Applied Analysis and Optimization
Volume 6, Number 3, 2022, 393–410

Yokohama Publishers
ISSN 2189-1664 Online Journal

NEW RECURSION FORMULAS FOR APPROXIMATING VARIATIONAL INEQUALITY AND FIXED POINT PROBLEMS

CHINEDU EZEA AND EJIKE AGBO

Abstract. In this study, we construct new recursion formulas and prove a strong convergence theorem for a common element of a variational inequality and fixed point of a *relatively nonexpansive* map in a 2-uniformly convex and uniformly smooth real Banach space. We extend our theorem to a *countable family* of relatively nonexpansive maps that solves fixed point and variational inequality problems. This fixed point problem is also a convex feasibility problem. Furthermore, we apply our theorem to approximate a zero of α-inverse strongly monotone map, solution of complementarity problem, and minimizer of a continuously Fr 迦 chet differentiable convex functional. Our theorems complement, improve, and extend the results of numerous authors in the literature.

1. Introduction

Let E be a real Banach space and E^* be the dual space of E. Let $\langle .,. \rangle$ denote the duality pairing between elements of E and those of E^*. Assume that $C \subset E$ is nonempty, closed, and convex. A map $A : C \to E^*$ is said to be

∗ *monotone* if the following inequality holds:
$$\Big\langle x - y, Ax - Ay \Big\rangle \geq 0, \ \forall \ x, y \in C.$$
∗ *k-Lipschitz* if there exists a constant $k \geq 0$ such that
$$||Ax - Ay|| \leq L||x - y||, \ \forall \ x, y \in C.$$
∗ *α-inverse strongly monotone* if there exists a positive real number α such that
$$\langle x - y, Ax - Ay \rangle \geq \alpha||Ax - Ay||^2, \ \forall \ x, y \in C.$$

A map $J : E \to E^*$ defined by
$$J(x) := \Big\{x^* \in E^* : \Big\langle x, x^* \Big\rangle = \|x\|^2, \ \|x^*\| = \|x\|, \ \forall \ x \in E\Big\}$$
is called the *normalized duality map* on E, where $\Big\langle \cdot, \cdot \Big\rangle$ denotes the duality pairing between the elements of E and E^*. The following are some properties of the normalized duality map that are needed in the sequel (see Ibaraki and Takahashi [25]).

∗ If E is uniformly convex, then J is one-to-one and onto.
∗ If E is uniformly smooth, then J is single valued.
∗ In particular, if a Banach space E is uniformly smooth and uniformly convex, the dual space E^* is also uniformly smooth and uniformly convex. Hence, the normalized duality map J on E and the normalized duality map J_*

2020 *Mathematics Subject Classification.* 47H06, 47H09, 47J05, 47J25.

Key words and phrases. Fixed point problems, generalized projection, variational inequality problems, monotone and Lipschitz maps, relatively nonexpansive maps.

on its dual space E^* are both uniformly continuous on bounded sets, and $J_* = J^{-1}$.

In the sequel, the following definitions and results are needed. Let E be a smooth real Banach space with dual space E^*. The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \forall\, x, y \in E, \tag{1.1}$$

where J is the normalized duality mapping from E into E^*. It was introduced by Alber and has been studied by Alber [3], Chidume *et al.* [17], Chidume [18], Chidume and Ezea [19], Chidume *et al.* [22], Chidume and Idu [23], and numerous authors.

If $E = H$, a real Hilbert space, equation (1.1) reduces to

$$\phi(x, y) = \|x - y\|^2, \quad \forall\, x, y \in H.$$

Consequently, it is clear from the definition of ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \quad \forall\, x, y \in E. \tag{1.2}$$

Define a map $V : E \times E^* \to \mathbb{R}$ by $V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2$. Then, it is easy to see that

$$V(x, x^*) = \phi(x, J^{-1}(x^*)),\ \forall\ x \in E,\ x^* \in E^*. \tag{1.3}$$

Let C be a nonempty, closed, and convex subset of a smooth, strictly convex, and reflexive real Banach space E. The *generalized projection* map introduced by Alber [3] is a map $\Pi_C : E \to C$ such that for any $x \in E$, there corresponds a unique element $x_0 := \Pi_C(x) \in C$ written as $\phi(x_0, x) = \min_{y \in C} \phi(y, x)$. If $E = H$ is a real Hilbert space, we remark that the generalized projection map Π_C coincides with the metric projection map from H onto C.

Definition 1.1. Let C be a nonempty, closed, and convex subset of E, and let $T : C \to E$ be a map. A point $x^* \in C$ is called a *fixed point* of T if $T(x^*) = x^*$. The set of fixed points of T is denoted by $F(T)$. A point $p \in C$ is said to be an *asymptotic fixed point* of T if C contains a sequence $\{x_n\}_{n=1}^{\infty}$ that converges *weakly* to p and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. The set of asymptotic fixed points of T is denoted by $\hat{F}(T)$.

Definition 1.2. A map $T : C \to E$ is said to be *relatively nonexpansive* if the following conditions hold (see, for example, Butnariu *et al.* [9], Matsushita and Takahashi ([45], [46]), Reich [49], and Yekini [53, 54]):

(1) $F(T) \neq \emptyset$,
(2) $\phi(p, Tx) \leq \phi(p, x),\ \forall\ x \in C$ and $p \in F(T)$,
(3) $\hat{F}(T) = F(T)$.

Let E be a smooth real Banach space. The Lyapunov functional is defined as follows:

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2,\ \forall\ x, y\ \in E. \tag{1.4}$$

Clearly, we have from the definition of ϕ that

(i) $(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2$,

(ii) $\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x-z, Jz - Jy\rangle$,
(iii) $\phi(x,y) = \langle x, Jx - Jy\rangle + \langle y - x, Jy\rangle \le \|x\|\|Jx - Jy\| + \|y-x\|\|y\|$.

Remark 1.3. From Remark 2.1 in Matsushita and Takahashi [46], if E is a strictly convex smooth Banach space, then we have that $\phi(x,y) = 0$ if and only if $x = y$, for any $x, y \in E$.

Variational inequality problem (VIP) is a problem of finding a point $u \in C$ such that

$$\langle v - u, Au\rangle \ge 0, \ \forall\ v \in C, \tag{1.5}$$

where $A : C \to E^*$ is a single-valued map. The set of solutions of the VIP is denoted by $VI(C,A)$. Lions and Stampacchia [40] in 1967 first studied VIP. Owing to its numerous applications in operations research, engineering design, and economic equilibrium, it has been widely studied. To solve a constrained VIP, the projection method can play a crucial role. Moreover, the simplest method is the gradient projection in which one projects onto the feasible set C at each iteration. However, this method employs the fact that map A is inverse strongly monotone, which is a restrictive assumption. Korpelecich [38] in 1976 introduced the extragradient method for solving the saddle point problem with the following recursion formula:

$$\begin{cases} x_0 \in C, \\ y_n = P_C(x_n - \lambda A(x_n)), \\ x_{n+1} = P_C(x_n - \lambda A(y_n)). \end{cases} \tag{1.6}$$

This method performs two projections onto the feasible subset C of a Banach space. Assuming $VI(C,A) \neq \emptyset$, the author demonstrated that the sequences generated by recursion formula (1.6) converge to some point $v \in VI(C,A)$. Various authors have investigated the extragradient method (see, for example, Ceng *et al.* [10], Censor *et al.* [12], Censor *et al.* [13], Fang *et al.* [26], Kraikaew and Saejung [39], Nedzehkina and Takahashi [47], Tufa and Zegeye [52], and the references therein). To obtain a common element of the set of fixed points and solutions of the VIP in Hilbert or Banach spaces, numerous authors proposed and studied several iterative recursion formulas (see, for example, Buong [8], Ceng *et al.* [10], Chen *et al.* [16], Chidume and Ezea [19], Chidume *et al.* [20], Iiduka and Takahashi [30], Kraikaew and Saejung [39], Nedzehkina and Takahashi [47], Takahashi and Toyoda [51], Tufa and Zegeye [52], and the references therein).

The ideas of studying a common solution problem arise from its applications to mathematical models whose constraints can be represented as VIP and/or fixed point problems. In particular, this occurs in some practical problems, such as signal processing, network resource allocation, and image recovery (see, for example, Censor, Gilbali and Reich [11], Iiduka [26, 27], and Iiduka and Yamada [28, 29]). Maing [43] proposed a hybrid-type method for finding an element of $F(T) \cap VI(C,A)$ in a 2-uniformly convex and uniformly smooth Banach space, where $T : C \to C$ is a relatively nonexpansive map and $A : C \to E^*$ is an α-inverse strongly monotone map satisfying the following condition:

$$\|Ay\| \le \|Ay - Au\|, \ \forall\ y \in C \ and \ u \in VI(C,A). \tag{1.7}$$

If A is an α-inverse strongly monotone, we remark that it is monotone and $\frac{1}{\alpha}$-Lipschitz. The following problems naturally arise:

(P1) How to relax the inverse strongly monotonicity of A to monotonicity and Lipschitz?
(P2) How can we drop condition (1.7)?

Nakajo [48] recently proposed the hybrid gradient projection method using the following recursion formula:

(1.8)
$$\begin{cases} x_0 = x \in E, \\ y_n = \Pi_C J^{-1}(Jx_n - \lambda_n A(x_n)), \\ z_n = Ty_n, \\ C_n = \{u \in C : \phi(u, x_n) \leq \phi(u, x_n) - \phi(y_n, x_n) - 2\lambda_n \langle y_n - u, Ax_n - Ay_n\rangle\}, \\ Q_n = \{u \in C : \langle x_n - u, Jx - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x, \end{cases}$$

where E is a 2-uniformly convex and uniformly smooth Banach space with the 2-uniform convexity constant c_1, $T : C \to C$ is a relatively nonexpansive map, $A : C \to E^*$ is a monotone and Lipschitz map, and $0 < \inf_{n\in\mathbb{N}} \lambda_n \leq \sup_{n\in\mathbb{N}} \lambda_n < \frac{c_1}{2L}$. The author proved that the sequence $\{x_n\}$ generated by recursion formula (1.8) converges strongly to $\Pi_{VI(C,A)\cap F(T)}x$. In recursion formula (1.8), condition (1.7) imposed by Liu [41] was removed and the inverse strong monotonicity of A was successfully weakened to monotonicity and Lipschitz. Thus, the study conducted by Nakajo [48] is of great significance. However, the set C_n in Nakajo's recursion formula appears to be difficult to compute.

Therefore, to solve problems (P1) and (P2), a new iterative algorithm that differs from recursion formula (1.8) was constructed by applying the idea of Nedezhkina and Takahashi [47]. By combining the hybrid and extragradient methods, Nedezhkina and Takahashi [47] constructed the following recursion formula:

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n A(x_n)), \\ z_n = \alpha_n x_n + (1-\alpha_n)TP_C(x_n - \lambda_n Ay_n), \\ C_n = \{v \in C : ||z_n - v|| \leq ||x_n - v||\}, \\ Q_n = \{v \in C : \langle x_n - v, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases} \tag{1.9}$$

In fact, the following theorem was proved by the authors.

Theorem (Nedzhkina and Takahashi [47])**.** *Let C be a nonempty, closed, and convex subset of a Hilbert space H. Let A be a monotone and k-Lipschitz map of C into H and T be a nonexpansive map of C into itself such that $F(T) \cap VI(C,A) \neq 0$. Let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be sequences generated by recursion formula (1.9). If $\{\lambda_n\} \subset [a,b]$ for some $a, b \in (o, \frac{1}{k})$ and $\alpha_n \in [0,c]$ for some $c \in [0,1)$, the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to $P_{F(T)\cap VI(C,A)}x$.*

We remark that map A in recursion formula (1.9) is only monotone and Lipschitz as well as does not require condition (1.7). Furthermore, the form of C_n in recursion formula (1.9) is simple compared to that of recursion formula (1.8), but the convergence result of recursion formula (1.9) is only in Hilbert spaces.

Thus, the following problem arises.

(P3) How can one employ recursion formula (1.9) in more general Banach spaces?

To solve these problems ((P1), (P2), and (P3)), the following recursion formula was studied by Liu and Kong [42]:

$$\begin{cases} x_0 = x \in C, \\ y_n = \Pi_C J^{-1}(Jx_n - \lambda_n A(x_n)), \\ t_n = \Pi_C J^{-1}(Jx_n - \lambda_n A(y_n)), \\ z_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTt_n), \\ C_n = \{v \in C : \phi(z, z_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0. \end{cases} \tag{1.10}$$

Moreover, the authors proved the following theorem.

Theorem (Liu and Kong [42]). *Let C be a nonempty, closed, and convex subset of 2-uniformly convex and uniformly smooth Banach space E with the 2-uniform convexity constant c_1. Let A be a monotone and k-Lipschitz map of C into E, and let T be a relatively nonexpansive map of C into itself such that $F(T) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, and $\{z_n\}_{n=1}^{\infty}$ be sequences generated by recursion formula (1.10). Then, $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, and $\{z_n\}_{n=1}^{\infty}$ converge strongly to $\Pi_{F(T)\cap VI(C,A)}x_1$*

Recursion formula (1.10) of Liu and Kong improves that of Nedzhkina and Takahashi (1.9) and solves problems (P1), (P2), and (P3). We remark that these aforementioned recursion formulas involve two sets, C_n and Q_n, as well as a relative nonexpansive map.

A convex feasibility problem is a problem of finding a point in the intersection of nonempty, closed, and convex sets. Numerous real-life problems can be modeled into these types of problems, such as radiation therapy treatment, image reconstruction, and image restoration (Censor [14]). The new recursion formula introduced in one of our theorems (Theorem 4.1) for a countable family of mappings is called *block interative algorithm*, and the block iterative algorithm is used in solving convex feasibility problems. See, for example, Aharoni and Censor [1], Aleyner and Reich [4], Bruck [7], Chidume et al. [15], Maingé [44], and Suzuki [50].

We ask the following question.

(P4) Can one obtain new recursion formulas, which involve only one set, C_n, in arbitrary Banach space that improve recursion formula (1.10) and solve fixed point and variational inequality problems? Moreoevr, the fixed point problem is also a convex feasibility problem involving a countable family of mappings.

Motivated and inspired by these studies, this study introduces recursion formulas that significantly improve (1.8), (1.9), and (1.10). Additionally, we solve problem

(P4) and prove a *strong convergence* theorem for a common element for variational inequality and fixed point of a *relatively nonexpansive* map in a 2-uniformly convex and uniformly smooth real Banach space. Moreover, we extend our theorem to a *countable family* of relatively nonexpansive maps that solves fixed point and variational inequality problems. The fixed point problem is also a convex feasibility problem. Furthermore, we apply our theorem to approximate a zero of α-inverse strongly monotone map, solution of complementarity problem, and minimizer of a continuously Fr 迣 chet differentiable convex functional.

Remark 1.4. We compare our theorems with some recent results.

(1) Recursion formula (3.1) studied in Theorem 3.1 is much simpler than recursion formulas (1.8), (1.9), and (1.10) studied in the theorems of Nakajo [48], Nedzhkina and Takahashi [47], and Liu and Kong [42], respectively. Recursion formula (3.1) requires fewer calculations. Moreover, at each stage of the iteration process, the recursion formulas examined by Nakajo [48], Nedzhkina and Takahashi [47], and Liu and Kong [42] compute two subsets of C, C_n and Q_n, and their intersection, $C_n \cap Q_n$, as well as project the initial vector onto this intersection. The subset Q_n has been dispensed with in our iteration process. Furthermore, recursion formulas (1.9) and (1.10) have 2 iteration parameters λ_n and α_n that are to be computed at each step of the iteration process. The iteration parameters in recursion formula (3.1) of Theorem 3.1 are *two* fixed arbitrary constants $\lambda \in \left(0, \frac{c_1}{k}\right)$ and $\alpha \in [0, 1)$ that are to be computed once and then used at each step of the iteration process. Consequently, these make recursion formula (3.1) more efficient, cost-effective, and applicable than recursion formulas (1.9) and (1.10).
(2) Theorem 4.1 is an extension of Theorem 3.1 from the case where T is a single relatively nonexpansive map to that of *a countable family* of relatively nonexpansive maps. Moreover, it solves fixed point and variational inequality problems. The fixed point problem is also a convex feasibility problem. Consequently, Theorem 4.1 further extends the Theorems of Nakajo [48], Nedzhkina and Takahashi, [47], and Liu and Kong [42] to *countable families* of relatively nonexpansive maps.

2. Preliminarie

Definition 2.1. Let E be a real Banach space with dual space E^*. A map $T : E \to E$ is said to be *Lipschitz* if for each $x, y \in E$, there exists $L \geq 0$ such that $\|Tx - Ty\| \leq L\|x - y\|$.

The *modulus of convexity* of a space E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) := \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : \|x\| = \|y\| = 1; \, \epsilon = \|x - y\| \right\}.$$

The space E is *uniformly convex* if $\delta_E(\epsilon) > 0$, for every $\epsilon \in (0, 2]$. If there exist a constant $c > 0$ and real number $p > 1$ such that $\delta_E(\epsilon) \geq c\epsilon^p$, then E is said to be *p-uniformly convex.* Typical examples of such spaces are L_p, ℓ_p , and Sobolev spaces, W_p^m , for $1 < p < \infty$, where

$$L_p \ (or\ l_p)\ or\ W_p^m \text{ is } \begin{cases} p-\text{uniformly convex}, & \text{if } 2 \le p < \infty, \\ 2-\text{uniformly convex}, & \text{if } 1 < p \le 2. \end{cases}$$

Let $S := \{z \in E : \|z\| = 1\}$. A space E is said to have a *Gâteaux differentiable norm* if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists, for all $x, y \in S$ and is said to have a *uniformly Gâteaux differentiable norm* if for each $y \in S$, limit (2.1) exists and is attained uniformly, for $x \in S$. The space E is said to have a *Fréchet differentiable norm* if, for each $x \in S$, limit (2.1) exists and is attained uniformly for $y \in S$.

Definition 2.2. Let E be a real normed space of dimension ≥ 2. The *modulus of smoothness* of E, $\rho_E : [0,\infty) \to [0,\infty)$, is defined by

$$\rho_E(\tau) := \sup\left\{ \frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau,\ \tau > 0 \right\}.$$

The space E is called *smooth* if $\rho_E(\tau) > 0\, \forall \tau > 0$, and is called *uniformly smooth* if $\lim_{t\to 0^+} \frac{\rho_E(t)}{t} = 0$.

In the sequel, we need the following lemmas.

Lemma 2.3 (Liu [41])**.** *Let E be a uniformly convex and smooth Banach space, and let $\{u_n\}$ and $\{v_n\}$ be sequences in E. If $\phi(u_n, v_n) \longrightarrow 0$ as $n \to \infty$ and either $\{u_n\}$ or $\{v_n\}$ is bounded, then $u_n - v_n \longrightarrow 0$ as $n \to \infty$.*

Lemma 2.4 (Alber [3])**.** *Let C be a nonempty, closed, and convex subset of a smooth, strictly convex, and reflexive real Banach space E. Then,*

$$\phi(y, \Pi_C(x)) + \phi(\Pi_C(x), x) \le \phi(y, x) \ \text{ for all } \ x \in E, \ \ y \in C.$$

Lemma 2.5 (Alber [2])**.** *Let E be a reflexive, strictly, convex, and smooth Banach space with E^* as its dual. Then,*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \le V(x, x^* + y^*), \tag{2.2}$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 2.6 (Alber [2])**.** *Let C be a nonempty, closed, and convex subset of a smooth real Banach space E write $x \in E$ and $x_0 \in C$. Then, $x_0 := \Pi_C x$ if and only if*

$$\langle y - x_0, Jx_0 - Jx\rangle \ge 0, \ \ \text{for all } \ y \in C.$$

Lemma 2.7 (Zegeye and Shahzad [55])**.** *Let C be a nonempty, closed, and convex subset of a real reflexive, strictly convex, and smooth Banach space E. If $A : C \to E^*$ is a continuous monotone map, then $VI(C, A)$ is closed and convex.*

Lemma 2.8 (Nilsrakoo and Saejung [37])**.** *Let C be a nonempty, closed, and convex subset of a uniformly convex and uniformly smooth real Banach space E, and let $\{T_i : C \to E\}_{i=1}^{\infty}$ be a sequence of map such that $\cap_{i=1}^{\infty} F(T_i) \ne \emptyset$ and $\phi(p, T_i x) \le$*

$\phi(p,x)\ \forall\ x\ \in C$ and $p \in \cap_{i=1}^{\infty}F(T_i),\ i \in N$. Suppose that $\{\beta_i\}_{i=1}^{\infty}$ is a sequence in $(0,1)$ such that $\sum_{i=1}^{\infty}\beta_i = 1$ and $T : C \to E$ is defined by

$$Tx = J^{-1}\Big(\sum_{i=1}^{\infty}\beta_i JT_i x)\Big) \quad for\ each\ x \in C,$$

let $\{x_n\}$ be a bounded sequence in C. Then,

(a) $x_n - Tx_n \to 0,$
(b) $x_n - T_i x_n \to 0\ for\ each\ i \in N,$
(c) $F(T) = \cap_{i=1}^{\infty}F(T_i).$

Remark 2.9. We remark that (c) is both fixed point and convex feasibility problems (see Matsushita and Takahashi [46]).

3. Main Results

Theorem 3.1. *Let E be a uniformly smooth and 2-uniformly convex real Banach space with dual E^*. Let C be a nonempty, closed, and convex subset of E with 2-uniform convexity constant c_1. Let $A : C \longrightarrow E^*$ be a monotone and k-Lipschitz map, and let $T : C \longrightarrow C$ be a relatively nonexpansive map. Assume that $W := F(T) \cap VI(C,A) \neq \emptyset$, for arbitrary $x_1 \in C$, let the sequence $\{x_n\}_{n=1}^{\infty}$ be iteratively defined by the following recursion formula:*

$$\begin{cases} x_1 \in C := C_1, \\ y_n = \Pi_C J^{-1}(Jx_n - \lambda A x_n), \\ t_n = \Pi_C J^{-1}(Jx_n - \lambda A y_n), \\ z_n = J^{-1}(\alpha J x_n + (1-\alpha)JTt_n), \\ C_{n+1} = \{v \in C_n : \phi(v, z_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \ \ \forall\ \ n \geq 1, \end{cases} \tag{3.1}$$

where Π_C denotes the generalized projection of E onto C, $J : E \to E^*$ is the normalized duality map, $\lambda \in (0, \frac{c_1}{k})$, $\alpha \in [0, c) \subset [0, 1)$, and k is the Lipschitz constant of A. Then, the sequences $\{x_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty}$, and $\{t_n\}_{n=1}^{\infty}$ converge strongly to some $x^* \in W$.

Proof. The proof is divided into 5 steps.

Step 1: $\Pi_{C_{n+1}}$ is well defined.
It is sufficient to show that C_{n+1} is closed and convex, for all $n \geq 1$. The proof follows by induction since $C_1 := C$ is closed and convex. Suppose C_n is closed and convex for some $n \geq 1$. Hence, $\phi(z, z_n) \leq \phi(z, x_n)$ if and only if $\langle z, Jx_n - Jz_n\rangle - \|x_n\|^2 + \|z_n\|^2 \leq 0$, so

$$C_{n+1} = \{z \in C_n : f(z) \leq 0\}$$

is closed and convex, where $f(z) := \langle z, Jx_n - Jy_n\rangle - \|x_n\|^2 + \|y_n\|^2$. Therefore, $\Pi_{C_{n+1}}$ is well defined.

Step 2: $\{\phi(x_{n+1}, x_n)\}_{n=1}^{\infty}$ *converges to* 0.
Let $v \in C_n$ for all $n \geq 1$. By applying $x_n = \Pi_{C_n} x_1$ and Lemma 2.4, we obtain that

$$\phi(x_n, x_1) = \phi(\Pi_{C_n} x_1, x_1) \leq \phi(v, x_1),$$

which yields that $\{\phi(x_n, x_1)\}_{n=1}^{\infty}$ is bounded. The utilization of inequality (1.2) gives that the sequence $\{x_n\}_{n=1}^{\infty}$ is also bounded. Moreover, for each $n \in \mathbb{N}$, $x_n = \Pi_{C_n} x_1$ and $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$; thus, Lemma 2.4 gives

$$\phi(x_n, x_1) \leq \phi(x_{n+1}, x_n) + \phi(x_n, x_1) \leq \phi(x_{n+1}, x_1). \tag{3.2}$$

Hence, $\phi(x_{n+1}, x_n) \longrightarrow 0$ *as* $n \longrightarrow 0$. By applying Lemma 2.3, we have that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.3}$$

<u>**Step 3**</u>: $W \subset C_n$, for each $n \in \mathbb{N}$.
This proof is by induction. Obviously, $W \subset C_1 = C$. Suppose that $W \subset C_n$ for some $n \geq 1$, let $u \in W$ be arbitrary. By utilizing Lemmas 2.4 and 2.5, monotonicity of A, $u \in VI(C, A)$, and $t_n = \Pi_C J^{-1}(Jx_n - \lambda Ay_n)$, we obtain that

$$\begin{aligned}
\phi(u, t_n) &\leq \phi(u, J^{-1}(Jx_n - \lambda Ay_n)) - \phi(t_n, J^{-1}(Jx_n - \lambda Ay_n)) \\
&= \phi(u, x_n) + \phi(x_n, J^{-1}(Jx_n - \lambda Ay_n)) + 2\langle u - x_n, Jx_n - (Jx_n - \lambda Ay_n)\rangle \\
&\quad -\phi(t_n, x_n) - \phi(x_n, J^{-1}(Jx_n - \lambda Ay_n)) - 2\langle t_n - x_n, \lambda Ay_n\rangle \\
&= \phi(u, x_n) + 2\langle u - x_n, \lambda Ay_n\rangle - \phi(t_n, x_n) - 2\langle t_n - x_n, \lambda Ay_n\rangle \\
&= \phi(u, x_n) + 2\lambda\langle u - t_n, Ay_n\rangle - \phi(t_n, x_n) \\
&= \phi(u, x_n) - \phi(t_n, x_n) + 2\lambda\langle u - y_n, Ay_n - Au\rangle + 2\lambda\langle u - y_n, Au\rangle \\
&\quad +2\lambda\langle y_n - t_n, Ay_n\rangle \\
&\leq \phi(u, x_n) - \phi(t_n, x_n) - 2\lambda\langle y_n - u, Ay_n - Au\rangle + 2\lambda\langle y_n - t_n, Ay_n\rangle \\
&\leq \phi(u, x_n) - \phi(t_n, x_n) + 2\lambda\langle y_n - t_n, Ay_n\rangle \\
&= \phi(u, x_n) - \phi(t_n, y_n) - \phi(y_n, x_n) - 2\langle t_n - y_n, Jy_n - Jx_n\rangle \\
&\quad +2\lambda\langle y_n - t_n, Ay_n\rangle \\
&= \phi(u, x_n) - \phi(t_n, y_n) - \phi(y_n, x_n) + 2\langle t_n - y_n, Jx_n - Jy_n - \lambda Ay_n\rangle. \qquad (3.4)
\end{aligned}$$

Additionally, since $y_n = \Pi_C J^{-1}(Jx_n - \lambda Ax_n)$, we have by applying Lemma 2.6 that

$$\langle t_n - y_n, Jx_n - \lambda Ax_n - Jy_n\rangle \leq 0. \tag{3.5}$$

The application of the fact that A is Lipschitz, Lemma 2.6, equation (3.5), and Cauchy–Schwartz inequality gives

$$\begin{aligned}
2\langle t_n - y_n, Jx_n - \lambda Ay_n - Jy_n\rangle &= 2\langle t_n - y_n, Jx_n - \lambda Ax_n - Jy_n\rangle \\
&\quad +2\lambda\langle t_n - y_n, Ax_n - Ay_n\rangle \\
&\leq 2\lambda\langle t_n - y_n, Ax_n - Ay_n\rangle \\
&\leq 2\lambda k||t_n - y_n||||x_n - y_n|| \\
&\leq \lambda k(||t_n - y_n||^2 + ||x_n - y_n||^2) \\
&\leq \lambda k(\frac{\phi(t_n, y_n)}{c_1} + \frac{\phi(y_n, x_n)}{c_1}) \\
&= \frac{\lambda k}{c_1}(\phi(t_n, y_n) + \phi(y_n, x_n)). \\
&\leq \phi(t_n, y_n) + \phi(y_n, x_n). \qquad (3.6)
\end{aligned}$$

The combination of inequalities (3.4) and (3.6) gives

$$\phi(u,t_n) \leq \phi(u,x_n). \tag{3.7}$$

In addition, since $z_n = J^{-1}(\alpha Jx_n + (1-\alpha)JTt_n)$ and $u \in F(T)$, we have from inequality (3.7) that

$$\begin{aligned}
\phi(u,z_n) &= \phi(u, J^{-1}(\alpha Jx_n + (1-\alpha)JTt_n)) \\
&= ||u||^2 - 2\langle u, \alpha Jx_n + (1-\alpha)JTt_n\rangle + ||\alpha Jx_n + (1-\alpha)JTt_n||^2 \\
&= \|u\|^2 - 2\alpha\langle u, Jx_n\rangle - 2(1-\alpha)\langle u, JTt_n\rangle + \|\alpha Jx_n + (1-\alpha)JTt_n\|^2 \\
&\leq \alpha\|u\|^2 - 2\alpha\langle u, Jx_n\rangle + \alpha\|Jx_n\|^2 + (1-\alpha)\|u\|^2 \\
&\quad -2(1-\alpha)\langle u, JTt_n\rangle + (1-\alpha)\|JTt_n\|^2 \\
&= \alpha\|u\|^2 - 2\alpha\langle u, Jx_n\rangle + \alpha\|x_n\|^2 + (1-\alpha)\|u\|^2 \\
&\quad -2(1-\alpha)\langle u, JTt_n\rangle + (1-\alpha)\|Tt_n\|^2 \\
&= \alpha\phi(u,x_n) + (1-\alpha)\phi(u,Tt_n) \\
&\leq \alpha\phi(u,x_n) + (1-\alpha)\phi(u,t_n) \\
&\leq \phi(u,x_n) - (1-\alpha)(1-\frac{\lambda k}{c_1})(\phi(t_n,y_n) + \phi(y_n,x_n)) \\
&\leq \phi(u,x_n) \qquad (3.8)
\end{aligned}$$

It follows that $u \in C_{n+1}$. Hence, $W := F(T) \cap VI(C,A) \subset C_n$ for all $n \in \mathbb{N}$

Step 4: $t_n \to x^* \in F(T)$.
Using the fact that $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$ and inequality (3.2), we get

$$\phi(x_{n+1}, z_n) \leq \phi(x_{n+1}, x_n) \to 0 \text{ as } n \to \infty. \tag{3.9}$$

It follows from Lemma 3.3 and inequality (3.9) that

$$\|x_{n+1} - z_n\| \to 0 \text{ as } n \to \infty. \tag{3.10}$$

By applying conditions (3.3) and (3.10), we observe that

$$\|x_n - z_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \to 0 \text{ as } n \to \infty. \tag{3.11}$$

The fact that J is norm-to-norm uniformly continuous on bounded subsets of E^* gives that

$$\|Jx_n - Jz_n\| \to 0 \text{ as } n \to \infty, \tag{3.12}$$

$$\|Jx_{n+1} - Jz_n\| \to 0 \text{ as } n \to \infty, \tag{3.13}$$

and

$$\|Jx_{n+1} - Jx_n\| \to 0 \text{ as } n \to \infty. \tag{3.14}$$

The utilization of conditions (3.11) and (3.12) gives

$$\begin{aligned}
\phi(u,x_n) - \phi(u,z_n) &= 2\langle u, Jz_n - Jx_n\rangle + \|x_n\|^2 - \|z_n\|^2 \\
&\leq 2\|u\|\|Jz_n - Jx_n\|| + (\|x_n - z_n\|)(\|x_n\| + \|z_n\|) \\
&\leq 2\|u\|\|Jz_n - Jx_n\| + \beta(\|x_n - z_n\|) \to 0 \text{ as } n \to \infty.
\end{aligned}$$

From inequalities (3.8) and (3.15), we have

$$(1-\alpha)(1-\frac{\lambda k}{c_1})\phi(t_n,x_n) \le \phi(u,x_n)-\phi(u,z_n) \to 0 \text{ as } n\to\infty \tag{3.15}$$

and

$$(1-\alpha)(1-\frac{\lambda k}{c_1})\phi(y_n,x_n) \le \phi(u,x_n)-\phi(u,z_n) \to 0 \text{ as } n\to\infty. \tag{3.16}$$

Thus, we get that

$$\phi(y_n,x_n)\to 0 \text{ as } n\to\infty \tag{3.17}$$

and

$$\phi(t_n,y_n)\to 0 \text{ as } n\to\infty. \tag{3.18}$$

Using Lemma 2.3, we see that

$$\|y_n-x_n\|\to 0 \text{ as } n\to\infty \tag{3.19}$$

and

$$\|t_n-y_n\|\to 0 \text{ as } n\to\infty. \tag{3.20}$$

By employing conditions (3.19) and (3.20), we get that

$$\|x_n-t_n\| \le \|y_n-x_n\|+\|t_n-y_n\| \to 0 \text{ as } n\to\infty. \tag{3.21}$$

Additionally, by utilizing conditions (3.11) and (3.21), we observe that

$$\|z_n-t_n\| \le \|z_n-x_n\|+\|x_n-t_n\| \to 0 \text{ as } n\to\infty. \tag{3.22}$$

The fact that J is norm-to-norm uniformly continuous on bounded subsets of E^* gives that

$$\|Jt_n-Jx_n\|\to 0 \text{ as } n\to\infty \tag{3.23}$$

and

$$\|Jt_n-Jz_n\|\to 0 \text{ as } n\to\infty. \tag{3.24}$$

Since $z_n = J^{-1}(\alpha Jx_n + (1-\alpha)JTt_n)$, we have

$$Jz_n - Jt_n = \alpha(Jx_n - Jt_n) + (1-\alpha)(JTt_n - Jt_n) \tag{3.25}$$

and

$$(1-\alpha)(JTt_n - Jt_n) = Jz_n - Jt_n - \alpha(Jx_n - Jt_n). \tag{3.26}$$

Using conditions (3.23), (3.24), and (3.26), as well as the fact that $0\le\alpha<1$, we see that

$$\begin{aligned}(1-\alpha)\|JTt_n - Jt_n\| &= \|Jz_n - Jt_n - \alpha(Jx_n - Jt_n)\| \\ &\le \alpha\|Jx_n - Jt_n\| + \|Jz_n - Jt_n\| \to 0 \text{ as } n\to\infty.\end{aligned}$$

It follows that

$$\lim_{n\to\infty}\|JTt_n - Jt_n\| = 0. \tag{3.27}$$

Since J^{-1} is norm-to-norm uniformly continuous on bounded subsets of E^*, we get

$$\lim_{n\to\infty}\|Tt_n - t_n\| = 0. \tag{3.28}$$

Besides, by applying conditions (3.28) and (3.21), we have

$$t_n \to x^* \text{ as } n \to \infty. \tag{3.29}$$

Since T is a relatively nonexpansive map, we obtain that $x^* \in F(T)$.

Step 5: $x_n \to x^* \in VI(C, A)$.
Let $x \in C$. Clearly,

$$\begin{aligned}
\langle x_n - x, \lambda A x_n\rangle &= \langle x_n - y_n, \lambda A x_n\rangle + \langle y_n - x, \lambda A x_n\rangle \\
&= \langle x_n - y_n, \lambda A x_n\rangle + \langle y_n - x, Jx_n - Jy_n\rangle \\
&\quad -\langle y_n - x, Jx_n - \lambda A x_n - Jy_n\rangle \\
&\le \lambda\|Ax_n\|\|x_n - y_n\| + \|Jx_n - Jy_n\|\|y_n - x\| \\
&\quad -\langle y_n - x, Jx_n - \lambda A x_n - Jy_n\rangle.
\end{aligned} \tag{3.30}$$

Additionally, using $y_n = \Pi_C J^{-1}(Jx_n - \lambda A x_n)$ and Lemma 2.6, we see that

$$\langle y_n - x, Jx_n - \lambda A x_n - Jy_n\rangle \ge 0.$$

Equation (3.30) becomes

$$\langle x_n - x, Ax_n\rangle \le \lambda\|Ax_n\|\|x_n - y_n\| + \|Jx_n - Jy_n\|\|y_n - x\|. \tag{3.31}$$

Further, owing to the boundedness of $\{A(x_n)\}$ and condition (3.19), we get

$$\limsup_{n\to\infty}\langle x_n - x, Ax_n\rangle \le 0.$$

Since A is monotone, we have that

$$\begin{aligned}
\langle x^* - x, Ax\rangle &= \limsup_{n\to\infty}\langle x_n - x, Ax\rangle \\
&\le \limsup_{n\to\infty}\langle x_n - x, Ax_n\rangle \le 0, \ \forall\, x \in C.
\end{aligned}$$

By combining the fact that $x^* \in C$ and inequality (1.5), we get that $x^* \in VI(C, A)$. This completes the proof. □

4. Convergence theorems concerning countable families

We now prove the following strong convergence theorem.

Theorem 4.1. *Let E be a uniformly smooth and 2-uniformly convex real Banach space with dual space E^*. Let C be a nonempty, closed, and convex subset of E; let $A : C \to E^*$ be a monotone and k-Lipschitz map. Let $T_i : C \to E$, $i = 1, 2, \ldots,$ be a countable family of relatively nonexpansive maps. Assume that $W := \cap_{i=1}^{\infty} F(T_i) \cap VI(C, A) \neq \emptyset$, for arbitrary $x_1 \in C$, let the sequence $\{x_n\}_{n=1}^{\infty}$ be iteratively defined by*

$$\begin{cases}
x_1 \in C := C_1, \\
y_n = \Pi_C J^{-1}(Jx_n - \lambda A x_n), \\
t_n = \Pi_C J^{-1}(Jx_n - \lambda A y_n), \\
z_n = \alpha J x_n + (1-\alpha) J(J^{-1}\sum_{i=1}^{\infty}\beta_i J T_i t_n), \\
C_{n+1} = \{v \in C_n : \phi(v, z_n) \le \phi(v, x_n)\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_1, \ \ \forall\ n \ge 1,
\end{cases} \tag{4.1}$$

where Π_C is the generalized projection of E onto C, $J : E \to E^*$ is the normalized duality map, $\lambda \in \left(0, \frac{c_1}{k}\right)$, $\alpha \in [0, c) \subset [0, 1)$, $k > 0$ denotes the Lipschitz constant of A, and $\{\beta_i\}_{i=1}^{\infty}$ is a sequence in $(0, 1)$ such that $\sum_{i=1}^{\infty} \beta_i = 1$. Then, the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ converge strongly to some $x^* \in W$.

Proof. We observe from Lemma 2.8 that the map $T : C \to E$ defined by

$$Tt_n := J^{-1} \sum_{i=1}^{\infty} \beta_i J T_i t_n$$

is relatively nonexpansive and $F(T) = \cap_{i=1}^{\infty} F(T_i)$. It follows by employing Theorem 3.1 that the sequences $\{x_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty}$, and $\{t_n\}_{n=1}^{\infty}$ converge strongly to some $x^* \in W := \cap_{i=1}^{\infty} F(T_i) \cap VI(C, A) = F(T) \cap VI(C, A)$. □

Corollary 4.2. *Let $E = L_p, \ell_p$, and W_m^p, $1 < p \leq 2$. Let C be a nonempty, closed, and convex subset of E. Let $A : C \to E^*$ be a monotone and k-Lipschitz map. Let $T_i : C \to E$, $i = 1, 2, \ldots,$ be a countable family of relatively nonexpansive maps. Assume that $W := \cap_{i=1}^{\infty} F(T_i) \cap VI(C, A) \neq \emptyset$, for arbitrary $x_1 \in C$, let the sequence $\{x_n\}_{n=1}^{\infty}$ be iteratively defined by*

$$(4.2) \qquad \begin{cases} x_1 \in C := C_1, \\ y_n = \Pi_C J^{-1}(Jx_n - \lambda A x_n), \\ t_n = \Pi_C J^{-1}(Jx_n - \lambda A y_n), \\ z_n = \alpha J x_n + (1 - \alpha) J(J^{-1} \sum_{i=1}^{\infty} \beta_i J T_i t_n), \\ C_{n+1} = \{v \in C_n : \phi(v, z_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \ \ \forall \ \ n \geq 1, \end{cases}$$

where Π_C is the generalized projection of E onto C, $J : E \to E^*$ be the normalized duality map, $\lambda \in \left(0, \frac{c_1}{k}\right)$, $\alpha \in [0, c) \subset [0, 1)$, $k > 0$ denotes the Lipschitz constant of A, and $\{\beta_i\}_{i=1}^{\infty}$ is a sequence in $(0, 1)$ such that $\sum_{i=1}^{\infty} \beta_i = 1$. Then, the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ converge strongly to some $x^* \in W$.

Proof. We observe that E is 2-uniformly convex and uniformly smooth. It follows from Theorem 3.1 that the sequences $\{x_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty}$, and $\{t_n\}_{n=1}^{\infty}$ converge strongly to some $x^* \in W := \cap_{i=1}^{\infty} F(T_i) \cap VI(C, A)$. □

We now consider more applications and state the theorems. Proofs of the theorems follow as those of similar applications given in Chidume *et al.* [21] as well as Iiduka and Takahashi [31]. For completeness, details of the sketches are provided.

5. Applications

5.1. **Approximating a zero of an α-inverse strongly monotone map.**

Theorem 5.1. *Let E be a 2-uniformly convex and uniformly smooth real Banach space with dual space E^*. Let $A : E \to E^*$ be an α-inverse strongly monotone map, and let $T_i : E \to E$, $i = 1, 2, \ldots,$ be a countable family of relatively nonexpansive maps. Assume that $W := \cap_{i=1}^{\infty} F(T_i) \cap A^{-1}0 \neq \emptyset$, where $A^{-1}0 = \{u \in E : Au =$*

$0\} \neq \emptyset$, for arbitrary $x_1 \in E$, let the sequence $\{x_n\}_{n=1}^{\infty}$ be iteratively defined by

$$(5.1)\qquad \begin{cases} x_1 \in C := C_1, \\ y_n = \Pi_C J^{-1}(Jx_n - \lambda A x_n), \\ t_n = \Pi_C J^{-1}(Jx_n - \lambda A y_n), \\ z_n = \alpha J x_n + (1-\alpha) J(J^{-1} \sum_{i=1}^{\infty} \beta_i J T_i t_n), \\ C_{n+1} = \{v \in C_n : \phi(v, z_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \ \ \forall \ n \geq 1, \end{cases}$$

where $J : E \to E^*$ is the normalized duality map, $\lambda \in \left(0, \frac{c_1}{k}\right)$, $\alpha \in [0, c) \subset [0, 1)$, $k > 0$ denotes the Lipschitz constant of A, and $\{\beta_i\}_{i=1}^{\infty}$ is a sequence in $(0, 1)$ such that $\sum_{i=1}^{\infty} \beta_i = 1$. Then, the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ converge strongly to some $x^* \in W := \cap_{i=1}^{\infty} F(T_i) \cap A^{-1}0$.

Proof. We observe from Theorem 4.1 that the map $T : E \to E$ defined by

$$Tt_n := J^{-1} \sum_{i=1}^{\infty} \beta_i J T_i t_n$$

is relatively nonexpansive and $F(T) = \cap_{i=1}^{\infty} F(T_i)$. Setting $C_1 = E$ and $\Pi_E = I$ in Theorem 3.1, we observe that

$$(5.2)\qquad t_n = J^{-1}(Jx_n - \lambda A y_n) = \Pi_E J^{-1}(Jx_n - \lambda A y_n), \ n \geq 1.$$

Further, $VI(E, A) = A^{-1}0$ and $t \in A^{-1}0$. It follows from Theorem 3.1 that $\{x_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ converge *strongly* to some $x^* \in W := \cap_{i=1}^{\infty} F(T_i) \cap A^{-1}0$. ☐

5.2. **Approximating a solution of complementarity problem.** Let C be a nonempty, closed, and convex subset of E; let $A : C \to E^*$ be a map. Let the polar in E^* be defined by the set $C^* = \{y^* \in E^* : \langle x, y^* \rangle \geq 0 \ \text{ for all } \ x \in C\}$. Then, we study the following problem: find $t \in C$ such that $At \in C^*$ and $\big\langle t, At \big\rangle = 0$. This problem is called the *complementarity* problem (see, for example, Blum and Oettli [6]). The set of solutions of the complementarity problem is denoted by $K(C, A)$.

Theorem 5.2. *Let E be a 2-uniformly convex and uniformly smooth real Banach space with dual space E^*. Let C be a nonempty, closed, and convex subset of E, and let $A : C \to E^*$ be an α-inverse strongly monotone map. Let $T_i : C \to E$, $i = 1, 2, \ldots,$ be a countable family of relatively nonexpansive maps. Assume that $W := \cap_{i=1}^{\infty} F(T_i) \cap K(C, A) \neq \emptyset$, for arbitrary $x_1 \in C$, let the sequence $\{x_n\}_{n=1}^{\infty}$ be iteratively defined by*

$$(5.3)\qquad \begin{cases} x_1 \in C := C_1, \\ y_n = \Pi_C J^{-1}(Jx_n - \lambda A x_n), \\ t_n = \Pi_C J^{-1}(Jx_n - \lambda A y_n), \\ z_n = \alpha J x_n + (1-\alpha) J(J^{-1} \sum_{i=1}^{\infty} \beta_i J T_i t_n), \\ C_{n+1} = \{v \in C_n : \phi(v, z_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \ \ \forall \ n \geq 1, \end{cases}$$

where Π_C is the generalized projection of E onto C, $J : E \to E^*$ is the normalized duality map, $\lambda \in \left(0, \frac{c_1}{k}\right)$, $\alpha \in [0, c) \subset [0, 1)$, $k > 0$ denotes the Lipschitz constant of

A and $\{\beta_i\}_{i=1}^{\infty}$ is a sequence in $(0,1)$ such that $\sum_{i=1}^{\infty}\beta_i = 1$. Then, the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ converge strongly to some $x^* \in W$.

Proof. We observe from Theorem 4.1 that the map $T : C \to E$ defined by

$$Tt_n := J^{-1}\sum_{i=1}^{\infty}\beta_i JT_i t_n$$

is relatively nonexpansive and $F(T) = \cap_{i=1}^{\infty}F(T_i)$. From Lemma 7.1.1 of Iiduka and Takahashi [31], we obtain that $VI(C,A) = K(C,A)$. It follows from Theorem 3.1 that the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ converge strongly to some $x^* \in W := F(T)\cap K(C,A) \neq \emptyset$. □

5.3. **Approximating a minimizer of a continuously Fréchet differentiable convex functional.**

Lemma 5.3 (Baillon and Haddad [5], see also Iiduka and Takahashi [31])**.** *Let E be a Banach space. Let f be a continuously Fréchet differentiable and convex functional, and let ∇f denote the gradient of f. If ∇f is $\frac{1}{\alpha}$-Lipschitz continuous, then ∇f is an α-inverse strongly monotone.*

Theorem 5.4. *Let E be a 2-uniformly convex and uniformly smooth real Banach space with dual space E^*. Let C be a nonempty, closed, and convex subset of E; let $T_i : C \to E$, $i = 1, 2, \dots$, be a countable family of relatively nonexpansive maps. Let $f : E \to \mathbb{R}$ be a map satisfying the following conditions:*

1. *f is a continuously Fréchet differentiable convex functional defined on E, and ∇f is a $\frac{1}{\alpha}$-Lipschitz map;*
2. *$K = arg\min_{y\in C} f(y) = \{x^* \in C : f(x^*) = \min_{y\in C} f(y)\} \neq \emptyset$.*

Assume that $W := \cap_{i=1}^{\infty}F(T_i)\cap K \neq \emptyset$. For arbitrary $x_1 \in C$, let the sequence $\{x_n\}_{n=1}^{\infty}$ be iteratively defined by

$$\begin{cases} x_1 \in C := C_1, \\ y_n = \Pi_C J^{-1}(Jx_n - \lambda\nabla f|_C x_n), \\ t_n = \Pi_C J^{-1}(Jx_n - \lambda\nabla f|_C y_n), \\ z_n = \alpha Jx_n + (1-\alpha)J(J^{-1}\sum_{i=1}^{\infty}\beta_i JT_i t_n), \\ C_{n+1} = \{v \in C_n : \phi(v, z_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_1, \ \ \forall \ n \geq 1, \end{cases} \tag{5.4}$$

where Π_C is the generalized projection of E onto C, $J : E \to E^*$ is the normalized duality map, $\lambda \in \left(0, \frac{c_1}{k}\right)$, $\alpha \in [0,c) \subset [0,1)$, $k > 0$ denotes the Lipschitz constant of A, and $\{\beta_i\}_{i=1}^{\infty}$ is a sequence in $(0,1)$ such that $\sum_{i=1}^{\infty}\beta_i = 1$. Then, the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ converge strongly to some $x^* \in W$.

Proof. We observe from Theorem 4.1 that the map $T : C \to E$ defined by

$$Tt_n := J^{-1}\sum_{i=1}^{\infty}\beta_i JT_i t_n$$

is relatively nonexpansive and $F(T) = \cap_{i=1}^{\infty}F(T_i)$. Using condition (1) of Theorem 5.4, it follows from Lemma 5.3 that $\nabla f|_C$ is an α-inverse strongly monotone map of

C into E^*. Since f is differentiable and convex, we have, as in Chidume *et al.* [21] and Iiduka and Takahashi [30], that $VI(C, \nabla f|_C) = K = arg \min_{y \in C} f(y)$. By applying Theorem 3.1, we obtain that the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ converge strongly to some $x^* \in W := F(T) \cap K \neq \emptyset$. □

Acknowledgment

The authors would like to thank the editors for their helpful remarks, which have significantly enhanced the quality of this study.

References

[1] R. Aharoni and Y. Censor, *Block-iterative projection methods for parallel computation of solutions to convex feasibility problem*, Linear Algebra Appl. **120** (1989), 165–175.

[2] Y. Alber and I. Ryazantseva, *Nonlinear Ill-Posed Problems of Monotone Type*, Springer, London, UK, (2006).

[3] Y. I. Alber and S. Reich, *An iterative method for solving a class of nonlinear operator equations in Banach spaces*, Panamer Math. J. **4** (1994), 39–54.

[4] A. Aleyner and S. Reich, *Block-iterative algoritms for solving convex feasibility problems in Hilbert and in Banach spaces*, J. Math. Anal. and Appl. **343** (2008), 427–435.

[5] J. B. Baillon and G. Haddad, *Quelques proprits des oprateurs angle-borns et n-cycliquement monotones*, Israel J. Math. **26** (1977), 137–150.

[6] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student **63** (1994), 123–145.

[7] R. E. Bruck, *Properties of fixed-point sets of nonexpansive mappings in Banach spaces*, Trans. Amer. Math. Soc. **179** (1973), 251–262.

[8] N. Buong, *Strong convergence theorem of an iterative method for variational inequalities and fixed point problems in Hilbert spaces*, Appl. Math. Comput. **217** (2010), 322–329.

[9] D. Butnariu, S. Reich, and A. J. Zaslavski, *Asymptotic behavior of relatively nonexpansive operators in Banach spaces*, J. Appl. Anal. **7** (2001), 151–174, DOI:10.1515/JAA.2001.151.

[10] L. C. Ceng, N. Hadjisavvas, and N. C. Wong, *Strong convergence theorem by hybrid extragradient-like approximation method for variational inequalities and fixed point problems*, J. Glob. Optim. **46** (2010),635–646.

[11] Y. Censor, A. Gibali, and S. Reich, *Algorithms for the split variational inequality problem, Numer. Algorithms* **59** (2012), 301–323.

[12] Y. Censor, A. Gibali, and S. Reich, *Strong convergence of subgradient extragradient methods for the variational inequality problem in Hilbert space*, Optim. Methods Softw. **26** (2011), 827–845.

[13] Y. Censor, A. Gibali, and S. Reich, *The subgradient extragradient method for solving variational inequalities in Hilbert spaces*, J. Optim. Theory Appl. **148** (2011), 318–335.

[14] Y. Censor, *Finite series-expansion reconstruction methods*, Proc. IEEE **71** (1983), 409–419.

[15] C. E. Chidume, S. I. Ikechukwu, and A. Adamu, *Inertial algorithm for approximating a common fixed point for a countable family of relatively nonexpansive maps*, Fixed Point Theory and Applications **9** (2018), https://doi.org/10.1186/s13663-018-0634-3.

[16] J. M. Chen, L. J.Zhang, and T. G. Fan, *Viscosity approximation methods for nonexpansive mappings and monotone mappings*, J. Math. Anal. Appl. **334** (2007), 1450–1461.

[17] C. E. Chidume, A. U. Bello, and B. Usman, *Iterative algorithms for zeros of strongly monotone Lipschitz maps in classical Banach Spaces*, SpringerPlus **9**,(2015), pp, DOI: 10.1186/s40064–015–1044–1.

[18] C. E. Chidume, *Geometric Properties of Banach Spaces and Nonlinear iterations vol. 1965 of Lectures Notes in Mathematics*, Springer, London, UK, (2009).

[19] C. E. Chidume and C. G Ezea, New algorithms for approximating zeros of inverse strongly monotone maps and J-fixed points, Fixed Point Theory and Applications, Springer **3** (2020). DOI: https://doi.org/10.1186/s13663-019-0668-1

[20] C. E. Chidume, C. G. Ezea, and E. E, Otubo, *A new algorithm for variational inequality problems with alpha inverse strongly monotone maps and common fixed points for a countable family of relatively weak nonexpansive maps, with applications*, Adv. Fixed point Theory **9** (2019), **3**, 214–238.

[21] C. E. Chidume, C. G. Ezea, and E. E, Otubo, *Strong convergence of generalized projection algorithm for variational inequality problems, with applications, Advances in Nonlinear Variational Inequalities* **20** (2017), 42–57.

[22] C. E. Chidume, C. O. Chidume, and A. U. Bello, *An algorithm for computing zeros of generalized phi-strongly monotone and bounded maps in classical Banach spaces*, Optimization **65**, (2016), 827–839, DOI:10.1080/02331934.2015.1074686.

[23] C. E. Chidume and K. O. Idu, *Approximation of zeros of bounded maximal monotone mappings, solutions of Hammerstein integral equations and convex minimization problems, Fixed Point Theory and Appl* (2016), DOI: 10.1186/s13663-016–0582–8.

[24] C. J. Fang, Y. Wang, and S. Yang, *Two algorithms for solving single-valued variational inequalities and fixed point problems*, J. Fixed Point Theory Appl. **18** (2016), 27–43.

[25] T. Ibaraki and W. Takahashi, *A new projection and convergence theorems for projections in Banach spaces*, Journal of Approximation Theory **149** (2007), 1–14.

[26] H. Iiduka, *Acceleration method for convex optimization over the fixed point set of a nonexpansive mapping*, Math. Program. Ser. A **149** (2015), 131–165.

[27] H. Iiduka, *A new iterative algorithm for the variational inequality problem over the fixed point set of a firmly nonexpansive mapping*, Optimization **59** (2010), 873–885.

[28] H. Iiduka and I. Yamada, *A subgradient-type method for the equilibrium problem over the fixed point set and its applications*, Optimization **58** (2009), 251–261.

[29] H. Iiduka and I. Yamada, *A use of conjugate gradient direction for the convex optimization problem over the fixed point set of a nonexpansive mapping*, SIAM J. Optim. **19** (2009), 1881–1893.

[30] H. Iiduka and W. Takahashi, *Strong convergence theorems for nonexpansive mappings and inverse strongly monotone mappings*, Nonlinear Anal-theor. **61** (2005), 341–350.

[31] H. Iiduka and W. Takahashi, *Weak convergence of a projection algorithm for variational inequalities in a Banach space*, J. Math. Anal. Appl. **339** (2008), 668–679.

[32] H. Iiduka, W. Takahashi, and M. Toyoda, *Approximation of solutions of variational inequalities for monotone mappings*, PanAmer. Math. J. **14** (2004), 49–61.

[33] S. Kamimura and W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space*, SIAM J. Optim. **13** (2002), 938–945.

[34] G. Kassay, S. Reich, and S. Sabach, *Iterative methods for solving systems of variational inequalities in reflexive Banach spaces*, SIAM J. Optim. **21** (2011), 1319–1344.

[35] D. Kinderlehrer and G. Stampacchia, *An introduction to variational inequalities and their applications*, Academic Press, New York, (1980).

[36] F. Kohasaka, *Strong convergence of an iterative sequence for maximal monotone operators in Banach spaces*, Abstr. Appl. Anal. **3** (2004), 239–249.

[37] W. Nilsrakoo and S. Saejung, *On the fixed-point set of relatively nonexpansive and generalized nonexpansive mappings*, Fixed Point Theory Appl. volume (2010) Article ID 414232, **14** pages, DOI:1155/2010/414232.

[38] G. M. Korpelevi'c, *An extragradient method for finding saddle points and for other problems*, 'E kon, Mat. Metody **12** (1976), 747–756, (In Russian).

[39] R. Kraikaew and S. Saejung, *Strong convergence of the Halpern subgradient extragradient method for solving variational inequalities in Hilbert spaces*, J. Optim. Theory Appl. **163** (2014), 399–412.

[40] J. L. Lions and G. Stampacchia, *Variational inequalities*, Comm. Pure Appl. Math. **20** (1967), 493–517.

[41] Y. Liu, *Strong convergence theorem for relatively nonexpansive mapping and inverse-strongly-monotone mapping in a Banach space*, Appl. Math. Mech-Engl. **30** (2009), 925–932.

[42] Y. Liu and H. Kong, *Strong convergence theorems for relatively nonexpansive mapping and Lipschitz-continuous monotone mapping in Banach spaces*, Indian J. Pure Appl. Math. **50(4)**: (2018) 1049–1065

[43] P. E. Maing, *A hybrid extragradient-viscosity method for monotone operators and fixed point problems*, SIAM J. Control Optim. **47** (2008), 1499–1515.

[44] P. Maingé, *Approximation methods for common fixed point of nonexpansive mappings in Hilbert spaces*, J. Math. Anal. and Appl. **325** (2007), 469–479.

[45] S. Matsushita and W. Takahashi, Weak and strong convergence theorems for relatively non-expansive mappings in Banach spaces, Fixed Point Theory Appl. **1** (2004), 37–47.

[46] S. Matsushita and W. Takahashi, *A strong convergence theorem for relatively nonexpansive mappings in a Banach space*, J. Approx. Theory **134** (2005), 257–266.

[47] N. Nadezhkina and W. Takahashi, *Strong convergence theorem by a hybrid method for nonexpansive mappings and Lipschitz continuous monotone mappings*, SIAM J. Optim. **16** (2006), 1230–1241.

[48] K. Nakajo, *Strong convergence for gradient projection method and relatively nonexpansive mappings in Banach spaces*, Appl. Math. Comput. **271** (2015), 251–258.

[49] S. Reich, *Constructive techniques for accretive and monotone operators, Applied Nonlinear Analysis*, in: Proceedings of the Third International Conference University of Texas, Arlington, TX, (1978), Academic Press, New York, (1979), 335–345.

[50] T. Suzuki, *Strong convergence theorems for infinite families of nonexpansive mappings in general Banach spaces*, Fixed Point Theory and Application **1** (2005), pp. 103–123.

[51] W. Takahashi and M. Toyoda, *Weak convergence theorem for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl. **118** (2003), 417–428.

[52] A. R. Tufa and H. Zegeye, *An algorithm for finding a common point of the solutions of fixed point and variational inequality problems in Banach spaces*, Arab. J. Math. **4** (2015), 199–213.

[53] S. Yekini, *A new iterative scheme for a countable family of relatively nonexpansive mappings and an equilibrium problem in Banach spaces*, Journal of Global Optimization **54** (2011). 1–17. 10.1007/s10898-011-9775-1

[54] S. Yekini, *Strong convergence theorems for infinite family of relatively quasi nonexpansive mappings and systems of equilibrium problems*, Applied Mathematics and Computation **218** (2012). 5146–5156. 10.1016/j.amc.2011.11.001

[55] H. Zegeye and N. Shahzad, *A hybrid scheme for finite families of equilibrium, variational inequality, and fixed point problems*, Nonlinear Analysis **12** (2011), 263–272.

[56] H. Zegeye and N. Shahzad, *Strong convergence theorems for monotone mappings and relatively weak nonexpansive mappings*, Nonlinear Anal. **70** (2009), 2707–2716.

[57] X. Zhang, Y. Su, and J. Kang, *New hybrid algorithm for weak relatively nonexpansive mapping and inverse-strongly monotone mapping in Banach space*, Appl. Math. and Informatics **29** (2011), 87–102.

Manuscript received 20 September 2022
revised 20 October 2022

C. Ezea
Department of Mathematics, Nnamdi Azikiwe University, Awka, Anambra-State, Nigeria
E-mail address: chinedu.ezea@gmail.com

E. Agbo
Department of Industrial Mathematics and Statistics, Enugu State University of Science and Technology, Agbani, Enugu-State, Nigeria
E-mail address: agk.ejike@gmail.com