then by Lemma 2.4 and 3.10, it suffices to show that $\limsup_{k\to\infty}\langle x^*, x^* - x_{n+1}\rangle \leq 0$ for any subsequence $\{\|x_{n_k} - x^*\|\}$ of $\{\|x_n - x^*\|\}$ satisfying $\liminf_{k\to\infty}(\|x_{n_k+1} - x^*\| - \|x_{n_k} - x^*\|) \geq 0$.

Now, suppose $\{\|x_{n_k} - x^*\|\}$ is a subsequence of $\{\|x_n - x^*\|\}$ satisfying

$$\liminf_{k\to\infty}(\|x_{n_k+1} - x^*\| - \|x_{n_k} - x^*\|) \geq 0.$$

Then

$$\begin{aligned} &\liminf_{k\to\infty}(\|x_{n_k+1} - x^*\|^2 - \|x_{n_k} - x^*\|^2) \\ &= \liminf_{k\to\infty}\Big((\|x_{n_k+1} - x^*\| + \|x_{n_k} - x^*\|)(\|x_{n_k+1} - x^*\| - \|x_{n_k} - x^*\|)\Big) \\ (3.19) \quad &\geq 0. \end{aligned}$$

Therefore, from Lemma 3.9, $(D3)$ and (3.19) we have

$$\begin{aligned} a^4\limsup_{k\to\infty}\|w_{n_k} - \mathcal{J}_{n_k}w_{n_k}\|^2 &\leq \limsup_{k\to\infty}(\|x_{n_k} - x^*\|^2 - \|x_{n_k+1} - x^*\|^2) \\ &+ \limsup_{k\to\infty}\Big\{\delta_{n_k}\Big[\frac{\theta_{n_k}}{\delta_{n_k}}||x_{n_k} - x_{n_k-1}||M_2 + \|x^*\|^2 - \|x_{n_k} - x^*\|^2\Big]\Big\} \\ &= -\liminf_{k\to\infty}(\|x_{n_k+1} - x^*\|^2 - \|x_{n_k} - x^*\|^2) \leq 0. \end{aligned}$$

This implies

$$\lim_{k\to\infty}\|w_{n_k} - \mathcal{J}_{n_k}w_{n_k}\| = 0. \tag{3.20}$$

Similarly, we have

$$\lim_{k\to\infty}\|w_{n_k} - \mathcal{J}_{n_k}y_{n_k}\| = 0. \tag{3.21}$$

$$\lim_{k\to\infty}\|w_{n_k} - \mathcal{J}_{n_k}z_{n_k}\| = 0. \tag{3.22}$$

$$\lim_{k\to\infty}\|A^*(I - T_{r_{n_k}}^{(f_2,g_2,B_2)})Az_{n_k}\| = 0. \tag{3.23}$$

From Algorithm 3.2, we get

$$\|x_{n_k+1} - w_{n_k}\| \leq \|\mathcal{J}_{n_k}w_{n_k} - w_{n_k}\| + \delta_{n_k}\|w_{n_k}\|.$$

Since $\lim_{k\to\infty}\delta_{n_k} = 0$, it follows from (3.20) that

$$\lim_{k\to\infty}\|x_{n_k+1} - w_{n_k}\| = 0. \tag{3.24}$$

Also,

$$\|w_{n_k} - x_{n_k}\| = \delta_{n_k}\Big(\frac{\theta_{n_k}}{\delta_{n_k}}\|x_{n_k} - x_{n_k-1}\|\Big) \to 0 \text{ as n} \to \infty. \tag{3.25}$$

Combining (3.24) and (3.25), we obtain

$$\lim_{k\to\infty}\|x_{n_k+1} - x_{n_k}\| = 0. \tag{3.26}$$

Furtheremore, from Algorithm 3.2 and (3.21) we get

$$\|z_{n_k} - w_{n_k}\| = \beta_{n_k}\|\mathcal{J}_{n_k}y_{n_k} - w_{n_k}\| \to 0 \text{ as n} \to \infty. \tag{3.27}$$

Observe that

$$\|x_{n_k} - z_{n_k}\| \leq \|x_{n_k} - w_{n_k}\| + \|w_{n_k} - z_{n_k}\|.$$

Using (3.25) and (3.27), we obtain

$$\lim_{k\to\infty}\|x_{n_k}-z_{n_k}\|=0. \tag{3.28}$$

From (3.22) and (3.27), it follows that

$$\|z_{n_k}-\mathcal{J}_{n_k}z_{n_k}\|\leq\|z_{n_k}-w_{n_k}\|+\|w_{n_k}-\mathcal{J}_{n_k}z_{n_k}\|\to 0 \text{ as n}\to\infty. \tag{3.29}$$

Let $t_{n_k}=T_{r_{n_k}}^{(f_1,g_1,B_1)}(z_{n_k}-\eta_{n_k}A^*(I-T_{r_{n_k}}^{(f_2,g_2,B_2)})Az_{n_k})$. Then by Lemma 3.6, we have

$$\begin{aligned}\|\mathcal{J}_{n_k}z_{n_k}-x^*\|^2 &= \|S_{\lambda_{n_k}}\Big(T_{r_{n_k}}^{(f_1,g_1,B_1)}(z_{n_k}-\eta_{n_k}A^*(I-T_{r_{n_k}}^{(f_2,g_2,B_2)})Az_{n_k})\Big)-x^*\|^2\\ &\leq \|T_{r_{n_k}}^{(f_1,g_1,B_1)}(z_{n_k}-\eta_{n_k}A^*(I-T_{r_{n_k}}^{(f_2,g_2,B_2)})Az_{n_k})-x^*\|^2\\ &\quad -\lambda_{n_k}(1-\tau-\lambda_{n_k})\|t_{n_k}-St_{n_k}\|^2\\ &\leq \|z_{n_k}-x^*-\eta_{n_k}A^*(I-T_{r_{n_k}}^{(f_2,g_2,B_2)})Az_{n_k}\|^2\\ &\quad -\lambda_{n_k}(1-\tau-\lambda_{n_k})\|t_{n_k}-St_{n_k}\|^2.\end{aligned}$$

From (3.14), we obtain

$$\begin{aligned}\|\mathcal{J}_{n_k}z_{n_k}-x^*\|^2 &\leq \|z_{n_k}-x^*\|^2-\eta_{n_k}\|A^*(I-T_{r_{n_k}}^{(f_2,g_2,B_2)})Az_{n_k}\|^2\\ &\quad -\lambda_{n_k}(1-\tau-\lambda_{n_k})\|t_{n_k}-St_{n_k}\|^2\\ &\leq \|z_{n_k}-x^*\|^2-\lambda_{n_k}(1-\tau-\lambda_{n_k})\|t_{n_k}-St_{n_k}\|^2.\end{aligned}$$

Hence,

$$\lambda_{n_k}(1-\tau-\lambda_{n_k})\|t_{n_k}-St_{n_k}\|^2\leq\|z_{n_k}-x^*\|^2-\|\mathcal{J}_{n_k}z_{n_k}-x^*\|^2. \tag{3.30}$$

Now using (2.2)

$$\begin{aligned}&\|z_{n_k}-\mathcal{J}_{n_k}z_{n_k}\|^2=\|(z_{n_k}-x^*)-(\mathcal{J}_{n_k}z_{n_k}-x^*)\|^2\\ &= \|z_{n_k}-x^*\|^2-\|\mathcal{J}_{n_k}z_{n_k}-x^*\|^2-2\langle z_{n_k}-\mathcal{J}_{n_k}z_{n_k},\mathcal{J}_{n_k}z_{n_k}-x^*\rangle,\end{aligned}$$

so that

$$\begin{aligned}&\|z_{n_k}-x^*\|^2-\|\mathcal{J}_{n_k}z_{n_k}-x^*\|^2\\ &= \|z_{n_k}-\mathcal{J}_{n_k}z_{n_k}\|^2+2\langle z_{n_k}-\mathcal{J}_{n_k}z_{n_k},\mathcal{J}_{n_k}z_{n_k}-x^*\rangle\\ &\leq \|z_{n_k}-\mathcal{J}_{n_k}z_{n_k}\|^2+2\|z_{n_k}-\mathcal{J}_{n_k}z_{n_k}\|\|\mathcal{J}_{n_k}z_{n_k}-x^*\|\\ &\leq \|z_{n_k}-\mathcal{J}_{n_k}z_{n_k}\|^2+2\|z_{n_k}-\mathcal{J}_{n_k}z_{n_k}\|\|z_{n_k}-x^*\|. \qquad (3.31)\end{aligned}$$

Since $\{z_{n_k}\}$ is bounded, it follows from (3.29) and (3.31) that

$$\lim_{k\to\infty}(\|z_{n_k}-x^*\|^2-\|\mathcal{J}_{n_k}z_{n_k}-x^*\|^2)=0. \tag{3.32}$$

Combining (3.30) and (3.32), we obtain

$$\lim_{k\to\infty}\|t_{n_k}-St_{n_k}\|=0. \tag{3.33}$$

Since $T_{r_{n_k}}^{(f_1,g_1,B_1)}$ is firmly nonexpansive and $I-\eta_{n_k}A^*(I-T_{r_{n_k}}^{(f_2,g_2,B_2)})A$ is nonexpansive, then we have the following estimate;

$$\|t_{n_k}-x^*\|^2 = \|T_{r_{n_k}}^{(f_1,g_1,B_1)}(z_{n_k}-\eta_{n_k}A^*(I-T_{r_{n_k}}^{(f_2,g_2,B_2)})Az_{n_k})-T_{r_{n_k}}^{(f_1,g_1,B_1)}x^*\|^2$$

$$\begin{aligned}
&\leq \langle t_{n_k} - x^*, z_{n_k} - \eta_{n_k} A^*(I - T_{r_{n_k}}^{(f_2,g_2,B_2)})Az_{n_k} - x^*\rangle \\
&= \frac{1}{2}\Big[\|t_{n_k} - x^*\|^2 + \|z_{n_k} - \eta_{n_k} A^*(I - T_{r_{n_k}}^{(f_2,g_2,B_2)})Az_{n_k} - x^*\|^2 \\
&\quad - \|t_{n_k} - z_{n_k} - \eta_{n_k} A^*(I - T_{r_{n_k}}^{(f_2,g_2,B_2)})Az_{n_k}\|^2\Big] \\
&\leq \frac{1}{2}\Big[\|t_{n_k} - x^*\|^2 + \|z_{n_k} - x^*\|^2 - (\|t_{n_k} - z_{n_k}\|^2 \\
&\quad + \eta_{n_k}^2 \|A^*(I - T_{r_{n_k}}^{(f_2,g_2,B_2)})Az_{n_k}\|^2 \\
&\quad + 2\eta_{n_k}\langle t_{n_k} - z_{n_k}, A^*(I - T_{r_{n_k}}^{(f_2,g_2,B_2)})Az_{n_k}\rangle)\Big] \\
&\leq \frac{1}{2}\Big[\|t_{n_k} - x^*\|^2 + \|z_{n_k} - x^*\|^2 - \|t_{n_k} - z_{n_k}\|^2 \\
&\quad + 2\eta_{n_k}\|z_{n_k} - t_{n_k}\|\|A^*(I - T_{r_{n_k}}^{(f_2,g_2,B_2)})Az_{n_k}\|\Big].
\end{aligned}$$

This implies

$$\begin{aligned}
\|t_{n_k} - x^*\|^2 &\leq \|z_{n_k} - x^*\|^2 - \|t_{n_k} - z_{n_k}\|^2 \\
&\quad + 2\eta_{n_k}\|z_{n_k} - t_{n_k}\|\|A^*(I - T_{r_{n_k}}^{(f_2,g_2,B_2)})Az_{n_k}\|. \qquad (3.34)
\end{aligned}$$

On the other hand, using Lemma 3.6 and (3.34) we have

$$\begin{aligned}
\|\mathcal{J}_{n_k} z_{n_k} - x^*\|^2 &= \|S_{\lambda_{n_k}}(t_{n_k}) - x^*\|^2 \\
&\leq |t_{n_k} - x^*\|^2 - \lambda_{n_k}(1 - \tau - \lambda_{n_k})\|t_{n_k} - St_{n_k}\|^2 \\
&\leq |t_{n_k} - x^*\|^2 \\
&\leq \|z_{n_k} - x^*\|^2 - \|t_{n_k} - z_{n_k}\|^2 \\
&\quad + 2\eta_{n_k}\|z_{n_k} - t_{n_k}\|\|A^*(I - T_{r_{n_k}}^{(f_2,g_2,B_2)})Az_{n_k}\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|t_{n_k} - z_{n_k}\|^2 &\leq \|z_{n_k} - x^*\|^2 - \|\mathcal{J}_{n_k} z_{n_k} - x^*\|^2 \\
&\quad + 2\eta_{n_k}\|z_{n_k} - t_{n_k}\|\|A^*(I - T_{r_{n_k}}^{(f_2,g_2,B_2)})Az_{n_k}\|. \qquad (3.35)
\end{aligned}$$

Using (3.35), it follows from the boundedness of $\{z_{n_k}\}$, (3.23) and (3.32) that

$$\lim_{k\to\infty}\|t_{n_k} - z_{n_k}\| = 0. \qquad (3.36)$$

From (3.28) and (3.36), we have

$$\|x_{n_k} - t_{n_k}\| \leq \|x_{n_k} - z_{n_k}\| + \|z_{n_k} - t_{n_k}\| \to 0 \text{ as n} \to \infty. \qquad (3.37)$$

From (3.5), we have

$$\|(I - T_{r_{n_k}}^{(f_2,g_2,B_2)})Az_{n_k}\|^2 \leq 2\|z_{n_k} - x^*\|\|A^*(I - T_{r_{n_k}}^{(f_2,g_2,B_2)})Az_{n_k}\|.$$

Using (3.23) and the boundedness of $\{z_{n_k}\}$, we obtain

$$\lim_{k\to\infty}\|(I - T_{r_{n_k}}^{(f_2,g_2,B_2)})Az_{n_k}\| = 0. \qquad (3.38)$$

Since $\{x_n\}$ is bounded, let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x'$ as n $\to \infty$ for some $x' \in H_1$. Since K_1 is closed and convex, we have $x' \in K_1$.

We now show that $x' \in \Omega$. Observe from (3.28) and (3.37), we have $z_{n_k} \rightharpoonup x'$ as n $\to \infty$ and $t_{n_k} \rightharpoonup x'$ as n $\to \infty$. Using (3.33) the assumption that $I - S$ is demiclosed at 0, we get $x' \in F(S)$. From $t_{n_k} = T_{r_{n_k}}^{(f_1,g_1,B_1)}(I - \eta_{n_k} A^*(I - T_{r_{n_k}}^{(f_2,g_2,B_2)})A)z_{n_k}$, we obtain

$$f_1(t_{n_k}, y) + g_1(t_{n_k}, y) + \frac{1}{r_{n_k}}\langle y - t_{n_k}, t_{n_k} - z_{n_k}\rangle + \langle B_1 t_{n_k}, y - t_{n_k}\rangle - g_1(t_{n_k}, t_{n_k})$$
$$+ \frac{1}{r_{n_k}}\langle y - t_{n_k}, \eta_{n_k} A^*(I - T_{r_{n_k}}^{(f_2,g_2,B_2)})Az_{n_k}\rangle + \psi_1(y) - \psi_1(t_{n_k}) \geq 0,\ \forall y \in K_1.$$

Using condition $(C2)$ of Assumption 3.1, we have

$$\frac{1}{r_{n_k}}\langle y - t_{n_k}, t_{n_k} - z_{n_k}\rangle + \frac{1}{r_{n_k}}\langle y - t_{n_k}, \eta_{n_k} A^*(I - T_{r_{n_k}}^{(f_2,g_2,B_2)})Az_{n_k}\rangle$$
$$\geq +\psi_1(t_{n_k}) - \psi_1(y) + f_1(y, t_{n_k}) - g_1(t_{n_k}, y) + g_1(t_{n_k}, t_{n_k})$$
$$+ \langle B_1 t_{n_k}, t_{n_k} - y\rangle,\ \forall y \in K_1. \tag{3.39}$$

From $(C4)$, $(C7$, $(C8))$, (3.23), (3.36) and allowing $k \to \infty$ in (3.39) we obtain

$$f_1(y, x') - g_1(x', y) + g_1(x', x') + \langle B_1 x', x' - y\rangle + \psi_1(x') - \psi_1(y) \leq 0\ \forall y \in K_1,$$

so that

$$f_1(y, x') + \langle B_1 x', x' - y\rangle + \psi_1(x') - \psi_1(y) \leq g_1(x', y) - g_1(x', x')\ \forall y \in K_1.$$

Let $t \in (0, 1]$. For each $y \in K_1$, let $y_t = ty + (1 - t)x'$. Then, $y_t \in K_1$ and so

$$f_1(y_t, x') + \langle B_1 x', x' - y_t\rangle + \psi_1(x') - \psi_1(y_t) \leq g_1(x', y_t) - g_1(x', x'). \tag{3.40}$$

Therefore using condition $(C1)$, $(C6)$ and (3.40) have

$$\begin{aligned}
0 &= f_1(y_t, y_t) + \langle B_1 x', y_t - y_t\rangle + \psi_1(y_t) - \psi_1(y_t)\\
&\leq t[f_1(y_t, y) + \langle B_1 x', y - y_t\rangle + \psi_1(y) - \psi_1(y_t)]\\
&\quad + (1 - t)[f_1(y_t, x') + \langle B_1 x', x' - y_t\rangle + \psi_1(x') - \psi_1(y_t)]\\
&\leq t[f_1(y_t, y) + \langle B_1 x', y - y_t\rangle + \psi_1(y) - \psi_1(y_t)]\\
&\quad + (1 - t)[g_1(x', y_t) - g_1(x', x')]\\
&\leq t[f_1(y_t, y) + \langle B_1 x', y - y_t\rangle + \psi_1(y) - \psi_1(y_t)]\\
&\quad + (1 - t)t[g_1(x', y) - g_1(x', x')].
\end{aligned}$$

The fact that $t > 0$, we obtain

$$f_1(x', y) + g_1(x', y) - g_1(x', x')\langle B_1 x', y - x'\rangle + \psi_1(y) - \psi_1(x') \geq 0,\ \forall y \in K_1,$$

which implies $x' \in GMEP(f_1, g_1, B_1)$.

We now show $Ax' \in GMEP(f_2, g_2, B_2)$. Since $z_{n_k} \rightharpoonup x'$ as $k \to \infty$ and A is bounded linear operator, then $Az_{n_k} \rightharpoonup Ax'$ as $k \to \infty$. Hence it follows from (3.38) that $T_{r_{n_k}}^{(f_2,g_2,B_2)} z_{n_k} \rightharpoonup Ax'$ as $k \to \infty$. Notice that from the Definition of $T_{r_{n_k}}^{(f_2,g_2,B_2)} z_{n_k}$ we have

$$f_2(T_{r_{n_k}}^{(f_2,g_2,B_2)}Az_{n_k}, v) + g_2(T_{r_{n_k}}^{(f_2,g_2,B_2)}Az_{n_k}, v) - g_2(T_{r_{n_k}}^{(f_2,g_2,B_2)}Az_{n_k}, T_{r_{n_k}}^{(f_2,g_2,B_2)}Az_{n_k})$$
$$+ \psi_2(v) - \psi_2(T_{r_{n_k}}^{(f_2,g_2,B_2)}Az_{n_k}) + \frac{1}{r_{n_k}}\langle v - T_{r_{n_k}}^{(f_2,g_2,B_2)}Az_{n_k}, T_{r_{n_k}}^{(f_2,g_2,B_2)}Az_{n_k} - Az_{n_k}\rangle$$

$$+ \langle B_2(T^{(f_2,g_2,B_2)}_{r_{n_k}} Az_{n_k}), v - T^{(f_2,g_2,B_2)}_{r_{n_k}} Az_{n_k}\rangle \geq 0, \ \forall v \in K_2.$$

Thus, we have from $(C7) - (C9)$, (3.38) and upper semicontinuity of f_2 in the first argument that

$$f_2(Ax', v) + g_2(Ax', v) - g_2(Ax', Ax') + \psi_2(v) - \psi_2(Ax') + \langle B_2(Ax'), v - Ax'\rangle \geq 0, \ \forall v \in K_2,$$

showing that $Ax' \in GMEP(f_2, g_2, B_2)$. Therefore, $x' \in F(S) \cap \Gamma = \Omega$. Next we show $\limsup_{k\to\infty}\langle x', x' - x_{n_k+1}\rangle \leq 0$. Since $\|x'\| \leq \|x^*\|$, $\forall x^* \in \Omega$, then $x^* = P_\Omega 0$. Now without loss of generality, for any $x^* \in \Omega$, there exists a subsequece $\{x_{n_{k_q}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_q}} \rightharpoonup x^*$ as $q \to \infty$. Hence using (2.5), we have

$$\limsup_{k\to\infty}\langle x', x' - x_{n_k}\rangle = \lim_{q\to\infty}\langle x', x' - x_{n_{k_q}}\rangle = \langle x', x' - x^*\rangle \leq 0. \tag{3.41}$$

Combining (3.26) and (3.41), we get

$$\begin{aligned}\limsup_{k\to\infty}\langle x', x' - x_{n_k+1}\rangle \ \leq \ & \limsup_{k\to\infty}\langle x', x' - x_{n_k}\rangle \\ & + \limsup_{k\to\infty}\langle x', x_{n_k} - x_{n_k+1}\rangle \leq 0.\end{aligned} \tag{3.42}$$

Now from Lemma 3.10, we have

$$\begin{aligned}\|x_{n_k+1} - x'\|^2 &\leq (1 - \delta_{n_k})\|x_{n_k} - x'\|^2 \\ &+ \delta_{n_k}\Big[\frac{\theta_{n_k}}{\delta_{n_k}}\|x_{n_k} - x_{n_k-1}\|M_2 + 2\langle x', x' - x_{n_k+1}\rangle\Big].\end{aligned} \tag{3.43}$$

Since $\lim_{k\to\infty}\frac{\theta_{n_k}}{\delta_{n_k}}\|x_{n_k} - x_{n_k-1}\| = 0$, it follows from (3.42) that

$$\limsup_{k\to\infty}\Big(\frac{\theta_{n_k}}{\delta_{n_k}}\|x_{n_k} - x_{n_k-1}\|M_2 + 2\langle x', x' - x_{n_k+1}\rangle\Big) \leq 0.$$

Therefore, by (3.43) and Lemma 2.4, we have $\lim_{n\to\infty}\|x_n - x'\| = 0$, i.e. $x_n \to x' \in \Gamma$. This completes the proof. $\square$

By Remark 1.1, we have the following Corollary:

Corollary 3.12. *Let $S : K_1 \to H_1$ be $\tau-$demicontractive Mapping. Let the Assumptions 3.1 hold and the sequence $\{x_n\}$ be as in Algorithm 3.2. Then $\{x_n\}$ converges strongly to an element $x' \in \Omega$ such that $\|x'\| = \{\min\|d\| : d \in \Omega\}$.*

If $\psi_1,\ \psi_2 = 0$ and $B_1,\ B_2 = 0$, then the split generalized mixed equilibrium problem, $SGMEP$ (1.10) reduces to the following generalized equilibrium problem:

$$f_1(z, y) + g_1(z, y) - g_1(z, z) \geq 0 \ \forall \ y \in K_1$$

and $v' = Az \in K_2$ solves

$$f_2(v', v) + g_2(v', v) - g_2(v', v') \geq 0 \ \forall \ v \in K_2. \tag{3.44}$$

Hence $\Gamma = \{z \in GEP(f_1, g_1) : Az \in GEP(f_2, g_2)\}$. Also if $S = I$, identity mapping, then $S_{\lambda_n} = I$. Therefore algorithm 3.1 reduces to the following:

Algorithm 3.13. Choose $x_0, x_1 \in H_1$. Given the iterates x_{n-1} and x_n for every $n \geq 1, \theta > 0$, select θ_n such that $0 \leq \theta_n \leq \bar{\theta_n}$ and

$$\bar{\theta_n} = \begin{cases} min\{\frac{\mu_n}{||x_n - x_{n-1}||}, \theta\}, & if\ x_n \neq x_{n-1} \\ \theta, \quad Otherwise, \end{cases} \tag{3.45}$$

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = (1-\alpha_n)w_n + \alpha_n \mathcal{J}_n w_n, \\ z_n = (1-\beta_n)w_n + \beta_n \mathcal{J}_n y_n, \\ x_{n+1} = (1-\gamma_n-\delta_n)w_n + \gamma_n \mathcal{J}_n z_n, \quad \forall n \geq 0, \end{cases} \tag{3.46}$$

where $\mathcal{J}_n = T_{r_n}^{(f_1,g_1)}(I - \eta_n A^*(I - T_{r_n}^{(f_2,g_2)})A)$, $r_n \in [\epsilon, \infty)$, $\epsilon > 0$. for $\eta,\ \xi > 0$, the step size η_n is chosen as follows:

$$0 < \xi \leq \eta_n = \begin{cases} min\{\frac{||(I-T_{r_n}^{(f_2,g_2)})Az_n||^2}{||A^*(I-T_{r_n}^{(f_2,g_2)})Az_n||^2}, \eta\}, & if\ T_{r_n}^{(f_2,g_2)}Az_n \neq Az_n, \\ \eta, \quad Otherwise. \end{cases} \tag{3.47}$$

Using Algorithm 3.13, Theorem 3.11 reduces to the following corollary:

Corollary 3.14. *Let $\{x_n\}$ be defined as in Algorithm 3.13 satisfying Assumptions 3.1. Then $\{x_n\}$ converges strongly to an element $x' \in \Gamma$ such that $\|x'\| = \{\min \|d\| : d \in \Gamma\}$.*

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