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A MODIFIED TSENG METHOD FOR SOLUTIONS OF A CLASS OF SPLIT VARIATIONAL INEQUALITY PROBLEM IN BANACH SPACE

BASHIR ALI AND A. M. HAMZA

Abstract. In this work, we study a modified Tseng's extragradient method in the setting of uniformly smooth which is also 2-uniformly convex real Banach space and 2-uniformly smooth real Banach space and prove its strong convergence to a solution of a variational inequality problem for a monotone $l-$Lipschitz continuous map whose image under a bounded linear operator is a fixed point of nonexpansive maps. The result in this paper is an extension or modification of some recently announce results.

1. Introduction

Let E be a real Banach space and E^* be its dual space. Let C be a nonempty, closed and convex subset of E, and $A : C \to E^*$ be a mapping.
The problem of finding a point $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \ \forall \ v \in C, \tag{1.1}$$

is called a *variational inequality problem* with respect to A and C. We denote the set of solutions of the variational inequality problem (1.1) by $VI(C, A)$.

Variational inequality problems VIs were formulated in the late 1960's by Lions and Stampacchia [25] and since then the problem captured various applications arising in many areas such as partial differential equations, optimal control, optimization e.t.c. (see, for example, [30, 37] and references there in), In numerous models for solving real-life problems, such as in signal processing, networking, resource allocation, image recovery, and so on, the constraints can be expressed as variational inequality problems and (or) as fixed point problems. Consequently, the problem of finding common element of the set of solutions of variational inequality problems and the set of fixed points of operators has become an interesting area of contemporary research for numerous mathematicians working in nonlinear operator theory (see, for example, [6, 26, 27] and the references contained in them).

Let $T : C \to E^*$ be a mapping, then T is said to be

- *l-Lipschitz continuous* if
$$\|Tx - Ty\| \leq l\|x - y\|, \ \forall x, y \in C, \ and \ for \ some \ l \geq 0.$$
- *η-strongly monotone* if
$$\big\langle x - y, Tx - Ty\big\rangle \geq \eta\|x - y\|^2, \ \forall x, y \in C \ for \ some \ \eta > 0.$$

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- *δ-inverse strongly monotone* if

$$\langle x-y, Tx-Ty\rangle \geq \delta\|Tx-Ty\|^2,\ \forall x,y\in C\ for\ some\ \delta>0.$$

- *Monotone* if

$$\langle x-y, Tx-Ty\rangle \geq 0,\ \forall x,y\in C. \tag{1.2}$$

Definition 1.1. A mapping $T : C \to C$ is said to be Demiclosed at zero if whenever a sequence $\{x_n\}$ in C converges weakly to p and $\|x_n - Tx_n\|$ converges strongly to 0, then $p \in F(T)$.

Korpelevich [24] (and also independently Antipin [1]) proposed a double projection method in Euclidean space, known as the extragradient method for solving VIs when A is monotone and $l-$ Lipschitz continuous as

$$\begin{cases} y_n = P_C(x_n - \lambda Ax_n) \\ x_{n+1} = P_C(x_n - \lambda A(y_n)). \end{cases} \tag{1.3}$$

The weak convergence of this method in infinite dimensional Hilbert spaces was studied in [7] under an additional assumption that a map A is a weakly convergent sequence to a strongly convergent sequence. This assumption is rather strong and is not satisfied even for a simple example when A is the identity operator. In [36], the author has weakened this assumption to sequentially weak - weak continuity of A. The extragradient method and its variants require (at least) two projections per iteration. Censor et al [12] proposed the following scheme, called subgradient extragradient method

$$\begin{cases} y_n = P_C(x_n - \lambda A(x_n)), \\ x_{n+1} = P_{T_n}(x_n - \lambda A(y_n))\ \forall n \geq 0, \\ \ where\ T_n = \{w \in H : \langle x_n - \lambda A(x_n) - y_n, w - y_n\rangle \leq 0\}. \end{cases} \tag{1.4}$$

Since the projection onto the half-space T_n can be explicitly calculated [2], the subgradient extragradient requires only one projection per iteration. This method converges for monotone VIs in infinite dimensional Hilbert spaces [11, 13].

An alternative method of the extragradient method is the following remarkable scheme studied by Tseng's [35], which also requires only one projection per iteration

$$\begin{cases} y_n = P_C(x_n - \lambda A(x_n)), \\ x_{n+1} = y_n + \lambda(A(x_n) - A(y_n))\ \forall n \geq 0. \end{cases} \tag{1.5}$$

The weak convergence of Tseng's extragradient method (also known as the Forward-Backward-Forward method) for solving monotone Lipschitz continuous VIs was established in [35].

The split feasibility problem (SFP) in a finite dimensional Hilbert space was introduced by Censor and Elfving [9] for modelling inverse problems which arise from phase retrivals and medical image reconstruction as, Let H_1 and H_2 be two Hilbert space, let C and Q be two nonempty closed and convex subset of H_1 and H_2 respectively. The split feasibility problem is to find

$$u \in C \text{ s.t } Au \in Q \tag{1.6}$$

Assuming that the SFP is consistent (i.e., (1.6) has a solution) it is easy to see that $x \in C$ solve 1.6 iff it solve the fixed point equation

$$x = P_C(I + \gamma A^*(P_Q - I)A)x. \tag{1.7}$$

where P_C and P_Q are the orthogonal projections onto C and Q, respectively, $\gamma_n > 0$, and A^* is the adjoint of A. To solve (1.7), Byrne [4] proposed the CQ algorithm which generates a sequence $\{x_n\}$ by

$$x_{n+1} = P_C(I + \gamma_n A^*(P_Q - I)Ax_n),$$

for each $n \in \mathbb{N}$, where $\gamma_n \in (0, (2/\lambda_n))$, λ_n being the spectral radius of the operator A^*A.

In 2010, Censor et al. [10] considered a new variational problem called split variational inequality problem ($SVIP$). It entails finding a solution of one variational inequality problem whose image under a bounded linear transformation is a solution of another variational inequality problem. The SVIP is formulated as find

$$u \in VI(C, f) \text{ such that } Au \in VI(Q, g),$$

where C and Q are nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ is a bounded linear operator. One can easily observe that split variational inequality has the split feasibility problem as a special case.

Recently, Tian and Jiang [34], based on the work of Censor et al. [10], considered a class of $SVIP$ which is to find

$$x \in VI(C, f) \;\text{ such that } Ax \in F(T)$$

where C is a nonempty closed convex subset of a real Hilbert space H_1, $f : C \to H_1$ is a monotone and $l-$Lipschitz continuous map, $A : H_1 \to H_2$ is a bounded linear map, and $T : H_2 \to H_2$ is a nonexpansive map. they proposed the following algorithm by combining the Korpelevich extragradient method [24] and Byrne CQ algorithm as

$$\begin{cases} x_1 = x \in C; \\ y_n = P_C(x_n - \gamma_n A^*(I - T)Ax_n); \\ t_n = P_C(y_n - \lambda_n f y_n); \\ x_{n+1} = P_C(y_n - \lambda_n f t_n) \end{cases} \tag{1.8}$$

they obtained the following result.

Theorem 1.2 (see [34])**.** *Let H_1 and H_2 be real Hilbert spaces. Let C be a nonempty, closed and convex subset of H_1, $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$, $f : C \to H_1$ be a monotone and l-Lipschitz continuous map, and $T : H_2 \to H_2$ be a nonexpansive map. Setting $\Gamma = \{z \in VI(C, f) : Az \in F(T)\}$, assume that $\Gamma \neq 0$. Let the sequence $\{x_n\}$ be generated by (1.8), where $\gamma_n \subset [a, b]$, $a, b \in (0, (1/\|A\|^2)$, and $\lambda_n \in (0, 1/k)$. Then, the sequence $\{x_n\}$ converges weakly to a point $z \in \Gamma$.*

To the best of our knowledge, we are not aware of using Tseng's method (also known as the Forward-Backward-Forward method) for solving a class of split variational inequality problem $SVIP$. One of the advantage of this Tseng's method is the implementation of Algorithm (1.5), the computation of the projection onto the feasible set C is done only once per iteration. This makes the Algorithm (1.5) more efficient and implementable than the extragradient algorithm (1.3), where the computation of the projection onto the feasible set is done twice per iteration and also outperforms the subgradient extragradient algorithm (1.4), which involve two computations of projections per iteration one onto the feasible set and the other onto the half-space. Therefore, algorithm (1.5) is much more desirable when computing the projection is a hard task during implementation. However, in implementing an Algorithm (1.5), one has to obtain the Lipschitz constant, l, of the cost function f (or an estimate of it). Inspired by the results of Tian and Jiang [34], the authors raised the following motivational questions:

- Can the result of Tian and Jiang hold in a more general setting of Banach space than Hilbert?
- Can the extragradient method be replaced by Tseng's method for solving a class of split variational inequality problem?
- Can strong convergence theorem be proved in this setting?

In this paper, the above questions are answered in affirmative. We study a modified Tseng's extragradient method in the setting of uniformly smooth which is also 2-uniformly convex real Banach space and 2-uniformly smooth real Banach space and prove its strong convergence to a solution of a variational inequality problem for a monotone $l-$Lipschitz continuous map whose image under a bounded linear operator is a fixed point of nonexpansive maps. Our theorems improve and extend the results of Tian and Jiang [34] and other recently announced results.

2. Preliminaries

The following results will play vital roles in establishing our main result.

Let E be a smooth real Banach space and $\phi : E \times E \to \mathbb{R}$ be define by

$$\phi(x, y) = \|x\|^2 - \langle x, Jy\rangle + \|y\|^2 \; \forall x, y \in E. \tag{2.1}$$

For all $x, y, z \in E$ and $\alpha \in (0, 1)$, then the following are satisfied

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \tag{2.2}$$

$$\phi(x, y) + \phi(y, x) = 2\langle y - x, Jy - Jx\rangle; \tag{2.3}$$

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle z - x, Jy - Jz\rangle; \tag{2.4}$$

$$\phi(z, \alpha x + (1 - \alpha)y) \leq \alpha\phi(z, x) + (1 - \alpha)\phi(z, y). \tag{2.5}$$

Also, we define the function $V : E \times E^* \to \mathbb{R}$ by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2.$$

That is, $V(x, x^*) = \phi(x, J^{-1}x^*)$ for all $x \in E$ and $x^* \in E^*$. It is well known that, if E is a reflexive, strictly convex and smooth Banach space with E^* as its dual, then

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V(x, x^* + y^*)$$

for all $x \in E$ and $x^*, y^* \in E^*$, see [33].

Lemma 2.1. *Let E be $2-$uniformly convex and smooth real Banach space and C be nonempty closed convex subset of E. Let $x, y \in E$ be arbitrary. $\Pi_C : E \to C$ be a generalised projection then the following hold*

$$\text{there exists a constant } c > 0 \text{ such that, } \langle x - y, Jx - Jy\rangle \geq c\|x - y\|^2; \tag{2.6}$$

$$\text{there exists a positive constant } \gamma \text{ such that, } \gamma\|x - y\|^2 \leq \phi(x, y) \ \forall\ x, y \in E; \tag{2.7}$$

$$z = \Pi_C x \text{ if and only if } \langle z - w, Jx - Jz\rangle \geq 0 \ \forall w \in C; \tag{2.8}$$

$$\phi(w, z) + \phi(z, x) \leq \phi(w, x) \ \forall w \in C. \tag{2.9}$$

Remark 2.2. The following clearly hold;

$$\begin{aligned} y_n &= \Pi_C(J^{-1}(Jw_n - \lambda_n f w_n)) \\ &\iff \langle Jw_n - \lambda_n f w_n - Jy_n, p - y_n\rangle \leq 0, \ \forall p \in \Omega \\ &\iff \langle Jy_n - Jw_n + \lambda_n f w_n, y_n - p\rangle \leq 0, \ \forall p \in \Omega \\ &\iff \langle Jy_n - Jw_n, y_n - p\rangle \leq -\lambda_n\langle f w_n, y_n - p\rangle, \ \forall p \in \Omega. \end{aligned} \tag{2.10}$$

Lemma 2.3 ([38])**.** *Let E be $q-$uniformly smooth Banach space, then there exisxs a constant $d_q > 0$ such that*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, jx\rangle + d_q\|y\|^q. \tag{2.11}$$

Lemma 2.4 ([18])**.** *Let E be a uniformly convex and smooth real Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\lim_{n\to\infty} \phi(y_n, x_n) = 0$, then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.5 ([38])**.** *Let C be a nonempty closed and convex subset of a reflexive Banach space E and $B : C \to E^*$ be a monotone, hemicontinuous map. Let $T : E \to 2^{E^*}$ be an operator defined by:*

$$Tu = \begin{cases} Bu + N_C(u), & u \in C, \\ \emptyset, & u \notin C, \end{cases} \tag{2.12}$$

where $N_C(u)$ is defined as follows:

$$N_C(u) = \{w^* \in E^* : \langle u - z, w^*\rangle \geq 0, \ \forall z \in C\}.$$

Then, T is maximal monotone and $T^{-1}0 = VI(C, B)$.

Lemma 2.6 ([39])**.** *Let $\{s_n\}$ be a sequence of nonnegative real number satisfying*

$$s_{n+1} = (1 - \alpha_n)s_n + \alpha_n\beta_n, \ \forall n \geq 0,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ satisfy the conditions

(1) $\alpha_n \subset [0, 1]$, $\sum \alpha_n = \infty$
(2) $\lim_{n\to\infty} sup\beta_n \leq 0$ *or* $\sum |\alpha_n\beta_n| < \infty$

Then $\lim_{n\to\infty} s_n = 0$.

3. Main Results

Theorem 3.1. *Let E_1 be a uniformly smooth and $2-$ uniformly convex real Banach space and E_2 a $2-$ uniformly smooth real Banach space with smoothness constant $d_2 \in (0,1)$. Let C be a nonempty, closed, and convex subset of E_1. Let $f : C \to E_1^*$ be a monotone and $l-$ Lipschitz continuous map and $A : E_1 \to E_2^*$ be a bounded linear operator with its adjoint A^* such that $A \neq 0$. Let $T : E_2 \to E_2$ be nonexpansive mapping. Set $\Omega = \{p \in VI(C,f) : Ap \in F(T) \}$ and assume $\Omega \neq \emptyset$. Let a sequence $\{x_n\}$ be generated by*

$$\begin{cases} x_0, x_1 \in E_1; \\ w_n = \Pi_C((J_1^{-1}(J_1x_n + \tau_n A^* J_2(I-T)Ax_n)); \\ y_n = \Pi_C(J_1^{-1}(J_1x_n - \lambda f w_n)); \\ z_n = (J_1^{-1}(J_1y_n - \lambda(fy_n - fx_n)); \\ x_{n+1} = (J_1^{-1}(\alpha_n J_1 x_0 + (1-\alpha_n)J_1 z_n) \end{cases} \tag{3.1}$$

where $\tau \in [a,b]$, $a, b \in (0, \frac{1}{d_2^* \|A\|^2)\|})$, d_2^* being the smoothness constant of E^* as in Lemma 2.3, with $d_2 < 1$ and $\lambda_n \subset (0, \sqrt{\frac{\gamma}{d_2^* l}})$, γ being a positive constant as in (2.7). Then the sequences $\{x_n\}$, $\{v_n\}$, $\{y_n\}$ and $\{w_n\}$ are well defined and converge strongly to a point $p \in \Omega$.

Proof. **Step 1.** We show that $\{x_n\}$ is bounded.

Let $p \in \Omega$, then we have

$$\begin{aligned} \phi(p, z_n) &= \phi(p, J^{-1}(Jy_n - \lambda(fy_n - fx_n)) \\ &= \|p\|^2 - 2\langle p, JJ^{-1}(Jy_n - \lambda(fy_n - fx_n)\rangle + \|J^{-1}(Jy_n - \lambda(fy_n - fx_n)\|^2 \\ &= \|p\|^2 - 2\langle p, Jy_n\rangle + 2\lambda\langle p, fy_n - fx_n\rangle + \|Jy_n - \lambda(fy_n - fx_n)\|^2 \\ &\leq \|p\|^2 - 2\langle p, Jy_n\rangle + 2\lambda\langle p, fy_n - fx_n\rangle \\ &\quad + \|y_n\|^2 - 2\lambda\langle y_n, fy_n - fx_n\rangle + d_2^*\lambda^2\|fy_n - fx_n\|^2 \\ &= \phi(p, y_n) - 2\lambda\langle y_n - p, fy_n - fx_n\rangle + d_2^*\lambda^2\|fy_n - fx_n\|^2. \end{aligned} \tag{3.2}$$

From (2.3), (2.4) and (2.10) we have

$$\begin{aligned} \phi(p, z_n) &\leq \phi(p, x_n) + \phi(x_n, y_n) + 2\langle x_n - p, Jy_n - Jx_n\rangle - 2\lambda\langle y_n - p, fy_n - fx_n\rangle \\ &\quad + d_2^*\lambda^2\|fy_n - fx_n\|^2 \\ &= \phi(p, x_n) + \phi(x_n, y_n) - 2\langle y_n - x_n, Jy_n - Jx_n\rangle + 2\langle y_n - p, Jy_n - Jx_n\rangle \\ &\quad - 2\lambda\langle y_n - p, fy_n - fx_n\rangle + d_2^*\lambda^2\|fy_n - fx_n\|^2 \\ &= \phi(p, x_n) - \phi(y_n, x_n) + 2\langle y_n - p, Jy_n - Jx_n\rangle - 2\lambda\langle y_n - p, fy_n - fx_n\rangle \\ &\quad + d_2^*\lambda^2\|fy_n - fx_n\|^2 \\ &\leq \quad \phi(p, x_n) - \phi(y_n, x_n) - 2\lambda\langle y_n - p, fx_n\rangle - 2\lambda\langle y_n - p, fy_n - fx_n\rangle \\ &\quad + d_2^*\lambda^2\|fy_n - fx_n\|^2 \\ &= \quad \phi(p, x_n) - \phi(y_n, x_n) - 2\lambda\langle y_n - p, fy_n\rangle + d_2^*\lambda^2\|fy_n - fx_n\|^2 \\ &\leq \phi(p, x_n) - \phi(y_n, x_n) + d_2^*\lambda^2\|fy_n - fx_n\|^2. \end{aligned} \tag{3.3}$$

Since f is monotone $l-$ Lipschitz, from (3.3) and (2.7) we have

$$\begin{aligned}\phi(p,z_n) &\le \phi(p,x_n)-\phi(y_n,x_n)+d_2^*\lambda^2\|fy_n-fx_n\|^2\\ &\le \phi(p,x_n)-\phi(y_n,x_n)+d_2^*\lambda^2 l\|y_n-x_n\|^2\\ &\le \phi(p,x_n)-\phi(y_n,x_n)+\frac{d_2^*\lambda^2 l}{\gamma}\phi(y_n,x_n)\\ &= \phi(p,x_n)-(1-\frac{d_2^*\lambda^2 l}{\gamma})\phi(y_n,x_n). \qquad (3.4)\end{aligned}$$

Therefore from the hypothesis in the theorem we have

$$\phi(p,z_n) \quad\le\quad \phi(p,x_n). \qquad (3.5)$$

Now since E_1^* is 2-uniformly smooth we have

$$\begin{aligned}\phi(p,w_n) &= \phi(p,\Pi_C(J^{-1}(Jx_n+\tau_n A^*J_2(I-T)Aw_n))\\ &= \|p\|^2-2\langle p,JJ^{-1}(Jx_n+\tau_n A^*J_2(T-I)Ax_n)\rangle\\ &\quad +\|J^{-1}(Jx_n+\tau_n A^*J_2(T-I)Ax_n\|^2\\ &= \|p\|^2-2\langle p,Jx_n\rangle-2\tau_n\langle p,A^*J_2(T-I)Ax_n\rangle\\ &\quad +\|Jx_n+\tau_n A^*J_2(T-I)Ax_n\|^2\\ &= \|p\|^2-2\langle p,Jx_n\rangle-2\tau_n\langle Ap,J_2(T-I)Ax_n\rangle\\ &\quad +\|Jx_n+\tau_n A^*J_2(T-I)Ax_n\|^2\\ &\le \|p\|^2-2\langle p,Jx_n\rangle-2\tau_n\langle Ap,J_2(T-I)Ax_n\rangle\\ &\quad +\|Jx_n\|^2+2\tau_n\langle x_n,A^*J_2(T-I)Ax_n\rangle\\ &\quad +d_2^*\tau_n^2\|A^*\|^2\|(T-I)Ax_n\|^2\\ &= \phi(p,x_n)-2\tau_n\langle Ap,J_2(T-I)Ax_n\rangle+2\tau_n\langle Ax_n,J_2(T-I)Ax_n\rangle\\ &\quad +d_2^*\tau_n^2\|A^*\|^2\|(T-I)Ax_n\|^2\\ &= \phi(p,x_n)+2\tau_n\langle Ax_n-Ap,J_2(T-I)Ax_n\rangle\\ &\quad +d_2^*\tau_n^2\|A^*\|^2\|(T-I)Ax_n\|^2\\ &= \phi(p,x_n)+2\tau_n\langle Ax_n-TAx_n+TAx_n-Ap,J_2(T-I)Ax_n\rangle\\ &\quad +d_2^*\tau_n^2\|A^*\|^2\|(T-I)Ax_n\|^2\\ &= \phi(p,x_n)+2\tau_n\langle (I-T)Ax_n,J_2(T-I)Ax_n\rangle\\ &\quad +2\tau_n\langle TAx_n-Ap,J_2(T-I)Ax_n\rangle\\ &\quad +d_2^*\tau_n^2\|A^*\|^2\|(T-I)Ax_n\|^2\\ &= \phi(p,x_n)-2\tau_n\langle (T-I)Ax_n,J_2(T-I)Ax_n\rangle\\ &\quad +2\tau_n\langle TAx_n-Ap,J_2(T-I)Ax_n\rangle\\ &\quad +d_2^*\tau_n^2\|A^*\|^2\|(T-I)Ax_n\|^2\\ &= \phi(p,x_n)-2\tau_n\|(T-I)Ax_n\|^2\\ &\quad +2\tau_n\langle TAx_n-Ap,J_2(T-I)Ax_n\rangle\end{aligned}$$

$$+ d_2^* \tau_n^2 \|A^*\|^2 \|(T-I)Ax_n\|^2 \tag{3.6}$$

So since E_2 is 2-uniformly smooth we have

$$\begin{aligned}\|Ax_n - Ap\|^2 &= \|Ax_n - TAx_n + TAx_n - Ap\|^2 \\ &= \|(TAx_n - Ap) - (TAx_n - Ax_n)\|^2 \\ &\leq \|(TAx_n - Ax_n)\|^2 - 2\langle TAx_n - Ap, J_2(T-I)Ax_n\rangle + d_2\|(TAx_n - Ap)\|^2\end{aligned}$$

Therefore

$$\begin{aligned}2\langle TAx_n - Ap, J_2(T-I)Ax_n\rangle &\leq \|(TAx_n - Ax_n)\|^2 - \|Ax_n - Ap\|^2 \\ &\quad + d_2\|(TAx_n - Ap)\|^2 \end{aligned} \tag{3.7}$$

Now from (3.6) and (3.7) we have

$$\begin{aligned}\phi(p, w_n) &\leq \phi(p, x_n) - 2\tau_n\|(T-I)Ax_n\|^2 + \tau_n\|(T-I)Ax_n)\|^2 - \tau_n\|Ax_n - Ap\|^2 \\ &\quad + d_2\tau_n\|(TAx_n - Ap)\|^2 + d_2^*\tau_n^2\|A^*\|^2\|(T-I)Ax_n\|^2 \\ &= \phi(p, x_n) - \tau_n\|(T-I)Ax_n\|^2 - \tau_n\|Ax_n - Ap\|^2 + d_2\tau_n\|(TAx_n - Ap)\|^2 \\ &\quad + d_2^*\tau_n^2\|A^*\|^2\|(T-I)Ax_n\|^2 \\ &= \phi(p, x_n) - \tau_n(1 - d_2^*\tau_n\|A^*\|^2)\|(T-I)Ax_n\|^2 \\ &\quad - \tau_n(1 - d_2)\|Ax_n - Ap\|^2 \end{aligned} \tag{3.8}$$

Hence

$$\phi(p, w_n) \leq \phi(p, x_n) \tag{3.9}$$

From the projection property we have

$$\begin{aligned}\phi(p, y_n) &\leq \phi(p, (J^{-1}(Jx_n - \lambda f w_n)) - \phi(y_n, (J^{-1}(Jx_n - \lambda f w_n)) \\ &= \|p\|^2 - 2\langle p, Jx_n - \lambda f w_n\rangle + \|Jx_n - \lambda f w_n\|^2 \\ &\quad - \|y_n\|^2 + 2\langle y_n, Jx_n - \lambda f w_n\rangle - \|Jx_n - \lambda f w_n\|^2 \\ &= \|p\|^2 - 2\langle p, Jx_n\rangle + 2\lambda\langle p, f w_n\rangle - \|Jy_n\|^2 + 2\langle y_n, Jx_n\rangle - 2\lambda\langle y_n, f w_n\rangle \\ &= \phi(p, x_n) - \phi(y_n, x_n) + 2\lambda\langle p - y_n, f w_n\rangle\end{aligned}$$

By Monotonicity property of f and the fact that $p \in \Omega$ we have

$$\phi(p, y_n) \leq \phi(p, x_n) - \phi(y_n, x_n) + 2\lambda\langle w_n - y_n, f w_n\rangle. \tag{3.10}$$

Since from the property of ϕ we have

$$\phi(y_n, x_n) = \phi(y_n, w_n) + \phi(w_n, x_n) + 2\langle w_n - y_n, Jw_n - Jx_n\rangle,$$

so,

$$\begin{aligned}\phi(p, y_n) &\leq \phi(p, x_n) - \phi(y_n, w_n) - \phi(w_n, x_n) \\ &\quad - 2\langle w_n - y_n, Jw_n - Jx_n\rangle + 2\lambda\langle w_n - y_n, f w_n\rangle \\ &= \phi(p, x_n) - \phi(y_n, w_n) - \phi(w_n, x_n) \\ &\quad + 2\langle w_n - y_n, Jx_n - \lambda f w_n - Jw_n\rangle \end{aligned} \tag{3.11}$$

Also from the fact that $y_n = \Pi_C(J_1^{-1}(J_1 x_n - \lambda f w_n))$ and $w_n \in C$. using Projection property we have

$$\begin{aligned}
2\langle w_n - y_n, Jx_n - \lambda f w_n - Jw_n\rangle &= 2\langle w_n - y_n, Jx_n - \lambda f w_n - Jy_n + Jy_n - Jw_n\rangle \\
&= 2\langle w_n - y_n, Jx_n - \lambda f w_n - Jy_n\rangle \\
&\quad + 2\langle w_n - y_n, Jy_n - Jw_n\rangle \\
&= 2\langle w_n - y_n, Jy_n - Jw_n\rangle \\
&= -2\langle y_n - w_n, Jy_n - Jw_n\rangle \\
&\leq 2| - \langle y_n - w_n, Jy_n - Jw_n\rangle| \\
&= 2\|y_n - w_n\|^2 \\
&\leq \frac{2\phi(y_n, w_n)}{\gamma}. \qquad (3.12)
\end{aligned}$$

From (3.11) and (3.12) we have

$$\begin{aligned}
\phi(p, y_n) &\leq \phi(p, x_n) - \phi(y_n, w_n) - \phi(w_n, x_n) + \frac{2\phi(y_n, w_n)}{\gamma} \\
&= \phi(p, x_n) - \phi(w_n, x_n) - (1 - \frac{2}{\gamma})\phi(y_n, w_n). \qquad (3.13)
\end{aligned}$$

Hence

$$\phi(p, y_n) \leq \phi(p, x_n). \qquad (3.14)$$

Also from (2.5) and (3.5)

$$\begin{aligned}
\phi(p, x_{n+1}) &= \phi(p, J^{-1}(\alpha_n Jx_0 + (1 - \alpha_n)Jz_n)) \\
&\leq \alpha_n \phi(p, x_0) + (1 - \alpha_n)\phi(p, z_n) \qquad (3.15) \\
&\leq \alpha_n \phi(p, x_0) + (1 - \alpha_n)\phi(p, x_n) \\
&\leq Max\{\phi(p, x_0), \phi(p, x_1)\}
\end{aligned}$$

Hence the sequence $\{\phi(p, x_n)\}_{n=1}^{\infty}$ is bounded, and from (2.2) $\{x_n\}$ is bounded. So also $\{y_n\}$ $\{w_n\}$ and $\{z_n\}$.

Step 2. We show that

$$\lim_{n\to\infty} \|x_n - w_n\| = \lim_{n\to\infty} \|(T - I)Ax_n\| = \lim_{n\to\infty} \|z_n - y_n\| = \lim_{n\to\infty} \|z_n - w_n\| = 0.$$

Consider the following two cases

Case 1. Suppose that $\{\phi(p, x_n)\}_{n=1}^{\infty}$ is nonincreasing sequence, then $\lim_{n\to\infty} \phi(p, x_n)$ exists. From (3.4), (3.9) and (3.15) we have

$$\phi(p, x_{n+1}) \leq \alpha_n[\phi(p, x_0) - \phi(p, x_n)] + \phi(p, x_n) - (1 - \alpha_n)(1 - \frac{d_2^* \lambda^2 l}{\gamma})\phi(y_n, x_n)].$$

This implies that

$$(1 - \frac{d_2^* \lambda^2 l}{\gamma})\phi(y_n, x_n) \leq \alpha_n[\phi(p, x_0) - \phi(p, x_n)] + [\phi(p, x_n) - \phi(p, x_{n+1})]$$

$$+\alpha_n(1-\frac{d_2^*\lambda^2 l}{\gamma})\phi(y_n,x_n)$$

From the fact that $\lim_{n\to\infty}\alpha_n = 0$, we have

$$\lim_{n\to\infty}\phi(y_n,x_n)=0 \tag{3.16}$$

From Lemma (2.4) we have

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{3.17}$$

From the definition of x_{n+1} we have

$$\|Jx_{n+1}-Jz_n\|=\alpha_n\|Jx_0-Jz_n\|\to 0\ as\ n\to\infty.$$

So

$$\lim_{n\to\infty}\|Jx_{n+1}-Jz_n\|=0,$$

and therefore since J^{-1} is uniformly norm to norm continuous on bounded sets, we obtained

$$\lim_{n\to\infty}\|x_{n+1}-z_n\|=0. \tag{3.18}$$

Also from the definition of z_n and (3.17) we have

$$\begin{aligned}\|Jz_n-Jy_n\| &= \lambda\|fy_n-fx_n\|\\ &\leq l\|y_n-x_n\|.\end{aligned}$$

So

$$\lim_{n\to\infty}\|Jz_n-Jy_n\|=0.$$

Since J^{-1} is uniformly norm to norm continuous on bounded sets, we obtained

$$\lim_{n\to\infty}\|z_n-y_n\|=0. \tag{3.19}$$

So from (3.13) , (3.17) and uniform continuity of J on bounded subset we have

$$\begin{aligned}\phi(w_n,x_n) &\leq \phi(p,x_n)-\phi(p,y_n)\\ &= \|x_n\|^2-\|y_n\|^2+\langle p,Jy_n-Jx_n\rangle\\ &= (\|x_n\ +\|y_n\|)(\|x_n\|-\|y_n\|)+\langle p,Jy_n-Jx_n\rangle\\ &\leq (\|x_n\ +\|y_n\|)(\|x_n-y_n\|)+\langle p,Jy_n-Jx_n\rangle. \end{aligned} \tag{3.20}$$

Therefore

$$\lim_{n\to\infty}\phi(w_n,x_n)=0. \tag{3.21}$$

From Lemma (2.4) we have

$$\lim_{n\to\infty}\|w_n-x_n\|=0. \tag{3.22}$$

and

$$\begin{aligned}\|z_n-x_n\| &= \|z_n-y_n+y_n-x_n\|\\ &\leq \|z_n-y_n\|+\|y_n-x_n\|.\end{aligned}$$

So

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.23}$$

From (3.19) and (3.23) we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|x_{n+1} - z_n + z_n - x_n\| \\ &\le \|x_{n+1} - z_n\| + \|z_n - x_n\|, \end{aligned}$$

Hence

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.24}$$

So from (3.8), (3.22) and and uniform continuity of J on bounded subset we have

$$\begin{aligned} &\tau_n(1 - d_2^*\tau_n\|A^*\|^2)\|(T-I)Ax_n\|^2 \le \phi(p, x_n) - \phi(p, w_n) \\ &= \|x_n\|^2 - \|w_n\|^2 + \langle p, Jw_n - Jx_n\rangle \\ &= (\|x_n + \|w_n\|)(\|x_n\| - \|w_n\|) + \langle p, Jw_n - Jx_n\rangle \\ &\le (\|x_n + \|w_n\|)(\|x_n - w_n\|) \\ &\quad + \langle p, Jw_n - Jx_n\rangle \to 0. \end{aligned} \tag{3.25}$$

Thus

$$\lim_{n\to\infty} \|(T-I)Ax_n\|^2 = 0. \tag{3.26}$$

Case 2. Suppose that the sequence $\{\phi(p, x_n)\}_{n=1}^{\infty}$ is not nonincreasing. Let $\eta : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \ge N$ values (where N is large enough). Now defined by $\eta_n := max\{k \in \mathbb{N} : \phi(p, x_k) \le \phi(p, x_{k+1})\}$. Then, $\eta_n \to \infty$ as $n \to \infty$ and $\phi(p, x_{\eta_n}) \le \phi(p, x_{\eta_{n+1}})$ and $\phi(p, x_n) \le \phi(p, x_{\eta_{n+1}})$ for all $n \ge N$. By using (3.4), (3.9) and (3.15) and the conditions of the sequence parameters for each $n \ge N$, the fact that $\lim_{n\to\infty} \alpha_{\eta_n} = 0$ we have

$$\begin{aligned} (1 - \frac{d_2^*\lambda^2 l}{\gamma})\phi(y_{\eta_n}, x_{\eta_n}) &\le \alpha_{\eta_n}[\phi(p, x_0) - \phi(p, x_{\eta_n})] + [\phi(p, x_{\eta_n}) - \phi(p, x_{\eta_{n+1}})] \\ &\quad + \alpha_{\eta_n}(1 - \frac{d_2^*\lambda^2 l}{\gamma})\phi(y_{\eta_n}, x_{\eta_n}) \end{aligned}$$

This implies that

$$\lim_{n\to\infty} \phi(y_{\eta_n}, x_{\eta_n}) = 0.$$

From Lemma (2.4) we have

$$\lim_{n\to\infty} \|y_{\eta_n} - x_{\eta_n}\| = 0.$$

Following the proof lines in **Case 1,** we can show that:

$$\lim_{n\to\infty} \|(T-I)Ax_{\eta_n}\|^2 = 0. \tag{3.27}$$

Step 3. Next. We show that $\{x_n\}$ converges weakly to an element of Ω. Since $\{x_n\}$ is bounded then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $p \in \Omega$ such that $x_{n_k} \rightharpoonup p$. From (3.17), (3.19) and (3.22) we have $w_{n_k} \rightharpoonup p$, $y_{n_k} \rightharpoonup p$ and $z_{n_k} \rightharpoonup p$.

As A is a bounded Linear operator we have $Ax_{n_k} \rightharpoonup Ap$. Also from (3.27) and the fact that T is nonexpansive, we have $(I-T)$ is demiclosed at 0. Therefore

$$Ap \in F(T).$$

Next We show that $p \in VI(C,f)$.
Define

$$Hv = \begin{cases} fv + N_C(v), & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then by Lemma 2.5, H is maximal monotone and $H^{-1}(0) = VI(C,f)$ i.e., $v \in H^{-1}(0)$ if and only if $v \in VI(C,f)$.
Claim: $(p,0) \in G(H)$.

Let $(v,x^*) \in G(H)$ then it is enough to show that $\langle v-p, x^*\rangle \geq 0$.
Now

$$\begin{aligned}(v,x^*) \in G(H) &\Rightarrow x^* \in Hv = fv + N_C(v)\\ &\Rightarrow x^* - fv \in N_C(v)\end{aligned}$$

Therefore $\langle v-y, x^*-fv\rangle \geq 0$ $\forall y \in C$. Since $y_{n_k} = \Pi_C(J^{-1}(Jx_{n_k} - \lambda f w_{n_k}))$ and $v \in C$ we have by generalise projection properties $\langle y_{n_k} - v, Jx_{n_k} - \lambda f w_{n_k} - Jy_{n_k}\rangle \geq 0$. Thus,

$$\langle v - y_{n_k}, \frac{Jy_{n_k} - Jfx_{n_k}}{\lambda} + fw_{n_k}\rangle \geq 0, \; n \geq 0.$$

Using the fact that $y_{n_k} \in C$ and $x^* - fv \in N_C(v)$, we have

$$\begin{aligned}\langle v - y_{n_k}, x^*\rangle &\geq \langle v - y_{n_k}, fv\rangle\\ &\geq \langle v - y_{n_k}, fv\rangle - \langle v - y_{n_k}, \frac{Jy_{n_k} - Jx_{n_k}}{\lambda} + fw_{n_k}\rangle\\ &= \langle v - y_{n_k}, fv - fy_{n_k}\rangle + \langle v - y_{n_k}, fy_{n_k} - fw_{n_k}\rangle\\ &\quad - \langle v - y_{n_k}, \frac{Jy_{n_k} - Jx_{n_k}}{\lambda}\rangle\\ &\geq \langle v - y_{n_k}, fy_{n_k} - fw_{n_k}\rangle - \langle v - y_{n_k}, \frac{Jy_{n_k} - Jx_{n_k}}{\lambda}\rangle.\end{aligned}$$

Using the fact that J is uniformly continuous on bounded set and f is Lipschitz continuous, hence, as $k \to \infty$, we have

$$\langle v - p, x^*\rangle \geq 0.$$

Since H is a maximal monotone $0 \in Hp$ and hence $p \in VI(C,f)$.
Therefore, we have $p \in \Omega$.

Step 4. We show that $\lim_{n\to\infty} sup\langle Jx_0 - Jw, x_n - w\rangle \leq 0$, where $w = \Pi_\Omega(x_0)$. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_{n\to\infty} sup\langle Jx_0 - Jw, x_n - w\rangle = \lim_{k\to\infty}\langle Jx_0 - Jw, x_{n_k} - w\rangle.$$

Since we have $p \in \Omega$,

$$\lim_{n\to\infty} sup\langle Jx_0 - Jw, x_n - w\rangle = \lim_{k\to\infty}\langle Jx_0 - Jw, x_{n_k} - w\rangle = \langle Jx_0 - Jw, p - w\rangle \leq 0.$$

Step 5. Finally, we show that $w = \lim_{n\to\infty} x_n$, where $w = \Pi_\Omega(x_0)$. Now

$$\begin{aligned}
\phi(w, x_{n+1}) &= \phi(w, J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jz_n)) \\
&= V(w, \alpha_n Jx_0 + (1-\alpha_n)Jz_n) \\
&\leq V(w, \alpha_n Jx_0 + (1-\alpha_n)Jz_n - \alpha_n(Jx_0 - Jw)) \\
&\quad + 2\alpha_n \langle Jx_0 - Jw, x_{n+1} - w\rangle \\
&\leq \alpha_n V(w, Jw) + (1-\alpha_n)V(w, Jz_n) + 2\alpha_n\langle Jx_0 - Jw, x_{n+1} - w\rangle \\
&\leq (1-\alpha_n)V(w, Jz_n) + 2\alpha_n\langle Jx_0 - Jw, x_{n+1} - w\rangle \\
&= (1-\alpha_n)\phi(w, z_n) + 2\alpha_n\langle Jx_0 - Jw, x_{n+1} - w\rangle \\
&\leq (1-\alpha_n)\phi(w, x_n) + 2\alpha_n\langle Jx_0 - Jw, x_{n+1} - w\rangle. \qquad (3.28)
\end{aligned}$$

From **Step 4**, (3.28), and Lemma 2.6, we have $w = \lim_{n\to\infty} x_n$. □

Corollary 3.2. *Let H_1 and H_2 be Hilbert space and let C be a nonempty, closed, and convex subset of H_1. Let $f : C \to H_1^*$ be a monotone and $l-$ Lipschitz continuous map and $A : H_1 \to H_2^*$ be a bounded linear operator with its adjoint A^* such that $A \neq 0$. Let $T : H_2 \to H_2$ be nonexpansive mapping. Set $\Omega = \{p \in VI(C, f) : Ap \in F(T)\ \}$ and assume $\Omega \neq \emptyset$. Let a sequence $\{x_n\}$ be generated by*

$$\begin{cases} x_0, x_1 \in E_1; \\ w_n = P_C(x_n + \tau_n A^*(I-T)Ax_n)); \\ y_n = P_C(x_n - \lambda f w_n)); \\ z_n = y_n - \lambda(fy_n - fx_n); \\ x_{n+1} = \alpha_n x_0 + (1-\alpha_n)z_n \end{cases} \qquad (3.29)$$

where $\tau \in [a, b]$, $a, b \in (0, \frac{1}{\|A\|^2\|})$, and $\lambda_n \subset (0, \sqrt{\frac{\gamma}{l}})$, γ being a positive constant as in (2.7). Then the sequences $\{x_n\}$, $\{v_n\}$, $\{y_n\}$ and $\{w_n\}$ are well defined and converge strongly to a point $p \in \Omega$.

4. Numerical example

In this section, we present a numerical example to show the convergence of a sequence generated by our algorithm in Theorem 3.1

Example 4.1. Let $E_1 = E_2 = \mathbb{R}$, $C = [0, \infty)$. and Let $T : C \to C$ be defined by $Tx = \frac{2}{3}x$, $\forall x \in C$. Let $f : C \to C$ be defined by $fx = \frac{1}{3}x$, $\forall x \in C$. and $VI(C, f) = 0$. Let $A : C \to C$ be defined by $Ax = \frac{1}{2}x$, $\forall x \in C$. and $\|A\|^2 = \frac{1}{4}$, $A^*y = \frac{1}{2}y$. When $z \in VI(C, f), Az = 0 \in F(T)$. So, $\Gamma = \{z \in VI(C, f) : Az \in F(T)\} \neq \emptyset$. Clearly, it satisfy the condition of theorem 3.1. So from the scheme we obtain the following

$$\begin{cases} w_n = Proj_C(x_n + \tau_n A^*(I-T)Ax_n)); \\ y_n = Proj_C(x_n - \lambda f w_n); \\ z_n = (y_n - \lambda(fy_n - fx_n)) \\ x_{n+1} = \alpha_n x_0 + (1-\alpha_n)z_n \end{cases} \qquad (4.1)$$

where $\lambda = \frac{3}{n^2}$, $\tau = \frac{4}{n^2}$ and $\alpha = \frac{5}{n^2}$. Then $\{x_n\}$ converges to $0 \in \Omega = \{0\}$.

Next, using Matlab software we have the following figures which shows that the sequence $\{x_n\}$ converges to 0.

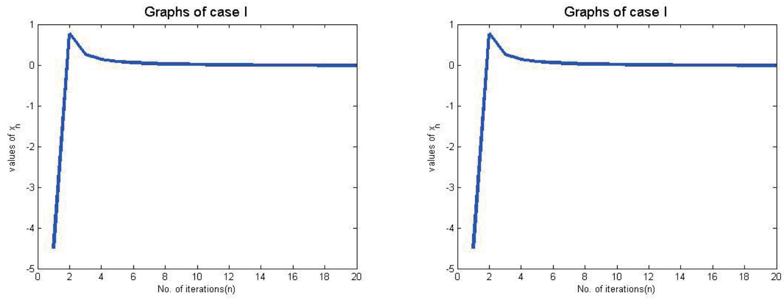

FIGURE 1. The diagram above is illustrating the convergence rate of the iterative algorithm.

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Bashir Ali
Department of Mathematical Sciences, Bayero University, Kano-Nigeria
E-mail address: bashiralik@yahoo.com

A. M. Hamza
School of Continuing Education Bayero University, Kano-Nigeria
E-mail address: amhamza.sce@buk.edu.ng