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STRONG CONVERGENCE RESULTS FOR SPLIT FEASIBILITY PROBLEM WITH MULTIPLE OUTPUT SETS IN BANACH SPACES WITH APPLICATIONS

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ABSTRACT. In this work, we prove strong convergence of a Halpern-Type algorithm to a solution of split feasibility problem with multiple output in 2-uniformly convex and uniformly smooth Banach spaces. We also give an application of our main result in approximating solutions of Fredholm integral equation of first kind. Our result compliments, extends and unifies several existing results in the literature.

1. Introduction

Let C and Q be nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $T : H_1 \to H_2$ be a bounded linear operator with adjoint $T^* : H_2 \to H_1$. The split feasibility problem introduced by Censor and Elfving [7], is to find

$$x^* \in C \text{ such that } Tx^* \in Q. \tag{1.1}$$

The split feasibility problem (SFP) arises naturally in different areas of applications such as road design [3, 6, 11–13], medical imaging reconstruction [14–16, 28], signal processing, optimization problems, radiation therapy [8], to mention but few. Several generalizations of the SFP have been considered by many researchers. Such generalizations include, for instance, the multiple-set split feasibility problem (MSSFP) (see eg., [8, 21]), the split common fixed point problem (SCFPP) (see e.g., [9, 22]), the split common null point problem (SCNPP) (see e.g., [5, 29, 30]).
One of the most common method of approximating solution of the SFP is the CQ algorithms introduced by Byrne see [4]. The fact that the CQ algorithms requires the computation of the orthogonal projection onto the sets C and Q per iterations, which can only be applied when the underlying sets are relatively simple, led to the several generalizations of the CQ algorithms see e.g., Fukushima [17], Yang [36], Censor and Segal [9] and Xu [34, 35].

In 2019, Reich and Tuyen [25] introduced the generalized split feasibility problem (GSFP) as follows: Let $H_i,\ i = 1, 2, 3, ..., N$, be real Hilbert spaces and let $C_i,\ i = 1, 2, 3, ..., N$, be closed and convex subsets of H_i, respectively. Let $A_i : H_i \to H_{i+1},\ i = 1, 2, ..., N-1$, be bounded linear operators on H_i, for each i.
The GSFP is to find

$$x^* \in C_1 \textit{ such that } A_1x^* \in C_2,\ A_2(A_1x^*) \in C_3..., A_{N-1}A_{N-2}...A_1x^* \in C_N.$$

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The GSFP has practical applications in line balancing problem, where the quantity of semi-finished products from the previous process has to be equal to that intended for the next process, (see e.g [25]).
Recently, Reich *et al.* [24] considered a more general problem which they call split feasibility problem with multiple output. The problem is formulated as follows: Let H, H_i, $i = 1, 2, 3, ..., N$ be a real Hilbert spaces and let $T_i : H \to H_i$, $i = 1, 2, 3, ..., N$ be bounded linear operators. Let C and Q_i be nonempty closed and convex subsets of H and H_i, $i = 1, 2, 3, ..., N$. The split feasibility problem with multiple output is to find

$$x^* \text{ such that } x^* \in S = C \cap (\cap_{i=1}^{N} T_i^{-1} Q_i) \neq \emptyset, \text{ for each } i = 1, 2, 3, ..., N, \tag{1.2}$$

i.e.,

$$x^* \in C \text{ and } T_i x^* \in Q_i, \text{ for each } i = 1, 2, 3, ..., N.$$

If we let $H = H_i$, $C = C_i$, and $Q_i = C_{i+1}$, $i = 1, 2, 3, ..., N$ with $T_1 = A_1$, $T_2 = A_2 A_1 ...$ and $T_{N-1} = A_{N-1} A_{N-2} ... A_1$ we see that GSFP is a special case of the split feasibility problem with multiple output.

It is easy to see that x^* solve (1.2) if and only if

$$0 \in \nabla g(x^*) + N_C(x^*) \tag{1.3}$$

where ∇ is the gradient of the function $g : H \to \mathbb{R}$ defined by $g(x) = \frac{1}{2} \sum_{i=1}^{n} ||(I - P_{Q_i}) T_i x||^2$ and $N_C(x)$ is the normal cone of the set C at x.
Further, we see that equation (1.3) holds if and only if

$$x^* = P_C \left[x^* - \gamma \sum_{i=1}^{N} T_i^* (I - P_{Q_i}) T_i x^* \right] \tag{1.4}$$

where $\gamma > 0$. This characterization of solution of the generalized split feasibility problem with multiple outputs led to the following iterative algorithms considered in Reich et al [24]; given $x_0, y_0 \in C$, let $\{x_n\}$ and $\{y_n\}$ be two sequences generated by the methods

$$x_{n+1} = P_C \left[x_n - \gamma_n \sum_{i=1}^{N} T_i^* (I - P_{Q_i}) T_i x_n \right] \tag{1.5}$$

$$y_{n+1} = \alpha_n f(y_n) + (1 - \alpha_n) P_C \left[y_n - \gamma_n \sum_{i=1}^{N} T_i^* (I - P_{Q_i}) T_i y_n \right]. \tag{1.6}$$

where $f : C \to C$ is a strict contraction of H_1 into itself, $\{\gamma_n\} \subset (0, +\infty)$ and $\{\alpha_n\} \subset (0, 1)$. Weak and strong convergence of (1.5) and (1.6) were both established.
Our Contribution: In this work, we consider the general split feasibility problem with multiple outputs in 2-uniformly convex and uniformly smooth Banach spaces. We apply our main result to approximate solution of some Fredholm integral equations of the first kind. We also give numerical examples to illustrate how our algorithms works in Banach spaces. Furthermore, our result, extends, unifies and compliments some existing results in the literature.

2. Preliminaries

In this section, we present some definitions and lemmas which we shall use in the proof of our main theorem. Throughout this paper, E is real normed space with its topological dual E^*. Let S_E and B_E denote the unit sphere and the closed unit ball of E, respectively. The modulus of smoothness of E, $\rho_E : [0, +\infty) \to [0, +\infty)$ is defined by

$$\rho_E(t) := \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : x \in S_E, \|y\| = t \right\}.$$

The space E is said to be smooth if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in S_E$. The space E is also said to be uniformly smooth if the limit in (2.1) converges uniformly for all $x, y \in S_E$; and E is said to be 2-uniformly smooth, if there exists a fixed constant $c > 0$ such that $\rho_E(t) \le ct^2$. It is well known that every 2-uniformly smooth space is uniformly smooth. A real normed space E is said to be strictly convex if

$$\left\| \frac{(x + y)}{2} \right\| < 1 \text{ for all } x, y \in S_E \text{ and } x \ne y.$$

E is said to be uniformly convex if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$, where δ_E is the modulus of convexity of E defined by

$$\delta_E(\epsilon) := \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : \ x, y \in B_E, \ \|x - y\| \ge \epsilon \right\}, \tag{2.2}$$

for all $\epsilon \in (0, 2]$. The space E is said to be 2-uniformly convex if there exists $c > 0$ such that $\delta_E(\epsilon) \ge c\epsilon^2$ for all $\epsilon \in (0, 2]$. It is obvious that every 2-uniformly convex Banach space is uniformly convex. It is known that all Hilbert spaces are uniformly smooth and 2-uniformly convex. It is also known that all the Lebesgue spaces L_p are uniformly smooth for $1 < p \le +\infty$, and 2-uniformly convex whenever $1 < p \le 2$ (see [10]).

Let E be a real normed space. The normalized duality mapping of E into E^* is defined by

$$Jx := \{x^* \in E^* : \langle x^*, x \rangle = \|x^*\|^2 = \|x\|^2\},$$

for all $x \in E$. The normalized duality mapping J has the following properties (see, e.g., [10]):

- if E is reflexive and strictly convex with the strictly convex dual space E^*, then J is single valued, one-to-one and onto mapping. In this case, we can define the single-valued mapping $J^{-1} : E^* \to E$ and we have $J^{-1} = J^*$, where J^* is the normalized duality mapping on E^*;
- if E is uniformly smooth, then J is norm-to-norm uniformly continuous on each bounded subset of E.

Definition 2.1. Let E be a smooth real Banach space with its dual E^*. The functional $\phi : E \times E \to \mathbb{R}$ defined by

$$\phi(x, y) = ||x||^2 - 2 \langle x, Jy \rangle + ||y||^2, \tag{2.3}$$

for all $x, y \in E$, where J is the normalized duality map from E to E^*, was introduced by (Alber, [1]). It is easy to see that in a real Hilbert space ,H, the function $\phi(x, y)$ reduces to $||x - y||^2$ for all $x, y \in H$.

Lemma 2.2. *(Min et al. [23]) The following inequalities hold:*

$$(||x|| - ||y||)^2 \leq \phi(x, y) \leq (||x|| + ||y||)^2 \text{ for all } x, y \in E \tag{2.4}$$

and

$$\phi(x, J^{-1}(\alpha_n Jy + (1 - \alpha_n)Jz) \leq \alpha_n \phi(x, y) + (1 - \alpha_n)\phi(x, z) \text{ for all } x, y, z \in E. \tag{2.5}$$

Lemma 2.3. *(Schöpter et al. [26]) For a p-uniformly convex space, the metric and the Bregman distance has the following relations for any $x, yinE$:*

$$\tau||x - y||^p \leq \phi(x, y) \leq \langle x - y, J^p(x) - J^p(y)\rangle \tag{2.6}$$

where $\tau > 0$ is a fixed number. In particular, for $p = 2$, we have in 2-uniformly convex spaces that

$$\tau||x - y||^2 \leq \phi(x, y) \leq \langle x - y, J(x) - J(y)\rangle .$$

Definition 2.4. (Schöpter, [27]): The Bregman projection defined by

$$\Pi_C x = \text{argmin}_{y \in C} \phi(x, y), \ x \in E$$

is the unique minimizer of the problem

$$\min_{y \in C} \phi(x, y).$$

The Bregman projection can also be characterized by the variational inequality:

$$\langle z - \Pi_C x, J(x - \Pi_C x)\rangle \leq 0, \text{for all } z \in C, \tag{2.7}$$

and

$$\langle z - \Pi_C x, Jx - J(\Pi_C x)\rangle \leq 0, \text{for all } z \in C. \tag{2.8}$$

Definition 2.5. (Alber, [1]): Let $V : E \times E^* \to \mathbb{R}$ defined by

$$V(x, y) = ||x||^2 - 2\langle x, y\rangle + ||y||^2 \text{ for all } x \in E \textit{ and } y \in E^*. \tag{2.9}$$

Then,

$$V(x, y) = \phi(x, J^{-1}y) \text{ for all } x \in E \textit{ and } y \in E^*. \tag{2.10}$$

Moreover, for all $x \in E$ *and* $\bar{x}, \ \bar{y} \in E^*$, we have

$$V(x, \bar{x}) + 2\left\langle J^{-1}\bar{x} - x, \bar{y}\right\rangle \leq V(x, \bar{x} + \bar{y}). \tag{2.11}$$

Lemma 2.6 (Xu, [32])**.** *Let $x, y \in E$. If E is a 2-uniformly smooth space, then there is a $C_2 > 0$ such that*

$$||x - y||^2 \leq ||x||^2 - 2\langle y, Jx\rangle + C_2||y||^2. \tag{2.12}$$

Lemma 2.7 (Xu, [33])**.** *Let $\{s_n\}$ be a sequence of nonnegative numbers, $\{\alpha_n\}$ be a sequence in $(0, 1)$ and $\{c_n\}$ be a sequence of real numbers satisfying the following conditions:*

(i) $s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n c_n$;
(ii) $\sum_{n=0}^{+\infty} \alpha_n = +\infty$, *limsup*$_{n \to +\infty} c_n \leq 0$;*endenumerateThen lim*$_{n \to +\infty} s_n = 0$.

Lemma 2.8 (Kamimura and Takahashi, [18]). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $||x_n - y_n|| \to 0$ as $n \to +\infty$.*

Lemma 2.9 (Maingé, [20]). *Let $\{s_n\}$ be a real sequence which does not decrease at infinity in the sense that there exists a subsequence $\{s_{n_k}\}$ such that $s_{n_k} \leq s_{n_{k+1}}$ for all $k \geq 0$. Define an integer sequence $\{\tau(n)\}$, by $\tau(n) := \max\{n_0 \leq k \leq n : s_{n_k} \leq s_{n_{k+1}}\}$, where $n > n_0$. Then, $\tau(n) \to +\infty$ as $n \to +\infty$ and for all $n > n_0$ and n_0 large enough, we have $\max\{s_{\tau(n)}, s_n\} \leq s_{\tau(n)+1}$.*

3. Main result

In this section, we state and prove the main result of this paper.

Theorem 3.1. *Let E and E_i, for each i=1,2,3,..., N, be 2-uniformly convex and uniformly smooth spaces. Let C and Q_i be nonempty closed and convex subsets of E and E_i, $i = 1, 2, 3, ..., N$. Let $T_i : E \to E_i$ be bounded linear operators and $T_i^* : E_i^* \to E^*$ be the adjoint operators of T_i, for each i=1,2,3, ..., N, where E and E^* are also uniformly smooth spaces. Suppose that problem (1.2) has a nonempty solution set say, S. Let the sequences $\{z_n\}$ and $\{x_n\}$ be generated by x_1, $u \in C$ and*

$$\begin{cases} z_n = J^{-1}(Jx_n - \gamma_n \sum_{i=1}^{N} T_i^* J_i(I - P_{Q_i})T_i x_n), \\ x_{n+1} = \Pi_C J^{-1}(\alpha_n Ju + (1-\alpha_n)Jz_n), \quad n \geq 1 \text{ and } i = 1, 2, 3, ..., N, \end{cases} \tag{3.1}$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\gamma_n\} \subset (0, +\infty)$ satisfy the following conditions

(i) $\lim_{n\to+\infty} \alpha_n = 0$,
(ii) $\sum_{n=1}^{+\infty} \alpha_n = +\infty$,
(iii) $0 < \gamma \leq \gamma_n \leq a < \frac{2}{C_2 N max_{1\leq i\leq N}\{\|T_i\|^2\}}$, *i=1,2,3,..., N.*

Then the sequence $\{x_n\}$ converges strongly to an element $\bar{z}$=$\Pi_S u$.

Proof. Let $x^* \in S$ and $w_{n,i} = (I - P_{Q_i})T_i x_n$, then using (3.1) we have

$$\phi(x^*, z_n) = \phi(x^*, J^{-1}(Jx_n - \gamma_n \sum_{i=1}^{N} T_i^* J_i w_{n,i})) \tag{3.2}$$

$$= V(x^*, Jx_n - \gamma_n \sum_{i=1}^{N} T_i^* J_i w_{n,i})$$

$$= ||x^*||^2 - 2\langle x^*, Jx_n - \gamma_n \sum_{i=1}^{N} T_i^* J_i w_{n,i}\rangle$$

$$+ ||Jx_n - \gamma_n \sum_{i=1}^{N} T_i^* J_i w_{n,i}||^2 \tag{3.3}$$

$$= ||x^*||^2 - 2\,\langle x^*, Jx_n\rangle + 2\gamma_n \sum_{i=1}^{N} \langle T_i x^*, J_i w_{n,i}\rangle$$

$$+ ||Jx_n - \gamma_n \sum_{i=1}^{N} T_i^* J_i w_{n,i}||^2.$$

Using Lemma 2.6, we get

(3.4)

$$||Jx_n - \gamma_n \sum_{i=1}^{N} T_i^* J_i w_{n,i}||^2 \leq ||x_n||^2 - 2\gamma_n \sum_{i=1}^{N} \langle T_i x_n, J_i w_{n,i}\rangle + C_2 N \gamma_n^2 \sum_{i=1}^{N} ||T_i^* J_i w_{n,i}||^2.$$

Substituting (3.4) in (3.3), we have

$$\begin{aligned}
\phi(x^*, z_n) &\leq ||x^*||^2 - 2\langle x^*, Jx_n\rangle + 2\gamma_n \sum_{i=1}^{N} \langle T_i x^*, J_i w_{n,i}\rangle \\
&\quad + ||x_n||^2 - 2\gamma_n \sum_{i=1}^{N} \langle T_i x_n, J_i w_{n,i}\rangle + C_2 N \gamma_n^2 \sum_{i=1}^{N} ||T_i^* J_i w_{n,i}||^2 \\
&= \phi(x^*, x_n) + 2\gamma_n \sum_{i=1}^{N} \langle T_i x^* - T_i x_n, J_i w_{n,i}\rangle + C_2 N \gamma_n^2 \sum_{i=1}^{N} ||T_i^* J_i w_{n,i}||^2. \qquad (3.5)
\end{aligned}$$

Now, we have for each $i = 1, 2, 3, ..., N$, that

$$\begin{aligned}
\langle T_i x^* - T_i x_n, J_i w_{n,i}\rangle &= \langle T_i x^* - T_i x_n, J_i(I - P_{Q_i})T_i x_n\rangle, \\
&= \langle T_i x^* - P_{Q_i} T_i x_n, J_i(I - P_{Q_i})T_i x_n\rangle - ||(I - P_{Q_i})T_i x_n||^2.
\end{aligned}$$

Applying (2.7), for each $i = 1, 2, 3..., N$, we have

$$\langle T_i x^* - T_i x_n, J_i(I - P_{Q_i})T_i x_n\rangle \leq -||(I - P_{Q_i})T_i x_n||^2. \qquad (3.6)$$

Using (3.6) in (3.5), we get

$$\begin{aligned}
\phi(x^*, z_n) &\leq \phi(x^*, x_n) - 2\gamma_n \sum_{i=1}^{N} ||(I - P_{Q_i})T_i x_n||^2 + C_2 N \gamma_n^2 \sum_{i=1}^{N} ||T_i^* J_i w_{n,i}||^2 \\
&= \phi(x^*, x_n) - 2\gamma_n \sum_{i=1}^{N} ||w_{n,i}||^2 + C_2 N \gamma_n^2 \max_{1\leq i\leq N}\{||T_i||^2\} \sum_{i=1}^{N} ||w_{n,i}||^2 \\
&= \phi(x^*, x_n) - \gamma_n \left(2 - C_2 N \gamma_n \max_{1\leq i\leq N}\{||T_i||^2\}\right) \sum_{i=1}^{N} ||w_{n,i}||^2. \qquad (3.7)
\end{aligned}$$

From condition (iii) we have

$$\left(2 - C_2 N \gamma_n \max_{1\leq i\leq N}\{||T_i||^2\}\right) \sum_{i=1}^{N} ||w_{n,i}||^2 \geq 0, \text{ for all } n \geq 1,$$

so that

$$\phi(x^*, z_n) \leq \phi(x^*, x_n), \text{ for all } n \geq 1. \qquad (3.8)$$

Also, using (3.1), we have

$$\begin{aligned}
\phi(x^*, x_{n+1}) &= \phi(x^*, \Pi_C J^{-1}(\alpha_n Ju + (1 - \alpha_n)Jz_n)) \\
&\leq \phi(x^*, J^{-1}(\alpha_n Ju + (1 - \alpha_n)Jz_n)) \\
&\leq \alpha_n \phi(x^*, u) + (1 - \alpha_n)\phi(x^*, z_n). \qquad (3.9)
\end{aligned}$$

Using (3.8) in (3.9), we get

$$\begin{aligned}\phi(x^*, x_{n+1}) &\leq \alpha_n \phi(x^*, u) + (1-\alpha_n)\phi(x^*, x_n)\\ &\leq \max\{\phi(x^*, u), \phi(x^*, x_n)\}\\ &\quad \vdots\\ &\leq \max\{\phi(x^*, u), \phi(x^*, x_1)\}.\end{aligned}$$

This implies, $\{\phi(x^*, x_n)\}$ is bounded. Hence, $\{x_n\}$ and $\{z_n\}$ are bounded.

Let $\bar{z} = \Pi_S u$. Now, we divide the proof into two cases:

Case 1: Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\phi(\bar{z}, x_n)\}$ is non-increasing. Then $\{\phi(\bar{z}, x_n)\}$ converges and $\phi(\bar{z}, x_n) - \phi(\bar{z}, x_{n+1}) \to 0$ as $n \to +\infty$. Then from (3.7), we obtain that

$$\gamma_n[2 - C_2 N\gamma_n \max_{1\leq i\leq N}\{||T_i||^2\}] \sum_{i=1}^{N} ||(I - P_{Q_i})T_i x_n||^2 \leq \phi(\bar{z}, x_n) - \phi(\bar{z}, z_n), \tag{3.10}$$

from (3.9), we have

$$\begin{aligned}\gamma_n[2 - C_2 N\gamma_n \max_{1\leq i\leq N}\{||T_i||^2\}] &\sum_{i=1}^{N} ||(I - P_{Q_i})T_i x_n||^2\\ &\leq \phi(\bar{z}, x_n) - \phi(\bar{z}, x_{n+1}) + \alpha_n[\phi(\bar{z}, u) - \phi(\bar{z}, z_n)].\end{aligned} \tag{3.11}$$

By condition (iii) we have

$$0 < \gamma \sum_{i=1}^{N} ||(I - P_{Q_i})T_i x_n||^2 \leq \phi(\bar{z}, x_n) - \phi(\bar{z}, x_{n+1}) + \alpha_n[\phi(\bar{z}, u) - \phi(\bar{z}, z_n)].$$

Using condition (i), we see that

$$\lim_{n\to+\infty} ||(I - P_{Q_i})T_i x_n|| = 0, \ \textit{for each } i = 1, 2, 3, ..., N. \tag{3.12}$$

Also, from (3.1), we have

$$\begin{aligned}0 \leq ||Jz_n - Jx_n||^2 &= ||J(J^{-1}(Jx_n - \gamma_n \sum_{i=1}^{N} T_i^* J_i(I - P_{Q_i})T_i x_n)) - Jx_n||^2\\ &= \gamma_n^2 ||\sum_{i=1}^{N} T_i^* J_i(I - P_{Q_i})T_i x_n)||^2\\ &\leq N\gamma_n^2 \sum_{i=1}^{N} ||T_i^*||^2 ||(I - P_{Q_i})T_i x_n)||^2,\end{aligned}$$

so that

$$0 \leq ||Jz_n - Jx_n||^2 \leq N\gamma_n^2 \sum_{i=1}^{N} ||T_i||^2 ||(I - P_{Q_i})T_i x_n)||^2. \tag{3.13}$$

Again, using condition (iii) in (3.13), we have

$$0 \leq ||Jz_n - Jx_n||^2 \leq \frac{2}{C_2^2 N(\max_{1\leq i\leq N}\{||T_i||^2\})^2} \sum_{i=1}^{N} ||T_i||^2 ||(I - P_{Q_i})T_i x_n)||^2,$$

by (3.12), we see that

$$0 \leq ||Jz_n - Jx_n||^2 \leq \frac{2}{C_2^2 N(\max_{1\leq i\leq N}\{||T_i||^2\})^2} \sum_{i=1}^{N} ||T_i||^2 ||(I - P_{Q_i})T_i x_n)||^2 \to 0 \quad as\ n \to +\infty.$$

Hence,

$$\lim_{n\to+\infty} ||Jz_n - Jx_n|| = 0. \tag{3.14}$$

Since J^{-1} is norm-to-norm uniformly continuous on bounded subsets of E^*, we obtain

$$\lim_{n\to+\infty} ||z_n - x_n|| = 0. \tag{3.15}$$

Moreover, from (3.1) we also have

$$\phi(z_n, x_{n+1}) \leq \alpha_n \phi(z_n, u) + (1 - \alpha_n)\phi(z_n, z_n), \tag{3.16}$$

which shows that

$$\lim_{n\to+\infty} \phi(z_n, x_{n+1}) = 0,$$

by lemma 2.8, we have

$$\lim_{n\to+\infty} ||x_{n+1} - z_n|| = 0.$$

Hence,

$$||x_{n+1} - x_n|| \to 0\ as\ n \to +\infty. \tag{3.17}$$

Now, since $\{x_n\}$ is bounded, let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup b$.

Applying Lemma 2.3, we have

$$\begin{aligned} \phi(b, \Pi_C b) &\leq \langle b - \Pi_C b, Jb - J\Pi_C b\rangle \\ &\leq \langle b - x_{n_k}, Jb - J\Pi_C b\rangle \\ &\quad + \langle x_{n_k} - \Pi_C b, Jb - J\Pi_C b\rangle . \end{aligned} \tag{3.18}$$

Now, we see from (3.1) that $\{x_n\} \subset C$ so is $\{x_{n_k}\}$, therefore applying (2.8) in (3.18), we get

$$\phi(b, \Pi_C b) \leq \langle b - x_{n_k}, Jb - J\Pi_C b\rangle .$$

So that we get $\phi(b, \Pi_C b) = 0$ as $k \to +\infty$. Consequently, $b = \Pi_C b$ and so $b \in C$.

By (3.12) we have

$$||(I - P_{Q_i})T_i x_{n_k}|| \to 0\ as\ k \to +\infty. \tag{3.19}$$

Now for each $i = 1, 2, 3, \ldots, N$, we have

$$||(I - P_{Q_i})T_i b||^2 = \langle T_i b - T_i x_{n_k}, J_i(T_i b - P_{Q_i} T_i b)\rangle \tag{3.20}$$

$$+\langle T_i x_{n_k} - P_{Q_i} T_i x_{n_k}, J_i(T_i b - P_{Q_i} T_i b)\rangle$$
$$+\langle P_{Q_i} T_i x_{n_k} - P_{Q_i} T_i b, J_i(T_i b - P_{Q_i} T_i b)\rangle$$

Since T_i is linear and bounded and $x_{n_k} \rightharpoonup b$, so by the continuity of T_i, for each $i = 1, 2, 3, ..., N$, we have $T_i x_{n_k} \rightharpoonup T_i b$ as $k \to +\infty$. Also, by using (3.19), (2.7) and letting $k \to +\infty$ in (3.20), we have

$$||T_i b - P_{Q_i} T_i b|| = 0 \ for\ each\ i = 1, 2, 3, ..., N.$$

Therefore, $T_i b = P_{Q_i} T_i b$, that is $T_i b \in Q_i$ for each $i = 1, 2, 3, ..., N$. Hence we have $b \in S$.

Let $t_n = J^{-1}(\alpha_n Ju + (1-\alpha_n)Jz_n), n \geq 1$. Then,

$$\phi(z_n, t_n) = \phi(z_n, J^{-1}(\alpha_n Ju + (1-\alpha_n)Jz_n))$$

again by (2.5), we have

$$\phi(z_n, t_n) = \alpha_n \phi(z_n, u) \to 0 \ as\ n \to +\infty.$$

Hence by Lemma 2.8, we have

$$||z_n - t_n|| \to 0 \ as\ n \to +\infty. \tag{3.21}$$

Consequently, $||x_n - t_n|| \to 0$ as $n \to +\infty$.

Without lost of generality, we suppose that $t_{n_k} \rightharpoonup b$ as $k \to +\infty$(otherwise we go down to another subsequence). Now by (2.8) and the fact that $\bar{z} = \Pi_S u$, we have

(3.22)
$$\limsup_{n\to+\infty} \langle t_n - \bar{z}, Ju - J\bar{z}\rangle = \lim_{j\to+\infty} \langle t_{n_k} - \bar{z}, Ju - J\bar{z}\rangle = \langle b - \bar{z}, Ju - J\bar{z}\rangle \leq 0.$$

Now, from (3.1), we have

$$\begin{aligned}\phi(\bar{z}, x_{n+1}) &= \phi(\bar{z}, \Pi_C J^{-1}(\alpha_n Ju + (1-\alpha_n)Jz_n)) \\ &\leq \phi(\bar{z}, J^{-1}(\alpha_n Ju + (1-\alpha_n)Jz_n)) \\ &= V(\bar{z}, \alpha_n Ju + (1-\alpha_n)Jz_n).\end{aligned}$$

So, from (2.11), we have

$$\begin{aligned}V(\bar{z}, \alpha_n Ju + (1-\alpha_n)Jz_n) &\leq V(\bar{z}, \alpha_n Ju + (1-\alpha_n)Jz_n - \alpha_n(Ju - J\bar{z})) \\ &\quad - 2\left\langle J^{-1}(\alpha_n Ju + (1-\alpha_n)Jz_n) - \bar{z}, -\alpha_n(Ju - J\bar{z})\right\rangle \\ &\leq \phi(\bar{z}, J^{-1}(\alpha_n Ju + (1-\alpha_n)Jz_n) - \alpha_n(Ju - J\bar{z})) \\ &\quad + \alpha_n \left\langle J^{-1}(\alpha_n Ju + (1-\alpha_n)Jz_n) - \bar{z}, (Ju - J\bar{z})\right\rangle\end{aligned}$$

(2.5) and (3.1) imply

$$\phi(\bar{z}, x_{n+1}) \leq (1-\alpha_n)\phi(\bar{z}, z_n) + 2\alpha_n \langle t_n - \bar{z}, Ju - J\bar{z}\rangle \tag{3.23}$$

which by (3.8) implies

$$\phi(\bar{z}, x_{n+1}) \leq (1-\alpha_n)\phi(\bar{z}, x_n) + 2\alpha_n \langle t_n - \bar{z}, Ju - J\bar{z}\rangle. \tag{3.24}$$

Applying Lemma 2.7 we see that $\lim_{n\to+\infty} \phi(\bar{z}, x_n) = 0$. Thus, $x_n \to \bar{z}$ as $n \to +\infty$.

Case 2: Suppose $\{\phi(\bar{z}, x_n)\}$ is not a decreasing sequence and set $A_n = \phi(\bar{z}, x_n)$ for all $n \geq 1$. Now using lemma 2.9, we define an integer sequence $\{\tau(n)\}$ for all $n \geq n_0$ (for some n_0 large enough) by
$\tau(n) := \max\{k \leq n : A_k \leq A_{k+1}\}$. Clearly $\tau(n)$ is non-decreasing sequence such that $\tau(n) \to +\infty$ as $n \to +\infty$ and

$$A_{\tau(n)} \leq A_{\tau(n)+1}, \ for\ all\ n \geq n_0. \tag{3.25}$$

From (3.25), we have $0 \leq A_{\tau(n)+1} - A_{\tau(n)}$ for all $n \geq n_0$.

Also, from (3.24), we deduce that

$$0 \leq A_{\tau(n)+1} - A_{\tau(n)} \leq \alpha_{\tau(n)}(C_{\tau(n)} - A_{\tau(n)}), \tag{3.26}$$

where $C_{\tau(n)} = \langle t_{\tau(n)} - \bar{z}, Ju - J\bar{z} \rangle \leq 0$, for all $\tau(n)$. Since $C_{\tau(n)}$ and $A_{\tau(n)}$ are bounded for all $\tau(n)$ and $\alpha_{\tau(n)} \to 0$ as $\tau(n) \to +\infty$, then

$$\lim_{\tau(n) \to +\infty} A_{\tau(n)+1} - A_{\tau(n)} = 0. \tag{3.27}$$

Using similar arguments as in case 1, we see from (3.11) with $L = \max_{1 \leq i \leq N}\{||T_i||^2\}$ that

$$\gamma_{\tau(n)}[2 - C_2 N \gamma_{\tau(n)} L] \sum_{i=1}^{N} ||(I - P_{Q_i})T_i x_{\tau(n)}||^2 \leq \phi(\bar{z}, x_{\tau(n)}) - \phi(\bar{z}, z_{\tau(n)}), \tag{3.28}$$

using (3.9), we have

$$\begin{aligned}\gamma_{\tau(n)}[2 - C_2 N \gamma_{\tau(n)} L] \sum_{i=1}^{N} ||(I - P_{Q_i})T_i x_{\tau(n)}||^2 &\leq \phi(\bar{z}, x_{\tau(n)}) - \phi(\bar{z}, x_{\tau(n+1)}) \\ &\quad + \alpha_{\tau(n)} \left[\phi(\bar{z}, u) - \phi(\bar{z}, z_{\tau(n)})\right].\end{aligned}$$

Applying condition (iii), we have

$$\begin{aligned}0 &< \gamma \sum_{i=1}^{N} ||(I - P_{Q_i})T_i x_{\tau(n)}||^2 \\ &\leq \phi(\bar{z}, x_{\tau(n)}) - \phi(\bar{z}, x_{\tau(n+1)}) + \alpha_{\tau(n)} \left[\phi(\bar{z}, u) - \phi(\bar{z}, z_{\tau(n)})\right]. \end{aligned} \tag{3.29}$$

Using condition (i) and (3.27) in (3.29), we have for each $i = 1, 2, 3, \ldots, N$ that

$$\lim_{n \to +\infty} ||(I - P_{Q_i})T_i x_{\tau(n)}|| = 0. \tag{3.30}$$

Since $z_{\tau(n)} = J^{-1}(Jx_{\tau(n)} - \gamma_{\tau(n)} \sum_{i=1}^{N} T_i^* J_i (I - P_{Q_i})T_i x_{\tau(n)})$. Then, we have

$$\begin{aligned}0 &\leq ||Jz_{\tau(n)} - Jx_{\tau(n)}||^2 \\ &\leq ||J(J^{-1}x_{\tau(n)} - \gamma_{\tau(n)} \sum_{i=1}^{N} T_i^* J_i (I - P_{Q_i})T_i x_{\tau(n)} - Jx_{\tau(n)}||^2 \\ &\leq N\gamma_{\tau(n)}^2 \sum_{i=1}^{N} ||T_i^* J_i (I - P_{Q_i})T_i x_{\tau(n)}||^2 \\ &\leq N\gamma_{\tau(n)}^2 \sum_{i=1}^{N} ||T_i||^2 ||(I - P_{Q_i})T_i x_{\tau(n)}||^2. \end{aligned} \tag{3.31}$$

Now, applying (3.30) and condition (iii) in (3.31), we get

$$0 \leq ||Jz_{\tau(n)} - Jx_{\tau(n)}||^2 \leq N\left(\frac{1}{C_2NL}\right)^2 \sum_{i=1}^{N} ||T_i||^2||(I - P_{Q_i})T_ix_{\tau(n)}||^2.$$

Hence,

$$\lim_{n\to+\infty} ||Jz_{\tau(n)} - Jx_{\tau(n)}|| = 0. \tag{3.32}$$

Since, J is norm-to-norm uniformly continuous on bounded subsets of E, we have

$$\lim_{n\to+\infty} ||z_{\tau(n)} - x_{\tau(n)}|| = 0. \tag{3.33}$$

Now, by (3.16) we see that

(3.34)

$$\phi(z_{\tau(n)}, x_{\tau(n)+1}) \leq \alpha_{\tau(n)}\phi(z_{\tau(n)}, u) + (1 - \alpha_{\tau(n)})\phi\left(z_{\tau(n)}, z_{\tau(n)}\right) \to 0 \text{ as } n \to +\infty.$$

By Lemma(2.8), we have

$$\lim_{n\to+\infty} ||z_{\tau(n)} - x_{\tau(n)+1}|| = 0, \tag{3.35}$$

by (3.33) and (3.35), we get

$$\lim_{n\to+\infty} ||x_{\tau(n)+1} - x_{\tau(n)}|| = 0. \tag{3.36}$$

Since $\{x_{\tau(n)}\}$ is bounded, there exists a subsequence $\{x_{\tau(n_k)}\}$ of $\{x_{\tau(n)}\}$ such that $x_{\tau(n_k)} \rightharpoonup b$. Following the same method as in case 1, we get $\phi(b, \Pi_C b) = 0$, so that $b = \Pi_C b$. Hence $b \in C$.

By (3.30) for each $i = 1, 2, 3, ..., N$, we have

$$||(I - P_{Q_i})T_ix_{\tau(n_k)}|| \to 0 \; as \; k \to +\infty. \tag{3.37}$$

Also,

$$\begin{aligned} ||(I - P_{Q_i})T_ib||^2 &= \langle T_ib - P_{Q_i}T_ib, J_i(T_ib - P_{Q_i}T_i)\rangle \\ &= \left\langle T_ib - T_ix_{\tau(n_k)}, J_i(T_i - P_{Q_i}T_ib)\right\rangle \\ &\quad + \left\langle T_ix_{\tau(n_k)} - P_{Q_i}T_ix_{\tau(n_k)}, J_i(T_ib - P_{Q_i}T_ib)\right\rangle \\ &\quad + \left\langle P_{Q_i}T_ix_{\tau(n_k)} - P_{Q_i}T_ib, J_i(T_ib - P_{Q_i}T_ib)\right\rangle, \end{aligned} \tag{3.38}$$

for each $i = 1, 2, 3, ..., N$. Since T_i is linear and bounded and $x_{\tau(n_k)} \rightharpoonup b$, so by the continuity of T_i, for each $i = 1, 2, 3, ..., N$, we have $T_ix_{\tau(n_k)} \rightharpoonup T_ib$ as $k \to +\infty$. Now, by letting $n \to +\infty$ in (3.38) and using (3.30) and (2.7), we have

$$||T_ib - P_{Q_i}T_ib|| = 0, \; for \; each \; i = 1, 2, 3, ..., N.$$

Therefore, $T_ib = P_{Q_i}T_ib$ and so $T_ib \in Q_i$ for each $i = 1, 2, 3, ..., N$. Hence we have $b \in S$.

Let $t_{\tau(n)} = J^{-1}(\alpha_{\tau(n)}Ju + (1 - \alpha_{\tau(n)})Jz_{\tau(n)}), n \geq n_0$. Then,

$$\phi(z_{\tau(n)}, t_{\tau(n)}) = \phi(z_{\tau(n)}, J^{-1}(\alpha_{\tau(n)}Ju + (1 - \alpha_{\tau(n)})Jz_{\tau(n)}))$$

again by (2.5), we have

$$\phi(z_{\tau(n)}, t_{\tau(n)}) = \alpha_{\tau(n)}\phi(z_{\tau(n)}, u) \to 0 \; as \; n \to +\infty.$$

Hence by Lemma(2.8), we have

$$||z_{\tau(n)} - t_{\tau(n)}|| \to 0 \ as \ n \to +\infty. \tag{3.39}$$

Hence (3.33) implies that $||x_{\tau(n)} - t_{\tau(n)}|| \to 0$ as $n \to +\infty$.
Following the same argument as in case 1, we get

$$\limsup_{n\to+\infty} \left\langle t_{\tau(n)} - \bar{z}, Ju - J\bar{z}\right\rangle = \lim_{j\to+\infty} \left\langle t_{\tau n_{k_j}} - \bar{z}, Ju - J\bar{z}\right\rangle = \langle b - \bar{z}, Ju - J\bar{z}\rangle \leq 0.$$

Also, using (3.1), we have

$$\begin{aligned}\phi(\bar{z}, x_{\tau(n)+1}) &= \phi(\bar{z}, \Pi_C J^{-1}(\alpha_{\tau(n)} Ju + (1-\alpha_{\tau(n)})Jz_{\tau(n)}))\\ &\leq \phi(\bar{z}, J^{-1}(\alpha_{\tau(n)} Ju + (1-\alpha_{\tau(n)})Jz_{\tau(n)}))\\ &= V(\bar{z}, \alpha_{\tau(n)} Ju + (1-\alpha_{\tau(n)})Jz_{\tau(n)}),\end{aligned}$$

so by (2.11), we have

$$\begin{aligned}&V(\bar{z}, \alpha_{\tau(n)} Ju + (1-\alpha_{\tau(n)})Jz_{\tau(n)})\\ &\quad\leq V(\bar{z}, \alpha_{\tau(n)} Ju + (1-\alpha_{\tau(n)})Jz_{\tau(n)} - \alpha_{\tau(n)}(Ju - J\bar{z}))\\ &\qquad - 2\left\langle J^{-1}(\alpha_{\tau(n)} Ju + (1-\alpha_{\tau(n)})Jz_{\tau(n)}) - \bar{z}, -\alpha_{\tau(n)}(Ju - J\bar{z})\right\rangle\\ &\quad\leq \phi(\bar{z}, J^{-1}(\alpha_{\tau(n)} Ju + (1-\alpha_{\tau(n)})Jz_{\tau(n)} - \alpha_{\tau(n)}(Ju - J\bar{z})))\\ &\qquad + \alpha_{\tau(n)}\left\langle J^{-1}(\alpha_{\tau(n)} Ju + (1-\alpha_{\tau(n)})Jz_{\tau(n)}) - \bar{z}, (Ju - J\bar{z})\right\rangle\end{aligned}$$

by (2.5) and (3.1), we have

$$\phi(\bar{z}, x_{\tau(n)+1}) \leq (1-\alpha_{\tau(n)})\phi(\bar{z}, z_{\tau(n)}) + 2\alpha_{\tau(n)}\left\langle t_{\tau(n)} - \bar{z}, Ju - J\bar{z}\right\rangle, \tag{3.40}$$

which by (3.8) implies that

$$\phi(\bar{z}, x_{\tau(n)+1}) \leq (1-\alpha_{\tau(n)})\phi(\bar{z}, x_{\tau(n)}) + 2\alpha_n\left\langle t_{\tau(n)} - \bar{z}, Ju - J\bar{z}\right\rangle. \tag{3.41}$$

Now, (3.41) is equivalent to

$$A_{\tau(n)+1} \leq (1-\alpha_{\tau(n)})A_{\tau(n)} + \alpha_{\tau(n)}C_{\tau(n)}, \tag{3.42}$$

with $A_{\tau(n)+1} = \phi(\bar{z}, x_{\tau(n)+1})$, $A_{\tau(n)} = \phi(\bar{z}, x_{\tau(n)})$ and $C_{\tau(n)} = \left\langle t_{\tau(n)} - \bar{z}, Ju - J\bar{z}\right\rangle$.
Now, applying Lemma 2.7 in (3.42), then $\lim_{n\to+\infty} A_{\tau(n)} = 0$. Thus, the sequence $\{x_n\}$ converges strongly to $\bar{z}$ as $n \to +\infty$, where $\bar{z} = \Pi_S u$. Hence the proof is completed. □

We now obtain the following Corollaries in Hilbert space and for a single operator, respectively.

Corollary 3.2. *Let H and H_i, for each i=1,2,3,..., N, be two Hilbert spaces. Let C and Q_i, for each i=1,2,3,..., N, be nonempty, closed and convex subsets of H and H_i, i=1, 2, 3,..., N, respectively. Let $T_i : H \to H_i$ be bounded linear operators and $T_i^* : H_i \to H$ be the adjoint operators of T_i, for each i=1,2,3, ..., N. Suppose that problem (1.2) has a nonempty solution set S. Let the sequences $\{z_n\}$ and $\{x_n\}$ be generated by x_1, $u \in C$ and*

$$\begin{cases} z_n = x_n - \gamma_n \sum_{i=1}^N T_i^*(I - P_{Q_i})T_i x_n,\\ x_{n+1} = P_C(\alpha_n u + (1-\alpha_n)z_n), \quad n \geq 1 \text{ and } i = 1, 2, ..., N,\end{cases} \tag{3.43}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\gamma_n\} \subset (0, +\infty)$ satisfy the following conditions:

(i) $\lim_{n\to+\infty} \alpha_n = 0$,
(ii) $\sum_{n=1}^{+\infty} \alpha_n = +\infty$,
(iii) $0 < \gamma \leq \gamma_n \leq a < \frac{1}{N max_{1\leq i\leq N}\{\|T_i\|^2\}}$, $i = 1, 2, 3, ..., N$.

Then the sequence $\{x_n\}$ *converges strongly to an element* $\bar{z}$=$\Pi_S u$.

Corollary 3.3. *Let* E_1 *and* E_2*, be 2-uniformly convex spaces. Let* C *and* Q*, be nonempty, closed and convex subsets of* E_1 *and* E_2*, respectively. Let* $T : E_1 \to E_2$ *be bounded linear operators and* $T^* : E_2^* \to E_1^*$ *be the adjoint operators of* T*, for each. Suppose that problem (1.2) has a nonempty solution set S. Let the sequences* $\{z_n\}$ *and* $\{x_n\}$ *be generated by* x_1, $u \in C$ *and*

$$\begin{cases} z_n = J^{-1}(Jx_n - \gamma_n T^* J(I - P_Q)Tx_n), \\ x_{n+1} = \Pi_C J^{-1}(\alpha_n Ju + (1 - \alpha_n)Jz_n), \quad n \geq 1 \text{ and } i = 1, 2, ..., N, \end{cases} \tag{3.44}$$

where $\{\alpha_n\} \subset (0, 1)$ *and* $\{\gamma_n\} \subset (0, +\infty)$ *satisfy the following conditions:*
(i) $\lim_{n\to+\infty} \alpha_n = 0$,
(ii) $\sum_{n=1}^{+\infty} \alpha_n = +\infty$,
(iii) $0 < \gamma \leq \gamma_n \leq a < \frac{1}{C_2 N max_{1\leq i\leq N}\{\|T_i\|^2\}}$, *i=1,2,3,..., N. Then the sequence* $\{x_n\}$ *converges strongly to an element* $\bar{\bar{z}}$=$\Pi_S u$.

4. Numerical experiment

In this section we give some examples in infinite dimensional spaces to illustrate how our algorithm works. Numerical experiments were carried out on MATLAB R2015a version. All programs were run on a 64-bit OS PC with Intel(R) Core(TM) i7-3540M CPU @ 1.00GHz 1.19 GHz and 3GB RAM. All figures were plotted using the log log plot command.

Example 4.1. Let $H = L_2([0, 1])$, with norm and inner product defined as

$$\|x\|_2 = \Big(\int_0^1 |x(t)|^2\, dt\Big)^{\frac{1}{2}} \quad \text{and} \quad \langle x, y\rangle = \int_0^1 x(t)y(t)\, dt, \quad \text{respectively.}$$

Consider the following Fredholm integral equation of the first kind,

$$\int_0^1 k(s,t)x(t)dt = f(s),\ 0 \leq s \leq 1, \tag{4.1}$$

where $k : [0, 1]\times[0, 1] \to \mathbb{R}$ is the kernel and f is a continuous function. Approximating solution of (4.1) is equivalent to approximating solution of the split feasibility problem with multiple output with $C = H$, $T_i = I$ and

$$Q_i = \{x \in H : \langle a_i, x\rangle = b_i\},$$

where $a_i(t) = k(s_i, t)$, $b_i = f(s_i)$ and $0 = s_1 < s_2 < s_3 < \cdots < s_M = 1$ (see, e.g., [19]). Now consider the following equation:

$$\int_0^1 e^{3s-4t}x(t)dt = (e - 1)e^{3s},\ 0 \leq s \leq 1, \tag{4.2}$$

which, under some appropiate conditions, has a solution, see, e.g., [2, 31]. To approximate solution of (4.2), we set $C = H$, $T_i = I$ and

$$Q_i = \{x \in H : \langle a_i, x\rangle = b_i\},$$

with $a_i(t) = e^{3s_i - 4t}$ and $b_i = (e-1)e^{3s_i}$, where $s_i = \frac{i-1}{M-1}$, $i = 1, 2, 3, \ldots, M$. The results of the experiment are displayed in Table 1 and Figures 1, 2 and 3.

Example 4.2. In this example, just as in example (4.1) above, we solve the following Fredholm integral equation

$$\int_0^1 stx(t)dt = 2s, \ 0 \le s \le 1. \tag{4.3}$$

Thus, $C = H$, $T_i = I$ and

$$Q_i = \{x \in H : \langle a_i, x \rangle = b_i\},$$

with $a_i(t) = s_i t$ and $b_i = 2s_i$, where $s_i = \frac{i-1}{M-1}$, $i = 1, 2, 3, \ldots, M$. The results of the experiment are displayed in Table 2 and Figures 4, 5 and 6.

Example 4.3. Let $H = L_2([0,1])$, with norm and inner product defined as

$$\|x\|_2 = \Big(\int_0^1 |x(t)|^2 \, dt \Big)^{\frac{1}{2}} \quad \text{and} \quad \langle x, y \rangle = \int_0^1 x(t)y(t) \, dt, \quad \text{respectively.}$$

Let

$$C = \left\{ x \in H : \langle a, x \rangle \le \frac{4}{3} \right\} \quad \text{and} \quad Q_i = \{x \in H : \langle b_i, x \rangle = c_i\},$$

and where

$$b_i(t) = e^{\frac{3(i-1)}{M-1} - 4t}, \qquad c_i = (e-1)e^{\frac{3(i-1)}{M-1}}, \text{ for all } t \in [0,1],$$

and

$$T_i x(t) = \frac{x(t)}{e^{it}}, \ i = 1, 2, 3, \ldots, M.$$

Clearly the solution set of (1.2) is not empty as it contains 0.

TABLE 1. Computational Results for Example (4.1).

Tolerance (TOL)	M	γ	No. of Iter.	Time(secs)
TOL $= 10^{-4}$	1000	$(n+10)^{-4}$	231	4.9658
		$(n+10)^{-4}$	1051	21.8111
TOL $= 10^{-4}$	2500	$(n+10)^{-4}$	355	18.6151
		$(n+10)^{-4}$	1407	73.4965
TOL $= 10^{-4}$	5000	$(n+10)^{-4}$	478	49.4227
		$(n+10)^{-4}$	1718	180.4342

Remark 4.4. From the values displayed in Tables 1, 2 and 3, it is clear that the convergence of Algorithm (3.1) depends on the choice of γ_n. The smaller the value of γ_n, the faster the algorithm converges.

Table 2. Computational Results for Example (4.2).

Tolerance (TOL)	M	γ	No. of Iter.	Time(secs)
TOL $= 10^{-4}$	1000	$(n+10)^{-4}$	193	4.0778
		$(n+10)^{-4}$	874	20.1912
TOL $= 10^{-4}$	2500	$(n+10)^{-4}$	297	15.6564
		$(n+10)^{-4}$	1171	61.9406
TOL $= 10^{-4}$	5000	$(n+10)^{-4}$	400	42.4260
		$(n+10)^{-4}$	1428	148.3921

Table 3. Computational Results for Example (4.3).

Tolerance (TOL)	M	γ	No. of Iter.	Time(secs)
TOL $= 10^{-4}$	500	$(n+10)^{-4}$	10	0.0.1711
		$(n+10)^{-4}$	617	7.1226
TOL $= 10^{-4}$	1000	$(n+10)^{-4}$	10	0.1964
		$(n+10)^{-4}$	833	31.6163

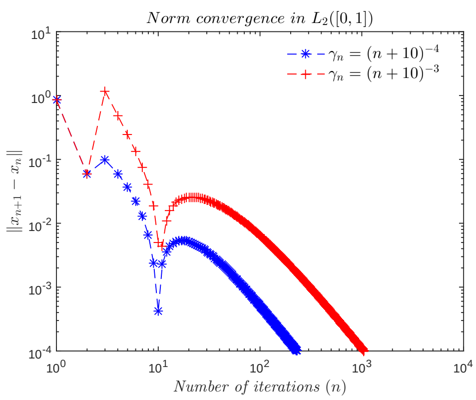

Figure 1. Example (4.1) with $M = 1000$.

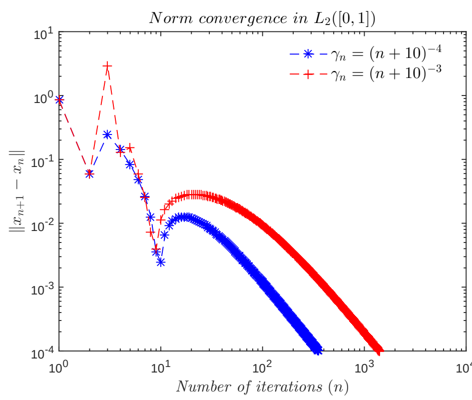

Figure 2. Example (4.1) with $M = 2500$.

5. Conclusion and further research

In this work, we have proved the strong convergence of Halpern-type iterative algorithm to a solution of split feasibility problem with multiple outputs in 2-uniformly convex and uniformly smooth Banach spaces. It is well known that these spaces do not cover L_p for $p > 2$. Thus, it would be desirable to obtain the results of this paper in p-uniformly convex Banach spaces for $p > 2$. We also gave an application of our main result to approximating solutions of Fredholm integral equations

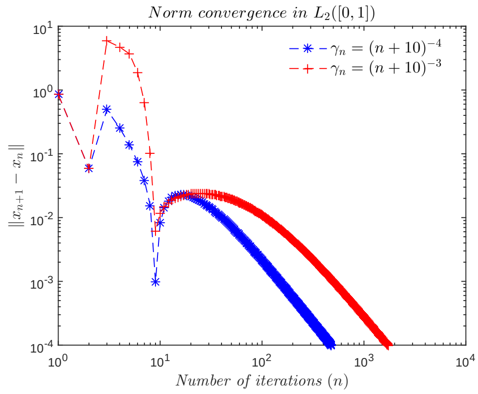

FIGURE 3. Example (4.1) with $M = 5000$.

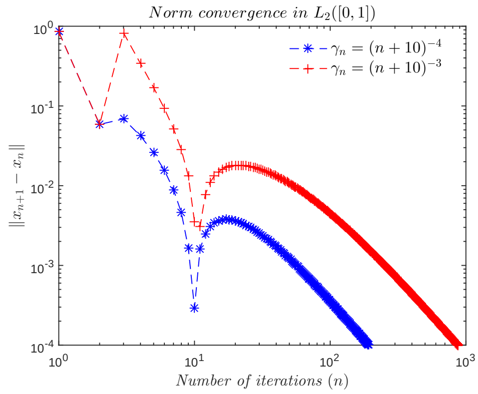

FIGURE 4. Example (4.2) with $M = 1000$.

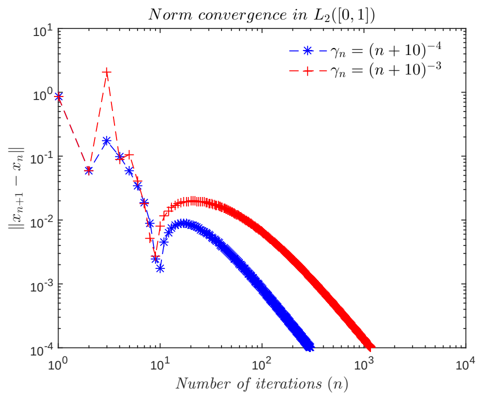

FIGURE 5. Example (4.2) with $M = 2500$.

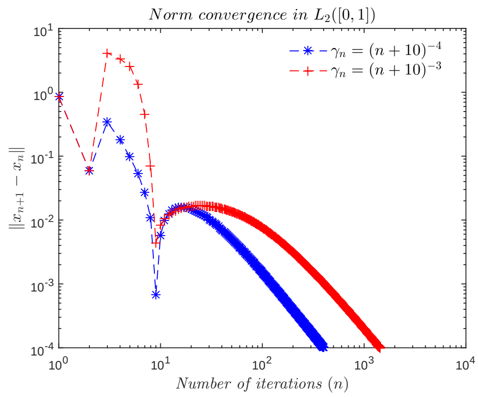

FIGURE 6. Example (4.2) with $M = 5000$.

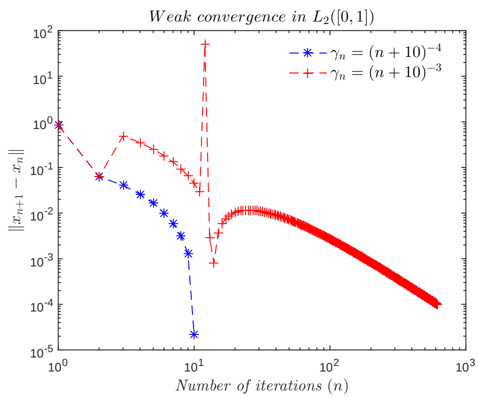

FIGURE 7. Example (4.3) with $M = 500$.

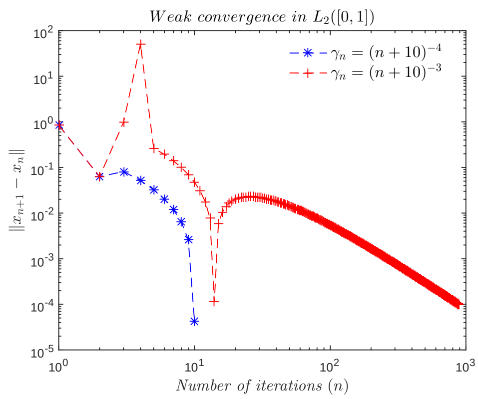

FIGURE 8. Example (4.3) with $M = 1000$.

of first kind in $L_2([0,1])$ space. Finally, our results generalize and complement some existing results in the literature.

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