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AN INERTIAL ALGORITHM FOR SOLVING SPLIT EQUALITY COMMON FIXED POINT PROBLEMS INVOLVING J-PSEUDOCONTRACTIONS

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ABSTRACT. In this paper, an inertial Halpern-type algorithm for approximating solutions of split equality common fixed point problems involving continuous J-pseudocontractive maps is introduced and studied in the setting of real Banach spaces that are 2-uniformly convex and uniformly smooth. Strong convergence of the sequence generated by the proposed algorithm is proved without imposing any compactness-type conditions on the operators. Furthermore, we give a numerical example on the classical Banach space $L_{\frac{3}{2}}([-2,2])$ to show that the proposed inertial algorithm is implementable in the setting of real Banach spaces. Finally, our proposed algorithm extends, improves and generalize many results in the literature.

1. Introduction

Let H_i, $i = 1, 2, 3$ be real Hilbert spaces and let C and D be nonempty closed and covex subsets of H_1 and H_2, respectively. The *split feasibility problem* (SFP) is to find

$$x \in C \quad \text{such that} \quad Ax \in D, \tag{1.1}$$

where $A : H_1 \to H_2$ is a bounded linear map. Problem (1.1) was first studied by Censor and Elfving [8] on finite-dimensional spaces for modeling inverse problems arising from phase retrievals and medical image reconstruction [7]. Extensions of problem (1.1) to infinite dimensional spaces have been studied by many authors (see, e.g., [9, 10, 12, 22, 23, 25, 30]).

In the year 2013, Moudafi [19] extended problem (1.1) to cover some models arising from game theory, intensity modulated radiation therapy and so on. The extension studied by Moudafi [19] is to find

$$x \in C, \; y \in D \quad \text{such that} \quad Ax = By, \tag{1.2}$$

where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are bounded linear maps. Problem (1.2) is the so-called *split equality feasibility problem.* Observe that if B is the identity map on H_2 and $H_2 = H_3$, problem (1.2) reduces to problem (1.1). Moudafi [19] introduced

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and studied the following extragradient-type algorithm for approximating solutions of (1.2):

$$\begin{cases} x_{n+1} = P_C\big(x_n - \gamma_n A^*(Ax_n - By_n)\big), \\ y_{n+1} = P_D\big(y_n + \gamma_n B^*(Ax_{n+1} - By_n)\big), \end{cases} \tag{1.3}$$

where A^* and B^* denote the adjoint operators A and B, respectively and $\{\gamma_n\}$ is a sequence of real numbers that satisfies some appropriate conditions. Later, in the same paper [19], the author replaced the arbitrary subsets C and D with the fixed point set of some nonlinear operators T and S in order to dispense with the projections required to implement algorithm (1.3). By making this replacements, problem (1.2) becomes:

$$\text{find } \ x \in F(T),\ y \in F(S) \ \text{ such that } \ Ax = By, \tag{1.4}$$

where $F(T) = \{x \in H_1 : Tx = x\}$ and $F(S) = \{x \in H_2 : Sx = x\}$. Problem (1.4) is the so-called *split equality fixed point problem.* Moudafi [19] introduced and studied the following algorithm for solving problem (1.4):

$$\begin{cases} x_{n+1} = T\big(x_n - \gamma_n A^*(Ax_n - By_n)\big), \\ y_{n+1} = S\big(y_n + \gamma_n B^*(Ax_{n+1} - By_n)\big), \end{cases} \tag{1.5}$$

where T and S are firmly quasi-nonexpansive, and $\{\gamma_n\}$ is a positive nondecreasing sequence such that $\gamma_n \in \big(\epsilon, \min\{\frac{1}{\lambda_A}, \frac{1}{\lambda_B}\} - \epsilon\big)$, for a small enough $\epsilon > 0$, λ_A and λ_B are the spectral radius of A^*A and B^*B, respectively. Moudafi [19] proved that the sequence generated by (1.5) converges weakly to a solution of problem (1.4).

Remark 1.1. Since the appearance of problem (1.4) in the literature, several authors have proposed C-Q versions of algorithm (1.5) to obtain strong convergence. Some authors have added some compactness-type conditions on the operators T and S to obtain strong convergence. Others have extended the class of operators to involve demicontractive, quasinonexpansive, quasi-pseudocontractive, quasi-phi-nonexpansive and so on, in the setting of Hilbert spaces and Banach spaces (see, e.g., [1, 5, 11, 13, 18, 30], for what has been done regarding problem (1.4)).

Our interest is on the recent generalization of problem (1.4) introduced and studied by Nnakwe et al. [21]. The setting is as follows:

Let X, Y and Z be real Banach spaces with dual spaces, X^*, Y^* and Z^*, respectively. Let C and D be nonempty closed and convex subsets of X and Y, respectively. Let $A : X \to Z$, $B : Y \to Z$ be bounded linear mappings and let $F_i : C \to X^*$, $i = 1, 2$ and $K_i : D \to Y^*$, $i = 1, 2$ be continuous J-pseudocontractive maps. The *split equality common fixed point problem* (SECFPP) is finding $(x, y) \in C \times D$ such that

$$x \in F_{J_X}(F_i),\ i = 1, 2 \quad \text{and} \quad y \in F_{J_Y}(K_i),\ i = 1, 2 \quad \text{with} \quad Ax = By, \tag{1.6}$$

where $F_{J_X}(F_i) = \{x \in X : F_i x \in J_X x\}$ and $F_{J_Y}(K_i) = \{x \in X : K_i x \in J_Y x\}$, J_X and J_Y are the normalized duality maps on X and Y, respectively. The solution set of the SECFPP will be denoted by

$$\Delta := \Big\{(x,y) \in C \times D : \ (x,y) \in \bigcap_{i=1}^{2} \big(F_J(F_i) \times F_J(K_i)\big) \ \text{ and } \ Ax = By\Big\}. \tag{1.7}$$

Nnakwe et al. [21] introduced and studied the following algorithm for approximating solutions of the SECFPP (1.6):

$$\begin{cases} (x_0, y_0) \in X \times Y, \\ a_n \in J_Z(Ax_n - By_n), \\ \theta_n = J_X^{-1}(J_X x_n - \mu A^* a_n) \\ \delta_n = J_Y^{-1}(J_Y y_n + \mu B^* a_n) \\ x_{n+1} = J_X^{-1}\big(\alpha_n J_X x_0 + (1-\alpha_n) J_X \mathcal{T}_{r_n}^{T_1} \circ \mathcal{T}_{r_n}^{T_2} \theta_n\big), \\ y_{n+1} = J_Y^{-1}\big(\alpha_n J_Y y_0 + (1-\alpha_n) J_Y \mathcal{F}_{r_n}^{S_1} \circ \mathcal{S}_{r_n}^{T_2} \delta_n\big), \ n \geq 1, \end{cases} \tag{1.8}$$

where X and Y are 2-uniformly convex and uniformly smooth real Banach spaces, Z is a real Banach space, $\mathcal{T}_{r_n}^{T_i}$ and $\mathcal{F}_{r_n}^{S_i}$ are resolvent maps of T_i and S_i, $i = 1, 2$, respectively, A and B bounded linear maps with adjoints A^* and B^*, respectively, $\{\alpha_n\} \subset (0,1)$ and μ is a positive constant satisfying some appropriate conditions.

Remark 1.2. It is worthy of mentioning that the class of J-pseudocontractive mappings were first introduced by Chidume and Idu [14]. They also gave some interesting motivations about J-pseudocontractive mappings and the notion of J-fixed point (see, e.g., [14]).

To honor the memory of the late Professor Charles Ejike Chidume, it is our purpose in this paper is to contribute our quota to the study of J-pseudocontraction mappings which he introduced. We incorporate the inertial acceleration strategy in algorithm (1.8) of Nnakwe et al. [21] and proved that the sequence generated by our our proposed inertial algorithm converges strongly to a solution of the SECFPP (1.6) in the setting of real Banach spaces that are 2-uniformly convex and uniformly smooth. Furthermore, we give a numerical example on the classical Banach space $L_{\frac{3}{2}}([-2,2])$ to show that the proposed inertial algorithm is implementable in the setting of real Banach spaces.

2. Preliminaries

The following definitions and lemmas will be needed in the proof of main theorem.

Definition 2.1. Let E be a strictly convex and smooth real Banach space. For $p > 1$, define $J_p : E \to 2^{E^*}$ by

$$J_p(x) := \{x^* \in E^* : \langle x, x^* \rangle = \|x\| \|x^*\|, \ \|x^*\| = \|x\|^{p-1}\}.$$

J_p is called the *generalized duality map on E*. If $p = 2$, J_2 is called the *normalized duality map* and is denoted by J. In a real Hilbert space H, J is the identity map on H. It is easy to see from the definition that

$$J_p(x) = \|x\|^{p-2} Jx, \quad \text{and} \quad \langle x, J_p x \rangle = \|x\|^p, \ \forall x \in E.$$

It is well-known that if E is smooth, then J is single-valued and if E is strictly convex, J is one-to-one, and J is surjective if E is reflexive.

The next definition is for the lyapunov functional ϕ introduced by Alber [3]. It is useful for estimations involving J and its inverse J^{-1} on smooth Banach space.

Definition 2.2. Let X be a real Banach space that is smooth and $\phi : X \times X \to \mathbb{R}$ be a map. The lyapunov functional ϕ is defined by

$$\phi(x, y) := \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \ \forall\, x, y \in X. \tag{2.1}$$

Observe that if X is a real Hilbert space, (2.1) reduces to $\phi(x, y) = \|x-y\|^2$, $\forall x, y \in X$.

Furthermore, given $x, y, z, u \in X$, ϕ has the following properties:

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2,$$

$$\phi(x, J^{-1}(\tau Jy + (1 - \tau)Jz) \leq \tau\phi(x, y) + (1 - \tau)\phi(x, z). \tag{2.2}$$

Also we shall use interchangeably the mapping $V : X \times X^* \to \mathbb{R}$ by

$$V(x, y) = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2$$

with ϕ since

$$V(x, y) = \phi(x, J^{-1}y), \forall x \in X, y \in X^*.$$

Next, we give the definition of the generalized projection operator which is defined in terms of ϕ.

Definition 2.3. Let X be a reflexive, strictly convex and smooth real Banach space. Let D be a nonempty convex and closed subset of X. The generalized projection $\Pi_D : X \to D$ is defined by $\tilde{u} = \Pi_D(u) \in D$ such that $\phi(\tilde{u}, u) = \inf_{v\in D} \phi(v, u)$.

Remark 2.4. On a real Hilbert space, the metric projection P_D coincides with the generalized projection Π_D.

The subsequent definitions are for the notions and operators which will be used in our main theorem. Except where we stated explicitly, the space X is assumed to be a reflexive, strictly convex and smooth real Banach space.

Definition 2.5. Let $T : X \to X^*$ be a map. A point $x \in X$ is called a J-fixed point of T if $Tx = Jx$, where J is the duality mapping on the real Banach space X.

Definition 2.6. A map $T : X \to X$ is called pseudocontractive if for all $x, y \in X$, we have

$$\langle Tx - Ty, J(x - y)\rangle \leq \|x - y\|^2,$$

where J is the normalized duality mapping on X.

Definition 2.7. A mapping $T : X \to X^*$ is called J-pseudocontractive if for all $x, y \in X$, we have

$$\langle x - y, Tx - Ty\rangle \leq \langle x - y, Jx - Jy\rangle.$$

Definition 2.8. The collection of linear and continuous maps $B : X_1 \to X_2$ is a normed linear space. The adjoint operator $B^* : X_2^* \longrightarrow X_1^*$ is defined by $\langle B^*x^*, v\rangle = \langle x^*, Bv\rangle$, $\forall\ v \in X_1$, $x^* \in X_2^*$, and $\|B^*\| = \|B\|$.

Now we state without proof the following lemmas which are central in establishing our main result.

Lemma 2.9 ([4]). *Let X be a smooth, strictly convex and reflexive real Banach space and X^* be its dual space . Then*

$$V(u,x^*)+2\langle J^{-1}x^*-u,y^*\rangle \le V(u,x^*+y^*),\quad \forall\ u\in X,\ x^*,y^*\in X^*.$$

Lemma 2.10 ([26]). *If X is a smooth and 2-uniformly convex real Banach space, then for all $u,v\in X^*$,*

$$\|J^{-1}u-J^{-1}v\| \le \frac{1}{\kappa}\|u-v\|,\ for\ some\ \ \kappa>0.$$

Lemma 2.11 ([3]). *Let C be a nonempty closed and convex subset of a smooth, strictly convex and reflexive real Banach space X. For any $x\in X$ and $y\in C$, $\tilde{x}=\Pi_C x$ if and only if $\langle \tilde{x}-y, Jx-J\tilde{x}\rangle \ge 0$, for all $y\in C$.*

Lemma 2.12 ([17]). *Let X be a uniformly convex and smooth real Banach space, and let $\{u_n\}$ and $\{v_n\}$ be two sequences of X. If either $\{u_n\}$ or $\{v_n\}$ is bounded and $\lim_{n\to\infty}\phi(u_n,v_n)=0$ then $\lim_{n\to\infty}\|u_n-v_n\|=0$.*

Lemma 2.13 ([20]). *Let X be a uniformly smooth and strictly convex real Banach space with dual space X^*. Let C be a nonempty closed and convex subset of X and $T:C\to X^*$ be a continuous J-pseudocontractive map. Let $r>0$ and $x\in X$. Then the following conditions hold:*

(1) *There exists $z\in C$ such that, $\langle w-z,Tz\rangle-\frac{1}{r}\langle w-z,(1+r)Jz-Jx\rangle \le 0,\ \forall w\in C$.*
(2) *Define a map $T_r: X\to C$ by*

$$\mathcal{T}^T(x):=\{z\in C:\langle w-z,Tz\rangle-\frac{1}{r}\langle w-z,(1+r)Jz-Jx\rangle\le 0,\ \forall w\in C\},\ x\in X.$$

Then the following conditions hold:
(a) *$\mathcal{T}_r^T$ is single valued;*
(b) *$\mathcal{T}_r^T$ is firmly nonexpansive-type map, i.e.,*

$$\forall\ x,y\in X,\ \langle \mathcal{T}_r^Tx-\mathcal{T}_r^Ty, J\mathcal{T}_r^Tx-J\mathcal{T}_r^Ty\rangle \le \langle \mathcal{T}_r^Tx-\mathcal{T}_r^Ty, Jx-Jy\rangle,$$

(c) *$F(\mathcal{T}_r^T)=F_J(T)$, where $F(\mathcal{T}_r^T)$ and $F_J(T)$ denote the fixed point set of $\mathcal{T}_r^T$ and J-fixed points of T, respectively.*
(d) *$F_J(T)$ is closed and convex,*
(e) *$\phi(u,\mathcal{T}_r^Tx)+\phi(\mathcal{T}_r^Tx,x)\le\phi(u,x),\ \forall\ u\in F(\mathcal{T}_r^T),\ x\in X.$*

Lemma 2.14 ([2]). *Let E be a 2-uniformly convex and uniformly smooth real Banach space and let $x_0,x_1,x\in E$. Let $\{w_n\}\subset E$ be a sequence defined by $w_n:=J^{-1}\big(Jx_n+\mu_n(Jx_n-Jx_{n-1})\big)$. Then,*

$$\begin{aligned}\phi(x,w_n)\le\ &\phi(x,x_n)+\kappa\mu_n^2\|Jx_n-Jx_{n-1}\|^2+\mu_n\phi(x_n,x_{n-1})\\ &+\mu_n\big(\phi(x,x_n)-\phi(x,x_{n-1})\big),\end{aligned}$$

where $\{\mu_n\}\subset(0,1)$ and κ is the constant appearing in Lemma 2.10.

Finally, the last two lemmas will play a vital role in concluding that the sequence generated by our proposed algorithm converges strongly.

Lemma 2.15 ([27]). *Let $\{a_n\}$ be a sequence of nonnegative numbers satisfying the condition*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\beta_n + c_n, \ n \geq 0,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{c_n\}$ are sequences of real numbers such that

(i) $\{\alpha_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\limsup_{n\to\infty} \beta_n \leq 0$;
(iii) $c_n \geq 0, \ \sum_{n=0}^{\infty} c_n < \infty$.

Then,

$$\lim_{n\to\infty} a_n = 0.$$

Lemma 2.16 ([6]). *Let the sequences $\{\Theta_n\}$,$\{\gamma_n\}$ and $\{\beta_n\}$ be in $[0,\infty)$ with*

$$\Theta_{n+1} \leq \Theta_n + \beta_n(\Theta_n - \Theta_{n-1}) + \gamma_n,$$

for all $n \geq 1$, $\sum_{n=1}^{\infty} \gamma_n < +\infty$ and there exists $\beta \in \mathbb{R}$ with $0 \leq \beta_n \leq \beta < 1$, for all $n \in \mathbb{N}$. Then the following hold:

(i) $\sum_{n\geq 1}[\Theta_n - \Theta_{n-1}]_+ < +\infty$, *where* $[r]_+ = \max\{r, 0\}$;
(ii) *there exists* $\Theta^* \in [0,\infty)$ *such that* $\lim_{n\to\infty} \Theta_n = \Theta^*$.

3. Main results

In this section, we will present the main result of this paper. First, we give the setting of our main algorithm 3.1.

The Setting of Algorithm 3.1.

(1) The spaces X and Y are 2-uniformly convex and uniformly smooth real Banach spaces, C and D are nonempty closed and convex subsets of X and Y, respectively and Z is a smooth real Banach space.
(2) The mappings $A : X \to Z$ and $B : Y \to Z$ (with $A, B \equiv 0$) are bounded linear maps with adjoints A^* and B^*, respectively. The mappings $F_i : C \to X^*$, $i = 1, 2$ and $K_i : D \to Y^*$, $i = 1, 2$ are continuous pseudo-contractive maps, with resolvents $\mathcal{T}_{\mu_n}^{F_i}$ and $\mathcal{S}_{\mu_n}^{K_i}$, $i = 1, 2$, as defined in Lemma 2.13, respectively.
(3) The solution set Δ as defined in (1.7) is nonempty.

Algorithm 3.1. Step 1: *Choose the sequences $\{\epsilon_n\}$ and $\{\alpha_n\}$ satisfying $\sum_{n=1}^{\infty} \epsilon_n < \infty$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\lim_{n\to\infty} \alpha_n = 0$. Furthermore, choose a positive constant γ such that $0 < \gamma < \frac{\rho}{\|A\|^2+\|B\|^2}$, $\rho = \min\{\rho_1, \rho_2\}$, where ρ_1, ρ_2 are constants as in Lemma 2.10 and choose $\{\mu_n\} \subset (0,\infty)$.*

Step 2: *Select arbitrarily the following initial points $x, x_0, x_1 \in X$, $y, y_0, y_1 \in Y$, $\beta \in (0,1)$ and choose β_n such that $0 \leq \beta_n \leq \bar{\beta_n}$, where*

$$\bar{\beta_n} = \begin{cases} \min\Big\{\beta, \epsilon_n\|J_X x_n - J_X x_{n-1}\|^{-2}, \ \epsilon_n\phi(x_n, x_{n-1})^{-1}, \\ \ \epsilon_n\|J_Y y_n - J_Y y_{n-1}\|^{-2}, \ \epsilon_n\phi(y_n, y_{n-1})^{-1}\Big\}, \quad x_n \neq x_{n-1}, y_n \neq y_{n-1}; \\ \beta, \quad \textit{otherwise}. \end{cases}$$

Step 3: *Compute*

$$\begin{cases} z_n = J_X^{-1}(J_X x_n + \beta_n(J_X x_n - J_X x_{n-1})), \\ w_n = J_Y^{-1}(J_Y y_n + \beta_n(J_Y y_n - J_Y y_{n-1})), \\ a_n = J_Z(Az_n - Bw_n), \\ u_n = J_X^{-1}(J_X z_n - \gamma A^* a_n), \\ v_n = J_Y^{-1}(J_Y w_n + \gamma B^* a_n). \end{cases}$$

Step 4: *Compute*

$$x_{n+1} = J_X^{-1}\big(\alpha_n J_X x + (1-\alpha_n) J_X \mathcal{T}_{\mu_n}^{F_1} \circ \mathcal{T}_{\mu_n}^{F_2} u_n\big),$$
$$y_{n+1} = J_Y^{-1}\big(\alpha_n J_Y y + (1-\alpha_n) J_Y \mathcal{S}_{\mu_n}^{K_1} \circ \mathcal{S}_{\mu_n}^{K_2} v_n\big).$$

Step 5: *Set $n = n+1$ and go to Step 2.*

Theorem 3.2. *Let $\{(x_n, y_n)\}$ be a sequence be generated by Algorithm 3.1, then $\{(x_n, y_n)\}$ converges strongly to a point in Δ.*

Proof. We first establish boundedness of the sequence $\{(x_n, y_n)\}$ generated by Algorithm 3.1 before we proceed to show its convergence. Let $(x^*, y^*) \in \Delta$ and set $p_n = \mathcal{T}_{\mu_n}^{F_2} u_n$ and $q_n = \mathcal{T}_{\mu_n}^{F_1} p_n$. Using inequality (2.2) and property (e) in Lemma 2.13, we have

$$\begin{aligned} \phi(x^*, x_{n+1}) &= \phi\big(x^*, J_X^{-1}(\alpha_n J_X x + (1-\alpha_n) J_X q_n)\big) \\ &\le \alpha_n \phi(x^*, x) + (1-\alpha_n)\phi(x^*, q_n) \\ &\le \alpha_n \phi(x^*, x) + (1-\alpha_n)\phi(x^*, u_n). \end{aligned} \tag{3.1}$$

Next, we estimate the last term in inequality (3.1) using Lemmas 2.9 and 2.14. Hence, we have

$$\begin{aligned} \phi(x^*, u_n) &= V(x^*, J_X z_n - \gamma A^* a_n) \\ &\le V(x^*, J_X z_n) - 2\gamma\langle J_X^{-1}(J_X z_n - \gamma A^* a_n) - x^*, A^* a_n\rangle \\ &= \phi(x^*, z_n) - 2\gamma\langle A(u_n - x^*), a_n\rangle \\ &\le \phi(x^*, x_n) + \rho_3 \beta_n^2 \|J_X x_n - J_X x_{n-1}\|^2 + \beta_n(x_n, x_{n-1}) \\ &\quad + \beta_n(\phi(x^*, x_n) - \phi(x^*, x_{n-1})) - 2\gamma\langle A(u_n - x^*), a_n\rangle. \end{aligned} \tag{3.2}$$

Substituting this in inequality (3.1), we have

$$\begin{aligned} \phi(x^*, x_{n+1}) \le\ & \alpha_n \phi(x^*, x) \\ & + (1-\alpha_n)\big(\phi(x^*, x_n) \\ & + \rho_3 \beta_n^2 \|J_X x_n - J_X x_{n-1}\|^2 \\ & + \beta_n(x_n, x_{n-1}) \\ & + \beta_n(\phi(x^*, x_n) - \phi(x^*, x_{n-1})) - 2\gamma\langle A(u_n - x^*), a_n\rangle\big). \end{aligned} \tag{3.3}$$

Following a similarly line of proof, we get

$$\begin{aligned} \phi(y^*, y_{n+1}) \le\ & \alpha_n \phi(y^*, y) + (1-\alpha_n)\big(\phi(y^*, y_n) \\ & + \rho_4 \beta_n^2 \|J_Y y_n - J_Y y_{n-1}\|^2 + \beta_n(y_n, y_{n-1}) \end{aligned}$$

$$+ \beta_n(\phi(y^*, y_n) - \phi(y^*, y_{n-1})) - 2\gamma\langle B(v_n + y^*), a_n\rangle\big). \tag{3.4}$$

Let $\Theta_n(x, y) = \phi(x, x_n) + \phi(y, y_n)$ and $\varrho = \max\{\rho_3, \rho_4\}$. Adding inequalities (3.3) and (3.4) and using the fact that $Ax^* = Bx^*$, we get

$$\begin{aligned}\Theta_{n+1}(x^*, y^*) \leq{}& \alpha_n\big(\phi(x^*, x) + \phi(y^*, y)\big)\\ &+ (1 - \alpha_n)\Big(\Theta_n(x^*, y^*) + \varrho\beta_n^2\big(\|J_X x_n - J_X x_{n-1}\|^2\\ &+ \|J_Y y_n - J_Y y_{n-1}\|^2\big) + \beta_n\Theta_{n-1}(x_n, y_n) + \beta_n\big(\Theta_n(x^*, y^*)\\ &- \Theta_{n-1}(x^*, y^*)\big) - 2\gamma\underline{\langle Au_n - Bv_n, a_n\rangle}\Big).\end{aligned} \tag{3.5}$$

Next we estimate the underlined term in inequality (3.5). Using the definition of a_n in Algorithm 3.1 and Lemma 2.10, we get

$$\begin{aligned}-\langle Au_n - Bv_n, a_n\rangle &= -\|Az_n - Bw_n\|^2 + \langle Az_n - Bw_n - (Au_n - Bv_n), a_n\rangle\\ &= -\|Az_n - Bw_n\|^2 + \langle A(z_n - u_n), a_n\rangle + \langle B(v_n - w_n), a_n\rangle\\ &\leq -\|Az_n - Bw_n\|^2 + \|J_X^{-1}(J_X z_n) - J_X^{-1}(J_X z_n - \gamma A^* a_n)\|\|A^* a_n\|\\ &\quad + \|J_Y^{-1}(J_Y w_n + \gamma B^* a_n) - J_Y^{-1}(J_Y w_n)\|\|B^* a_n\|\\ &\leq -\|Az_n - Bw_n\|^2 + \frac{\gamma}{\rho_1}\|A^* a_n\|^2 + \frac{\gamma}{\rho_2}\|B^* a_n\|^2\\ &\leq -\Big(1 - \frac{\gamma(\|A\|^2 + \|B\|^2)}{\rho}\Big)\|Az_n - Bw_n\|^2.\end{aligned} \tag{3.6}$$

Substituting this inequality in (3.5), we get

$$\begin{aligned}\Theta_{n+1}(x^*, y^*) \leq{}& \alpha_n\big(\phi(x^*, x) + \phi(y^*, y)\big)\\ &+ (1 - \alpha_n)\Bigg(\Theta_n(x^*, y^*) + \varrho\beta_n^2\big(\|J_X x_n - J_X x_{n-1}\|^2\\ &+ \|J_Y y_n - J_Y y_{n-1}\|^2\big) + \beta_n\Theta_{n-1}(x_n, y_n)\\ &+ \beta_n\big(\Theta_n(x^*, y^*) - \Theta_{n-1}(x^*, y^*)\big)\\ &- \Big(1 - \frac{\gamma(\|A\|^2 + \|B\|^2)}{\rho}\Big)\|Az_n - Bw_n\|^2\Bigg)\\ \leq{}& \max\Big\{\phi(x^*, x) + \phi(y^*, y), \Theta_n(x^*, y^*)\\ &+ \varrho\beta_n^2\big(\|J_X x_n - J_X x_{n-1}\|^2 + \|J_Y y_n - J_Y y_{n-1}\|^2\big)\\ &+ \beta_n\Theta_{n-1}(x_n, y_n) + \beta_n\big(\Theta_n(x^*, y^*) - \Theta_{n-1}(x^*, y^*)\big)\Big\}.\end{aligned}$$

If $\Theta_{n+1}(x^*, y^*) \leq \phi(x^*, x) + \phi(y^*, y),\ \forall n \geq 1,$ then

$$\phi(x^*, x_{n+1}) \leq \phi(x^*, x) + \phi(y^*, y) \text{ and } \phi(y^*, y_{n+1}) \leq \phi(x^*, x) + \phi(y^*, y).$$

Thus, $\{x_n\}$ and $\{y_n\}$ are bounded. Else, there exists $n_0 \geq 1$ such that for all $n \geq n_0$,

$$\begin{aligned}\Theta_{n+1}(x^*, y^*) \leq{}& \Theta_n(x^*, y^*) + \varrho\beta_n^2\big(\|J_X x_n - J_X x_{n-1}\|^2\\ &+ \|J_Y y_n - J_Y y_{n-1}\|^2\big) + \beta_n\Theta_{n-1}(x_n, y_n)\\ &+ \beta_n\big(\Theta_n(x^*, y^*) - \Theta_{n-1}(x^*, y^*)\big).\end{aligned}$$

By Step 2 of Algorithm 3.1 and Lemma 2.16 we have that $\{\Theta_n(x,y)\}$ is convergent. Consequently, $\{x_n\}$ and $\{y_n\}$ are bounded.

Next, we show $\{(x_n,y_n)\}$ converges strongly to a point in Δ. Let $(x^*,y^*)\in\Delta$. Using Lemma 2.9, inequality (2.2), Lemma 2.13(e), inequality (3.2)

$$\begin{aligned}
\phi(x^*,x_{n+1}) &= V(x^*,\alpha_n J_X x+(1-\alpha_n)Jq_n)\\
&\le V(x^*,\alpha_n J_X x^*+(1-\alpha_n)Jq_n)+2\alpha_n\langle x_{n+1}-x^*,Jx-Jx^*\rangle\\
&\le (1-\alpha_n)\phi(x^*,\mathcal{T}^{F_1}_{\mu_n}p_n)\\
&\quad+2\alpha_n\langle x_{n+1}-x^*,Jx-Jx^*\rangle\\
&\le (1-\alpha_n)\big(\phi(x^*,u_n)-\phi(p_n,u_n)\\
&\quad-\phi(q_n,p_n)\big)+2\alpha_n\langle x_{n+1}-x^*,Jx-Jx^*\rangle\\
&\le (1-\alpha_n)\big(\phi(x^*,x_n)+\rho_3\beta_n^2\|J_X x_n-J_X x_{n-1}\|^2+\beta_n\phi(x_n,x_{n-1})\\
&\quad+\beta_n(\phi(x^*,x_n)-\phi(x^*,x_{n-1}))\\
&\quad-2\gamma\langle A(u_n-x^*),a_n\rangle-\phi(p_n,u_n)-\phi(q_n,p_n)\big)\\
&\quad+2\alpha_n\langle x_{n+1}-x^*,Jx-Jx^*\rangle\\
&\le (1-\alpha_n)\big(\phi(x^*,x_n)+\rho_3\beta_n^2\|J_X x_n-J_X x_{n-1}\|^2+\beta_n\phi(x_n,x_{n-1})\\
&\quad+\beta_n(\phi(x^*,x_n)-\phi(x^*,x_{n-1}))-2\gamma\langle A(u_n-x^*),a_n\rangle\\
&\quad-\phi(p_n,u_n)-\phi(q_n,p_n)\big)\\
&\quad+2\alpha_n\langle x_n-x^*,Jx-Jx^*\rangle+2\alpha_n\|x_{n+1}-x_n\|c_0,\quad\text{for some } c_0>0.
\end{aligned}\tag{3.7}$$

Similarly,

$$\begin{aligned}
\phi(y^*,y_{n+1}) &\le (1-\alpha_n)\big(\phi(y^*,y_n)+\rho_4\beta_n^2\|J_Y y_n-J_Y y_{n-1}\|^2+\beta_n\phi(y_n,y_{n-1})\\
&\quad+\beta_n(\phi(y^*,y_n)-\phi(y^*,y_{n-1}))-2\gamma\langle B(v_n+y^*),a_n\rangle\\
&\quad-\phi(r_n,v_n)-\phi(s_n,r_n)\big)+2\alpha_n\langle y_n-y^*,Jy-Jy^*\rangle\\
&\quad+2\alpha_n\|y_{n+1}-y_n\|c_1,\quad\text{for some } c_1>0,
\end{aligned}\tag{3.8}$$

where $r_n=\mathcal{S}^{K_2}_{\mu_n}v_n$ and $s_n=\mathcal{S}^{K_1}_{\mu_n}r_n$. Adding inequalities (3.7) and (3.8) and using inequality (3.6), we get

$$\begin{aligned}
\Theta_{n+1}(x^*,y^*) &\le (1-\alpha_n)\Theta_n(x^*,y^*)\\
&\quad+\varrho\beta_n^2(1-\alpha_n)(\|J_X x_n-J_X x_{n-1}\|^2+\|J_Y y_n-J_Y y_{n-1}\|^2)\\
&\quad+\beta_n(1-\alpha_n)\Theta_{n-1}(x_n,y_n)+\beta_n(1-\alpha_n)\big(\Theta_n(x^*,y^*)-\Theta_{n-1}(x^*,y^*)\big)\\
&\quad-(1-\alpha_n)\Big(1-\frac{\gamma(\|A\|^2+\|B\|^2)}{\rho}\Big)\|Az_n-Bw_n\|^2\\
&\quad-(1-\alpha_n)\big(\phi(p_n,u_n)\\
&\quad+\phi(q_n,p_n)+\phi(r_n,v_n)+\phi(s_n,r_n)\big)+2\alpha_n\big(\langle x_n-x^*,Jx-Jx^*\rangle\\
&\quad+\langle y_n-y^*,Jy-Jy^*\rangle\big)+2\alpha_n(\|x_{n+1}-x_n\|c_0+\|y_{n+1}-y_n\|c_1)
\end{aligned}\tag{3.9}$$

$$\le (1-\alpha_n)\Theta_n(x^*,y^*)+\varrho\beta_n^2(1-\alpha_n)(\|J_X x_n-J_X x_{n-1}\|^2$$

$$
\begin{aligned}
&+ \|J_Y y_n - J_Y y_{n-1}\|^2) \\
&+ \beta_n(1-\alpha_n)\Theta_{n-1}(x_n, y_n) + \beta_n(1-\alpha_n)\big(\Theta_n(x^*, y^*) - \Theta_{n-1}(x^*, y^*)\big) \\
&+ 2\alpha_n\big(\langle x_n - x^*, Jx - Jx^*\rangle + \langle y_n - y^*, Jy - Jy^*\rangle\big) \\
&+ 2\alpha_n(\|x_{n+1} - x_n\|c_0 + \|y_{n+1} - y_n\|c_1). \quad (3.10)
\end{aligned}
$$

To complete the proof, we will consider the following two cases:

Case 1. Suppose there exists $n_1 \geq 1$ such that $\Theta_{n+1}(x^*, y^*) \leq \Theta_n(x^*, y^*)$, $\forall\, n \geq n_1$. Then, $\{\Theta_n(x^*, y^*)\}$ is convergent. Thus, from inequality (3.9), by rearranging the terms and using the convergence of $\{\Theta_n(x^*, y^*)\}$, boundedness of $\{x_n\}$, $\{y_n\}$, Steps 1 and 2 of Algorithm 3.1, we get that

$$
\begin{aligned}
\lim_{n\to\infty} \|Az_n - Bw_n\| &= \lim_{n\to\infty} \phi(p_n, u_n) = \lim_{n\to\infty} \phi(q_n, p_n) = \lim_{n\to\infty} \phi(r_n, v_n) \\
&= \lim_{n\to\infty} \phi(s_n, r_n) = 0. \quad (3.11)
\end{aligned}
$$

By Lemma 2.12, it follows that

$$
\lim_{n\to\infty} \|p_n - u_n\| = \lim_{n\to\infty} \|q_n - p_n\| = \lim_{n\to\infty} \|r_n - v_n\| = \lim_{n\to\infty} \|s_n - r_n\| = 0.
$$

Furthermore, by the uniform continuity J on bounded sets, we have

$$
\begin{aligned}
\lim_{n\to\infty} \|J_X p_n - J_X u_n\| &= \lim_{n\to\infty} \|J_X q_n - J_X p_n\| = \lim_{n\to\infty} \|J_Y r_n - J_Y v_n\| \\
&= \lim_{n\to\infty} \|J_Y s_n - J_Y r_n\| = 0. \quad (3.12)
\end{aligned}
$$

Observe that by using equation (3.11), we deduce that

$$
\lim_{n\to\infty} \|J_X z_n - J_X u_n\| = \|J_Y w_n - J_Y v_n\| = 0. \quad (3.13)
$$

Also, since $\lim_{n\to\infty} \alpha_n = 0$,

$$
\lim_{n\to\infty} \|J_X p_n - J_X x_{n+1}\| = \lim_{n\to\infty} \|J_Y s_n - J_Y y_{n+1}\| = 0. \quad (3.14)
$$

By equation (3.12), (3.13) and (3.14), we get

$$
\lim_{n\to\infty} \|J_X z_n - J_X x_{n+1}\| = \lim_{n\to\infty} \|J_Y w_n - J_Y y_{n+1}\| = 0.
$$

Hence,

$$
\lim_{n\to\infty} \|J_X x_n - J_X x_{n+1}\| = \lim_{n\to\infty} \|J_Y y_n - J_Y y_{n+1}\| = 0.
$$

Thus,

$$
\lim_{n\to\infty} \|x_n - x_{n+1}\| = \lim_{n\to\infty} \|y_n - y_{n+1}\| = 0. \quad (3.15)
$$

The next step is to show that the solution set Δ is contained in the set of weak subsequential limit $\Omega_\omega(x_n, y_n)$. However, the proof is standard we will not include the proof here to avoid unnecessary repetitions (see, e.g. [20,21] for a proof of this).

Now, we show that $\{(x_n, y_n)\}$ converges strongly to the point $(x^*, y^*) = \mathcal{P}_\Delta(x, y)$, where

$$
\mathcal{P}_\Delta(x, y) = \Big(\Pi_{\Delta_X} x, \Pi_{\Delta_Y} y\Big), \ \Delta_X = \bigcap_{i=1}^{2} F_{J_X}(F_i) \ \text{ and } \ \Delta_Y = \bigcap_{i=1}^{2} K_{J_Y}(F_i).
$$

Let (u,v) be a weak limit of $\{(x_n,y_n)\}$. Then, there exists $\{(x_{n_j},y_{n_j})\}\subset\{(x_n,y_n)\}$ such that

$$\limsup_{n\to\infty}\langle x_n-x^*,J_Xx-J_Xx^*\rangle=\lim_{j\to\infty}\langle x_{n_j}-x^*,J_Xx-J_Xx^*\rangle=\langle u-x^*,J_Xx-J_Xx^*\rangle \tag{3.16}$$

and

$$\limsup_{n\to\infty}\langle y_n-y^*,J_Yy-J_Yy^*\rangle=\lim_{j\to\infty}\langle y_{n_j}-y^*,J_Yy-J_Yy^*\rangle=\langle v-y^*,J_Yy-J_Yy^*\rangle. \tag{3.17}$$

Since $(x^*,y^*)=\Big(\Pi_{\Delta_X}x,\Pi_{\Delta_Y}y\Big)$ and $(u,v)\in\Delta$, by Lemma 2.11, we have that

$$\langle u-x^*,J_Xx-J_Xx^*\rangle\le 0 \text{ and } \langle v-y^*,J_Yy-J_Yy^*\rangle\le 0.$$

From inequality (3.10), using the condition of Case 1, we have

$$\begin{aligned}
\Theta_{n+1}(x^*,y^*)&\le(1-\alpha_n)\Theta_n(x^*,y^*)\\
&\quad+\varrho\beta_n^2(1-\alpha_n)(\|J_Xx_n-J_Xx_{n-1}\|^2+\|J_Yy_n-J_Yy_{n-1}\|^2)\\
&\quad+\beta_n(1-\alpha_n)\Theta_{n-1}(x_n,y_n)+\beta_n(1-\alpha_n)\big(\Theta_n(x^*,y^*)-\Theta_{n-1}(x^*,y^*)\big)\\
&\quad+2\alpha_n\big(\langle x_n-x^*,Jx-Jx^*\rangle+\langle y_n-y^*,Jy-Jy^*\rangle\big)\\
&\quad+2\alpha_n(\|x_{n+1}-x_n\|c_0+\|y_{n+1}-y_n\|c_1)\\
&\le(1-\alpha_n)\Theta_n(x^*,y^*)\\
&\quad+\varrho\beta_n^2(1-\alpha_n)(\|J_Xx_n-J_Xx_{n-1}\|^2+\|J_Yy_n-J_Yy_{n-1}\|^2)\\
&\quad+\beta_n(1-\alpha_n)\Theta_{n-1}(x_n,y_n)\\
&\quad+2\alpha_n\big(\langle x_n-x^*,Jx-Jx^*\rangle+\langle y_n-y^*,Jy-Jy^*\rangle\big)\\
&\quad+2\alpha_n(\|x_{n+1}-x_n\|c_0+\|y_{n+1}-y_n\|c_1).
\end{aligned}$$

Using Steps 1 and 2 of Algorithm 3.1, equation (3.15), inequalities (3.16) and (3.17) it follows by Lemma 2.15 that

$$\lim_{n\to\infty}\Theta_n(x^*,y^*)=0.\ \text{ Thus, }\ \lim_{n\to\infty}\phi(x^*,x_n)=\lim_{n\to\infty}\phi(y^*,y_n)=0.$$

Therefore, by Lemma 2.12, $\lim_{n\to\infty}x_n=x^*$ and $\lim_{n\to\infty}y_n=y^*$.

Case 2. If Case 1 does not hold, since $\{\Theta_n(x^*,y^*)\}\subset\mathbb{R}$ and every sequence in $\mathbb{R}$ has a monotone subsequence, there exists $\{\Theta_{n_j}(x^*,y^*)\}\subset\{\Theta_n(x^*,y^*)\}$ with $\Theta_{n_j+1}(x^*,y^*)>\Theta_{n_j}(x^*,y^*)$, for all $j\in\mathbb{N}$. By Lemma 2.14, there exists a nondecreasing sequence $\{m_j\}\subset\mathbb{N}$ such that $\lim_{j\to\infty}m_j=\infty$ and the following inequalities hold:

$$\Theta_{m_j}(x^*,y^*)\le\Theta_{m_j+1}(x^*,y^*)\ \text{ and }\ \Theta_j(x^*,y^*)\le\Theta_{m_j+1}(x^*,y^*),\ \forall j\in\mathbb{N}.$$

By replacing n by m_j and rearranging the terms in inequality (3.7), following a similar argument as in Case 1 above, we obtain that

$$\begin{aligned}
\lim_{j\to\infty}\|Az_{m_j}-Bw_{m_j}\|&=\lim_{j\to\infty}\|J_Xp_{m_j}-J_Xu_{m_j}\|=\lim_{j\to\infty}\|J_Xq_{m_j}-J_Xp_{m_j}\|\\
&=\lim_{j\to\infty}\|J_Yr_{m_j}-J_Yv_{m_j}\|=\lim_{j\to\infty}\|J_Ys_{m_j}-J_Yr_{m_j}\|
\end{aligned}$$

$$= \lim_{j\to\infty} \|x_{m_j} - x_{m_j+1}\| = \lim_{j\to\infty} \|y_{m_j} - y_{m_j+1}\| = 0. \tag{3.18}$$

Furthermore, using similar arguments as in Case 1, we get

$$\limsup_{n\to\infty} \langle x_{m_j} - x^*, J_X x - J_X x^*\rangle \le 0, \quad \limsup_{n\to\infty} \langle y_{m_j} - y^*, J_Y y - J_Y y^*\rangle \le 0 \tag{3.19}$$

Moreover, from inequality (3.10),

$$\begin{aligned}\Theta_{m_j+1}(x^*, y^*) \le{}& (1-\alpha_{m_j})\Theta_{m_j}(x^*, y^*)\\ &+ \varrho\beta_{m_j}^2(1-\alpha_{m_j})(\|J_X x_{m_j} - J_X x_{n-1}\|^2 + \|J_Y y_{m_j} - J_Y y_{m_j-1}\|^2)\\ &+ \beta_{m_j}(1-\alpha_{m_j})\Theta_{n-1}(x_{m_j}, y_{m_j})\\ &+ \beta_{m_j}(1-\alpha_{m_j})\big(\Theta_{m_j}(x^*, y^*) - \Theta_{m_j-1}(x^*, y^*)\big)\\ &+ 2\alpha_{m_j}\big(\langle x_{m_j} - x^*, Jx - Jx^*\rangle + \langle y_{m_j} - y^*, Jy - Jy^*\rangle\big)\\ &+ 2\alpha_{m_j}(\|x_{m_j+1} - x_n\|c_0 + \|y_{m_j+1} - y_n\|c_1).\end{aligned} \tag{3.20}$$

Using Steps 1 and 2 of Algorithm 3.1, equation (3.18), inequalities (3.19) and (3.17) it follows by Lemma 2.15 that

$$\lim_{j\to\infty} \Theta_{m_j+1}(x^*, y^*) = 0.$$

Since

$$\Theta_j(x^*, y^*) \le \Theta_{m_j+1}(x^*, y^*), \ \limsup_{j\to\infty} \Theta_j(x^*, y^*) = 0.$$

Thus,

$$\lim_{j\to\infty} \phi(x*, x_j) = \phi(y^*, y_j) = 0.$$

Therefore, by Lemma 2.12 $\lim_{j\to\infty} x_j = x^*$ and $\lim_{j\to\infty} y_j = y^*$. This and the conlusion obtained in Case 1 completes the proof.

□

Corollary 3.3. *Algorithm 3.1 can be extended to a finite family of mappings by letting i used in the setting of the algorithm to be $i = 1, 2, \ldots, m$, for some $m \ge 3$.*

4. Numerical description

In this section, we give a numerical description on how to implement our proposed inertial algorithm using MATLAB, on the classical 2-uniformly convex and uniformly smooth real Banach space $L_{\frac{3}{2}}([-2, 2])$ with dual space $L_3([-2, 2])$. By Alber and Ryazantseva [4] p. 36, the normalized duality map $J_{\frac{3}{2}}$ and its inverse J_3 are computed as follows:

$$J_{\frac{3}{2}} z(t) = \|z\|_{L_{\frac{3}{2}}}^{0.5} |z(t)|^{-0.5} z(t), \quad \text{and} \quad J_3 z(t) = \|z\|_{L_3}^{-1} |z(t)| z(t), \ t \in [-2, 2],$$

$$\text{where } \|z\|_{L_p} = \Big(\int_{-1}^{1} |z(t)|^p\Big)^{\frac{1}{p}}, \ L_p := L_p([-2, 2]), \ p > 1.$$

Furthermore, we shall describe how to compute the resolvent operator which we use in Step 4 of our Algorithm 3.1 before we choose the control parameters. By a result of Chidume and Idu [14], we deduce that a mapping $\mathcal{A} : L_{\frac{3}{2}}([-2, 2]) \to L_3([-2, 2])$

is monotone if and only if $J-\mathcal{A}$ is pseudocontractive. Let $\mathcal{A},\mathcal{B}:L_{\frac{3}{2}}([-2,2])\to L_3([-2,2])$ be defined by

$$\mathcal{A}z(t)=Jz(t) \quad \text{and} \quad \mathcal{B}z(t)=(1+t)Jz(t), \quad \text{respectively}.$$

It is not difficult to show that $\mathcal{A}$ and $\mathcal{B}$ are monotone continuous. Define

$$F_1=J-\mathcal{A},\ F_2=J-\mathcal{B},\ K_1=J-\mathcal{B} \ \text{ and } \ K_2=J-\mathcal{A}.$$

Therefore, F_1,F_2,K_1 and K_2 are continuous J-pseudocontractions with the solution set $\Delta=\{(0,0)\}$. Furthermore, from Lemma 2.13,

$$\mathcal{T}_r^T(x):=\{z\in C:\langle w-z,Tz\rangle-\frac{1}{r}\langle w-z,(1+r)Jz-Jx\rangle\le 0,\ \forall w\in C\},\ x\in X.$$

Thus,

$$\mathcal{T}_\mu^{F_1}x(t)=\frac{x(t)}{1+\mu},\ \mathcal{T}_\mu^{F_2}x(t)=\frac{x(t)}{1+t\mu} \ \text{ and } \ \mathcal{T}_\mu^{F_1}\circ\mathcal{T}_\mu^{F_2}x(t)=\frac{x(t)}{(1+\mu)(1+t\mu)}.$$

Also,

$$\mathcal{S}_\mu^{K_1}x(t)=\frac{x(t)}{1+t\mu},\ \mathcal{S}_\mu^{K_2}x(t)=\frac{x(t)}{1+\mu} \ \text{ and } \ \mathcal{S}_\mu^{K_1}\circ\mathcal{S}_\mu^{K_2}x(t)=\frac{x(t)}{(1+\mu)(1+t\mu)}.$$

Having established the computational values of these functions, we are ready to implement our proposed algorithm.

In Algorithm 3.1, set $X=Y=Z=L_{\frac{3}{2}}([-2,2])$. Let $A:X\to Z$ and $B:Y\to Z$ be define by

$$Ax(t)=2x(t) \ \text{ and } \ Bx(t)=x(t). \ \text{ Then } \ A^*=A \ \text{ and } \ B^*=B.$$

For the control parameters, we choose $\alpha_n=\frac{1}{100n}$, $\gamma_n=\mu_n=0.1$, $x=\sin t$ and $y=\cos t$. From Step 2 of Algorithm 3.1, since $\beta_n\le\bar{\beta_n}\le\beta$, we choose $\beta=0.5$ and set $\beta_n=0.00001$. For the integration in MATLAB, we use the trapezoidal rule with domain of integration '-2:0.1:2'. We terminate the iteration process when $\|x_{n+1}-0\|+\|y_{n+1}-0\|<10^{-7}$ or $n>10$. Below is table of the numerical performance of our proposed algorithm with different initial points.

Table of values choosing $x_0(t)=2t$, $x_1(t)=t$, $y_0(t)=\sin t$ and $y_1(t)=t$

	Algorithm 3.1	
n	$\|x_n-0\|$	$\|y_n-0\|$
0	5.4826	1.8728
1	2.7413	2.7413
2	0.4857	0.5421
3	0.0225	0.0280
4	3.2E-04	1.79E-04
5	9.32E-7	1.24E-06

Table of values choosing $x_0(t) = t^2$, $x_1(t) = Exp(t)$, $y_0(t) = 2t + \sin t$ and $y_1(t) = \frac{1}{2+\cos t}$		
	Algorithm 3.1	
n	$\|x_n - 0\|$	$\|y_n - 0\|$
0	4.0266	7.3382
1	5.6577	1.0781
2	1.0715	0.135
3	0.0633	0.0016
4	2.22E-04	2.45E-06
5	9.22E-07	1.25E-06

5. Conclusion

This paper presents an inertial Halpern-type algorithm for approximating solutions of the split common equality fixed point problem involving continuous J-pseudocontractions. Without any compactness-type requirements on the operators as it was the case in [15, 16, 24, 29]. The sequence generated by the algorithm is proved to converge strongly to a solution of the SECFPP (1.6). Numerical implementation of the proposed algorithm is presented in the setting of the classical Banach space $L_{\frac{3}{2}}([-2,2])$. The proposed algorithm appears to be robust because it converges in few iterations even as we vary the initial points. Finally, the numerical implementation of the proposed algorithm in $L_{\frac{3}{2}}([-2,2])$ shows that the problem studied by Nnakwe et al. [21] is interesting and their proposed algorithm is implementable.

References

[1] A. Adamu and A. A. Adam, *Approximation of solutions of split equality fixed point problems with applications*, Carpathian J. Math. **37** (2021), 381–392.

[2] A. Adamu, D. Kitkuan, P. Kumam, A. Padcharoen, and T. Seangwattana, *Approximation method for monotone inclusion problems in real Banach spaces with applications,* J. Inequal. Appl. **70** (2022), 1–20.

[3] Ya. Alber, *Metric and generalized projection operators in Banach spaces: properties and applications*, in: Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Kartsatos, A.G. (ed.), Marcel Dekker, New York, 1996.

[4] Ya. Alber and I. Ryazantseva, *Nonlinear Ill Posed Problems of Monotone Type*, Springer, London, 2006.

[5] S. M. Alsulami, A. Latif and W. Takahashi The split common fixed point problem and strong convergence theorems by hybrid methods for new demimetric mappings in Hilbert spaces, Appl. Anal. Optim. **2** (2018), 11–26.

[6] F. Alvarez, *Weak convergence of a relaxed and inertial hybrid projection-proximal point algorithm for maximal monotone operators in Hilbert space*, SIAM J. Optim. **14** (2004), 773–782.

[7] C. Byrne, *Iterative oblique projection onto convex sets and the split feasibility problem*, Inverse Problems **18** (2002), 441–453.

[8] Y. Censor and T. Elfving, *A multiprojection algorithm using Bregman projections in a product space,* Numer. Algor. **8** (1994), 221–239.

[9] Y. Censor and A. Segal, *The split common fixed point problem for directed operators,* J. Convex Anal. **16** (2009), 587–600.

[10] S. Chang and R. P. Agarwal, *Strong convergence theorems of general split equality problems for quasi-nonexpansive mappings,* J. Inequal. Appl. (2014), 1–14.

[11] C. E. Chidume and A. Adamu, *On split generalized mixed equality equilibrium and split equality fixed point problems,* Appl. Set-Valued Anal. Optim. **2** (2020), 273–283.

[12] C. E. Chidume and A. Adamu, *A new iterative algorithm for split feasibility and fixed point problem,* J. Nonlinear Var. Anal. **5** (2021), 201–210.

[13] C. E. Chidume and A. Adamu, *Solving split equality fixed point problem for quasi-phi-nonexpansive mappings,* Thai J. Math. **19** (2021), 1699–1717.

[14] C. E. Chidume and K. O. Idu, *Approximation of zeros of bounded maximal monotone maps, solutions of Hammerstein integral equations and convex minimization problems*, Fixed Point Theory Appl. **2016** (2016): Article number 97.

[15] C.E. Chidume, P. Ndambomve and A.U. Bello, *The split equality fixed point problem for demicontractive mappingpings*, J. Nonlinear Anal. Optim. **6** (2015), 61–69.

[16] C. E. Chidume, O. M. Romanus and U. V. Nyaba, *An iterative algorithm for solving split equality fixed point problems for a class of nonexpansive-type mappings in Banach spaces*, Numer. Algor. **82** (2019), 987–1007.

[17] S. Kamimura and W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space*, SIAM J. Optim. **13** (2002), 938–945.

[18] A. Moudafi, *A relaxed alternating CQ-algorithm for convex feasibility problems,* Nonlinear Anal. Theory, Methods & Appl. **79** (2013), 117–121.

[19] A. Moudafi, *Alternating CQ-algorithm for convex feasibility and split fixed-point problems*, J. Nonlinear Convex Anal. **15** (2014), 809–818.

[20] M. O. Nnakwe, *An algorithm for approximating a common solution of variational inequality and convex minimization problems,* Optimization **21** (2020), 1–20.

[21] M. O. Nnakwe, T. I. Aroh and F. A. Yakasai *Approximating a common solution of split equality fixed point problems in Banach spaces with applications*, Thai J. Math. (Accepted).

[22] Y. Shehu, O. S. Iyiola, and C. D. Enyi, *An iterative algorithm for solving split feasibility problems and fixed point problems in Banach spaces,* Numer. Algor. **72** (2016), 835–864.

[23] Y. Shehu and O. S. Iyiola, *Strong convergence result for proximal split feasibility problem in Hilbert spaces*, Optimization, **66** (2017), 2275-2290.

[24] Y. Shehu, F. U. Ogbuisi and O. S. Iyiola, *Strong convergence theorem for solving split equality fixed point problem which does not involve the prior knowledge of operator norms,* Bull Iranian Math. Soc. **43** (2017), 349–371.

[25] S. Suantai, Y. Shehu, P. Cholamjiak and O. S. Iyiola, *Strong convergence of a self-adaptive method for the split feasibility problem in Banach spaces*, J. Fixed Point Theory Appl. **20** (2018), 1–21.

[26] H. K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal. **16** (1991), 1127–1138.

[27] H. K. Xu, *Iterative algorithms for nonlinear operators*, J. Lond. Math. Soc. **66** (2002), 240–256.

[28] Q. Yang, *The relaxed CQ algorithm for solving the split feasibility problem,* Inverse Problems **20** (2004), 1261–1266.

[29] J. Zhao, *Solving split equality fixed-point problem of quasi-nonexpansive mappings without prior knowledge of operators norms*, Optimization, 2014, DOI:10.1080/02331934.2014.883515.

[30] J. Zhao, H. Zong, G. Liu and H. Zhang, *Solving variational inequality and split equality common fixed-point problem without prior knowledge of operator norms*, J. Nonlinear Sci. Appl. **9** (2016), 5428–5440.

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