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A CERTAIN GENERAL FAMILY OF SUBCLASSES OF MEROMORPHICALLY MULTIVALENT FUNCTIONS ASSOCIATED WITH THE HYPERGEOMETRIC FUNCTION

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Abstract. In this article, the authors make appropriate use of the Choi-Saigo-Srivastava operator for meromorphic functions in order to introduce and investigate the following two general subclasses:

$$\mathcal{MSK}_{p,k,\nu}^{\alpha.e_1}(q,s,\xi:w) \qquad \text{and} \qquad \mathcal{MC}_{p,k,\nu}^{\alpha,e_1}(q,s,\eta,\xi,w,\phi)$$

of meromorphically multivalent (p-valent) functions in the punctured unit disk, which have a pole of order p at the origin ($z=0$). Inclusion properties and other results for each of these subclasses, which are associated with an integral operator F_μ, are presented. Some sufficient conditions for $F_\mu f$, in which the function f belongs to the aforementioned subclasses, to be member of the subclasses:

$$\mathcal{MSK}_{p,k,\nu}^{\alpha,e_1}(q,s,\xi:w) \qquad \text{and} \qquad \mathcal{MC}_{p,k,\nu}^{\alpha,e_1}(q,s,\eta,\xi,w,\phi)$$

are established. Relevant connections of the developments reported here with those in some earlier works on the subject are also considered briefly.

1. Introduction, definitions and preliminaries

We denote by $\mathcal{M}_{p,k}$ the class of functions of the form:

$$f(z) = z^{-p} + \sum_{m=k}^{\infty} a_m z^m \qquad (p \in \mathbb{N} = \{1,2,3,\cdots\};\ 1-p \leqq k \in \mathbb{Z}), \tag{1.1}$$

which are analytic and p-valent in the *punctured* unit disk

$$\mathbb{U}^* = \{z : z \in \mathbb{C} \quad \text{and} \quad 0 < |z| < 1\} = \mathbb{U} \setminus \{0\},$$

$\mathbb{Z}$ being, as usual, the set of integers. We also set $\mathcal{M}_{p,1-p} =: \mathcal{M}_p$.

For $0 \leqq \tau < p$, we denote by

$$\mathcal{M}_S(p;\tau), \quad \mathcal{M}_K(p;\tau) \qquad \text{and} \qquad \mathcal{M}_C(p;\tau)$$

the subclasses of $\mathcal{M}_p$ consisting of all meromorphic functions which are, respectively, starlike, convex and close-to-convex of order τ.

If f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g, written $f \prec g$ or (more precisely)

$$f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

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if there exists a Schwarz function ω, analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \qquad \text{and} \qquad |\omega(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = g\big(\omega(z)\big) \qquad (z \in \mathbb{U}).$$

In particular, if g is univalent in $\mathbb{U}$, then we have the following equivalence (see, for example, [16]; see also [17]):

$$f(z) \prec g(z) \;\; (z \in \mathbb{U}) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

For a function $f \in \mathcal{M}_{p,k}$, given by (1.1), and $g \in \mathcal{M}_{p,k}$ defined by

$$g(z) = z^{-p} + \sum_{m=k}^{\infty} b_m z^m \qquad (p \in \mathbb{N};\; 1-p \leqq k \in \mathbb{Z}),$$

we define the Hadamard product (or convolution) of f and g by

$$f(z) * g(z) = (f * g)(z) = z^{-p} + \sum_{m=k}^{\infty} a_m b_m z^m \quad (p \in \mathbb{N};\; 1-p \leqq k \in \mathbb{Z}).$$

For real or complex numbers

$$e_1, e_2, \cdots, e_q \quad \text{and}$$

$$d_1, d_2, \cdots, d_s \;\; \big(d_j \notin \mathbb{Z}_0^- = \{0, -1, -2, \cdots\} := \mathbb{Z} \setminus \mathbb{N};\; j = 1, 2, \cdots, s\big),$$

we consider the generalized hypergeometric function ${}_qF_s$ defined as follows (see, for example, [25, p. 19]) :

$$(1.2) \qquad {}_qF_s(e_1, \cdots, e_q; d_1, \cdots, d_s; z) = \sum_{k=0}^{\infty} \frac{(e_1)_m \cdots (e_q)_m}{(d_1)_m \cdots (d_s)_m} \, \frac{z^k}{k!}$$

$$(q \leqq s+1;\; q, s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\; z \in \mathbb{U}),$$

where $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ denotes the general Pochhammer symbol or the *shifted factorial*, since

$$(1)_n = n! \qquad (n \in \mathbb{N}_0),$$

which is defined, in terms of the familiar (Euler's) Gamma function Γ, by

$$(1.3) \quad (\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\; \lambda \in \mathbb{C}), \end{cases}$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ-quotient exists.

Corresponding to the function $\mathfrak{F}_p(e_1, \cdots, e_q; d_1. \cdots, d_s; z)$ given by

$$(1.4) \qquad \mathfrak{F}_p(e_1, \cdots, e_q; d_1, \cdots, d_s; z) := z^{-p} \, {}_qF_s(e_1, \cdots, e_q; d_1, \cdots, d_s; z),$$

here we first introduce a function $\mathfrak{F}_{p,\alpha}(e_1, \cdots, e_q; d_1, \cdots, d_s; z)$ defined by the following convolution:

$$\mathfrak{F}_p(e_1, \cdots, e_q; d_1, \cdots, d_s; z) * \mathfrak{F}_{p,\alpha}(e_1, \cdots, e_q; d_1, \cdots, d_s; z)$$

$$(1.5) \qquad = \frac{1}{z^p(1-z)^{\alpha+p}} \qquad (\alpha > -p;\; z \in \mathbb{U}^*).$$

We then define a linear operator:

$$\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1,\cdots,e_q;d_1,\cdots,d_s):\mathcal{M}_{p,k}\longrightarrow\mathcal{M}_{p,k}$$

by

$$\begin{aligned}\mathcal{Q}_{p,q,s}^{k,\alpha}&(e_1,\cdots,e_q;d_1,\cdots,d_s)f(z)\\&=\mathfrak{F}_{p,\alpha}(e_1,\cdots,e_q;d_1,\cdots,d_s;z)*f(z)\end{aligned}\tag{1.6}$$

$$\big(e_i,d_j\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ i=1,2\cdots,q;\ j=1,2,\cdots,s;\\ \alpha>-p;\ f\in\mathcal{M}_{p,k};\ z\in\mathbb{U}^*\big).$$

We also make use of the following notational abbreviations and conventions:

$$\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1,\cdots,e_q;d_1,\cdots,d_s)=:\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)$$

and

$$\mathcal{Q}_{p,q,s}^{1-p,\alpha}(e_1)=:\mathcal{Q}_{p,q,s}^{\alpha}(e_1).$$

Remark 1.1. The usage of the generalized hypergeometric function ${}_qF_s$, defined by (1.2), in Geometric Function Theory of Complex Analysis was initiated by Owa and Srivastava [19] in their systematic study of univalent and starlike generalized hypergeometric functions. Subsequently, the widely-investigated Dziok-Srivastava convolution operator, which is based upon the generalized hypergeometric function ${}_qF_s$, defined by (1.2), was used by Dziok and Srivastava (see, for example, [5] and [6]; see also [30]). More recently, a much more general Fox-Wright function than the generalized hypergeometric function ${}_qF_s$, defined by (1.2), was used in order to introduce and investigate what is popularly known as the Srivastava-Wright convolution operator (see, for details, [21]; see also [11], and the recent works [22] and [23]) for other interesting usages and applications of such general families of higher transcendental functions.

Now, by using (1.1) and (1.6), we can write

$$\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)f(z)=z^{-p}+\sum_{k=m}^{\infty}\frac{(\alpha+p)_{p+k}(d_1)_{p+k}\cdots(d_s)_{p+k}}{(e_1)_{p+k}\cdots(e_q)_{p+k}}a_kz^k\tag{1.7}$$

$$(\alpha>-p;\ z\in\mathbb{U}^*).$$

We can also easily verify each of the following two relations on $z\in\mathbb{U}^*$ by using (1.7):

$$\begin{aligned}z\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)f\right)'(z)&=(\alpha+p)\mathcal{Q}_{p,q,s}^{k,\alpha+1}(e_1)f(z)\\&\quad-(\alpha+2p)\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)f(z)\end{aligned}\tag{1.8}$$

and

$$\begin{aligned}z\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1+1)f\right)'(z)&=e_1\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)f(z)\\&\quad-(p+e_1)\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1+1)f(z).\end{aligned}\tag{1.9}$$

We note that the linear operator $\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)$ is closely related to the Choi-Saigo-Srivastava operator (see, for details, [4]) for analytic functions and is essentially motivated by the operators defined and studied in [3]. The linear operator $\mathcal{Q}^{0,\mu}_{1,q,s}(\alpha_1)$ was investigated by Cho and Kim [2], where as

$$\mathcal{Q}^{1-p}_{p,2,1}(c,1;a;z) =: \mathcal{L}_p(a,c) \qquad (c \in \mathbb{R};\ a \notin \mathbb{Z}_0^-)$$

is the operator which was introduced and studied by Liu and Srivastava [13].

Remark 1.2. The operator $\mathcal{Q}^{k,\alpha}_{p,q,s}(e_1)$ is a generalization of several known operators in earlier works, some of which are being recalled below.

1. $\mathcal{Q}^{k,0}_{p,s+1,s}(p+1, d_1, \cdots, d_s; d_1, d_2, \cdots, d_s)f(z) = \dfrac{p}{z^{2p}} \displaystyle\int_0^z t^{2p-1} f(t)\, \mathrm{d}t.$
2. $\mathcal{Q}^{k,0}_{p,s+1,s}(p, d_1, \cdots, d_s; d_1, d_2, \cdots, d_s)f(z)$
$$= \mathcal{Q}^{k,1}_{p,s+1,s}(p+1, d_1, \cdots, d_s; d_1, d_2, \cdots, d_s)f(z)$$
$$= f(z).$$
3. $\mathcal{Q}^{k,1}_{p,s+1,s}(p, d_1, \cdots, d_s; d_1, d_2, \cdots, d_s)f(z) = \dfrac{zf'(z) + 2pf(z)}{p}.$
4. $\mathcal{Q}^{m,2}_{p,s+1,s}(p+1, d_1, \cdots, d_s; d_1, d_2, \cdots, d_s)f(z) = \dfrac{zf'(z) + (2p+1)f(z)}{p+1}.$
5. $\mathcal{Q}^{1-p,n}_{p,s+1,s}(d_1, d_2, \cdots, d_s, 1; d_1, d_2, \cdots, d_s)f(z) = \dfrac{1}{z^p(1-z)^{n+p}} = \mathcal{D}^{n+p-1}f(z)$
$(-p < n \in \mathbb{N})$ (see [8]).
6. $\mathcal{Q}^{k,1-p}_{p,s+1,s}(\delta+1, d_2, \cdots, d_s, 1; \delta, d_2, \cdots, d_s)f(z) = \dfrac{\delta}{z^{\delta+p}} \displaystyle\int_0^z t^{\delta+p-1} f(t)\, \mathrm{d}t$
$(\delta > 0; z \in \mathbb{U}^*)$ (see [3] and [8]).
7. $\mathcal{Q}^{k,\alpha}_{p,2,1}(e_1)f(z) = z^{-p} + \displaystyle\sum_{k=m}^{\infty} \frac{(\alpha+p)_{p+k}(d_1)_{p+k}}{(e_1)_{p+k}(e_2)_{p+k}} a_k z^k$

(see [18] for $e_2 = 1$, $\alpha + p = \mu$, $d_1 = \alpha + \beta$ and $e_1 = \beta$).

Let $\mathcal{P}$ be the class of all functions w which are analytic and univalent in $\mathbb{U}$, and for which $w(\mathbb{U})$ is convex with

$$w(0) = 1 \qquad \text{and} \qquad \Re\{w(z)\} > 0 \quad (z \in \mathbb{U}).$$

Next, by making use of the linear operator $\mathcal{Q}^{k,\alpha}_{p,q,s}(e_1)$, we introduce the following subclasses of the meromorphically multivalent (p-valent) function class $\mathcal{M}_{p,k}$.

Definition 1.3. A function $f \in \mathcal{M}_{p,k}$ is said to be in the meromorphically multivalent (p-valent) function class $\mathcal{MSK}^{\alpha,e_1}_{p,k,\nu}(q,s,\xi : w)$ if f satisfies the following subordination condition:

$$-\frac{1}{p-\xi}\left(\frac{z\left(\mathcal{Q}^{k,\alpha}_{p,q,s}(e_1)f\right)'(z) + \nu z^2\left(\mathcal{Q}^{k,\alpha}_{p,q,s}(e_1)f\right)''(z)}{(1-\nu)\mathcal{Q}^{k,\alpha}_{p,q,s}(e_1)f(z) + \nu z\left(\mathcal{Q}^{k,\alpha}_{p,q,s}(e_1)f\right)'(z)} + \xi\right) \prec w(z) \tag{1.10}$$

$$(w \in \mathcal{P};\ 0 \leqq \xi < p;\ 0 \leqq \nu \leqq 1;\ \alpha > -p;\ z \in \mathbb{U}).$$

Definition 1.4. A function $f \in \mathcal{M}_{p,k}$ is said to be in the meromorphically multivalent (p-valent) function class $\mathcal{MC}^{\alpha,e_1}_{p,k,\nu}(q,s,\eta,\xi,w,\phi)$ if there exists another function $g \in \mathcal{MSK}^{\alpha,e_1}_{p,k,\nu}(q,s,\xi : w)$ such that the following subordination condition holds true:

$$-\frac{1}{p-\eta}\left(\frac{z\left(\mathcal{Q}^{k,\alpha}_{p,q,s}(e_1)f\right)'(z)+\nu z^2\left(\mathcal{Q}^{k,\alpha}_{p,q,s}(e_1)f\right)''(z)}{(1-\nu)\mathcal{Q}^{k,\alpha}_{p,q,s}(e_1)g(z)+\nu z\left(\mathcal{Q}^{k,\alpha}_{p,q,s}(e_1)g\right)'(z)}+\eta\right) \prec \phi(z) \tag{1.11}$$

$$(\phi \in \mathcal{P};\ 0 \leqq \eta < p;\ 0 \leqq \nu \leqq 1;\ \alpha > -p;\ z \in \mathbb{U}).$$

Remark 1.5. By choosing several particular values of the parameters involved in Definition 1.3 and Definition 1.4 above, we get several known function classes which are recorded below.

1. $\mathcal{MSK}^{\alpha,e_1}_{p,k,0}(q,s,\xi : w) = \mathcal{MS}^{\alpha.e_1}_{p,k}(q,s,\xi : w)$ (Patel and Palit [20]).

2. $\mathcal{MS}^{\alpha,e_1}_{p,k,1}(q,s,\xi : w) = \mathcal{MK}^{\alpha.e_1}_{p,k}(q,s,\xi : w)$.

3. $\mathcal{MS}^{\alpha,e_1}_{1,0}(q,s,\xi : w) = \mathcal{MS}^{\alpha.e_1}(q,s,\xi : w)$ (Cho and Kim [2]).

4. $\mathcal{MK}^{\alpha,e_1}_{1,0}(q,s,\xi : w) = \mathcal{MK}^{\alpha.e_1}(q,s,\xi : w)$ (Cho and Kim [2]).

5. $\mathcal{MSK}^{\alpha,e_1}_{p,k,\nu}\left(q,s,\xi : \dfrac{1+Az}{1+Bz}\right) = \mathcal{MS}^{\alpha,e_1}_{p,k}(q,s,\xi : A,B)$
$(-1 < B < A \leqq 1)$.

6. $\mathcal{MC}^{\alpha,e_1}_{p,k,\nu}\left(q,s,\eta,\xi;\dfrac{1+Az}{1+Bz}\right)(q,s,\eta,\xi : A,B)$
$(-1 < B < A \leqq 1)$.

7. $\mathcal{MS}^{\alpha,e_1}_{p,k}(2,1,\xi;w) = \Sigma_p S^{\alpha}_{e_1,\alpha+p}(\xi;w)$ (Mostafa [18]).

8. $\mathcal{MK}^{\alpha,e_1}_{p,k}(2,1,\xi;w) = \Sigma_p K^{\alpha}_{e_1,\alpha+p}(\xi;w)$ (Mostafa [18]).

9. $\mathcal{MC}^{\alpha,e_1}_{p,k,0}(2,1,\xi,\eta;w,\phi) = \Sigma_p C^{\alpha}_{e_1,\alpha+p}(\xi,\eta;w,\phi)$ (Mostafa [18]).

10. $\mathcal{MC}^{\alpha.e_1}_{p,k,1}(2,1,\xi,\eta;w,\phi) = \Sigma_p C^{\alpha\star}_{e_1,\alpha+p}(\xi,\eta;w,\phi)$ (Mostafa [18]).

We need each of the following lemmas in our present investigation.

Lemma 1.6. (see [7]) *Let the function w be convex univalent in $\mathbb{U}$ with*

$$w(0) = 1 \qquad \text{and} \qquad \Re\{\zeta w(z)+\eta\} > 0 \quad (\eta,\zeta \in \mathbb{C}).$$

If the function ϕ is analytic in $\mathbb{U}$ with $\phi(0) = 1$, then the following subordination:

$$\phi(z) + \frac{z\phi'(z)}{\zeta\phi(z)+\eta} \prec w(z) \qquad (z \in \mathbb{U})$$

implies that

$$\phi(z) \prec w(z) \qquad (z \in \mathbb{U}).$$

Lemma 1.7. (see [16]) *Let $\mathfrak{w}$ be a convex univalent function in $\mathbb{U}$. Also let the function $\mathfrak{p}$ be analytic in $\mathbb{U}$ with*

$$\Re\{\mathfrak{p}(z)\} > 0 \qquad (z \in \mathbb{U}).$$

If the function $\mathfrak{w}$ is analytic in $\mathbb{U}$ and $\mathfrak{p}(0) = \mathfrak{w}(0)$, then the following subordination:

$$\mathfrak{w}(z) + z\mathfrak{p}(z)\, \mathfrak{w}'(z) \prec \mathfrak{w}(z) \qquad (z \in \mathbb{U})$$

implies that

$$\mathfrak{p}(z) \prec \mathfrak{w}(z) \qquad (z \in \mathbb{U}).$$

2. A set of main inclusion relations

Unless otherwise mentioned, we assume throughout this article that

$$e_1 > 0, \quad e_i, d_j \in \mathbb{R} \setminus \mathbb{Z}_0^- \ \ (i = 2, 3, \cdots, q;\ j = 1, 2, \cdots, s), \quad \nu > 0,$$

$$\alpha > -p \qquad \text{and} \qquad -1 \leqq B < A \leqq 1.$$

We now state our first main result as Theorem 2.1 below.

Theorem 2.1. *For $w \in \mathcal{P}$, if*

$$\max_{z\in\mathbb{U}} \Re\{w(z)\} < \min_{z\in\mathbb{U}} \left\{ \frac{\alpha + 2p - \xi}{p - \xi}, \frac{e_1 + p - \xi}{p - \xi} \right\} \qquad (0 \leqq \xi < p),$$

then each of the following inclusion relations holds true:

$$\mathcal{MSK}_{p,k,\nu}^{\alpha+1,e_1}(q, s, \xi; w) \subset \mathcal{MSK}_{p,k,\nu}^{\alpha,e_1}(q, s, \xi; w) \subset \mathcal{MSK}_{p,k,\nu}^{\alpha,e_1+1}(q, s, \xi; w). \tag{2.1}$$

Proof. We consider a function $h(z)$ given by

$$h(z) = -\frac{1}{p-\xi} \left(\frac{z\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)f\right)'(z) + \nu z^2 \left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)f\right)''(z)}{(1-\nu)\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)f(z) + \nu z\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)f\right)'(z)} + \xi \right). \tag{2.2}$$

The function $h(z)$ is analytic in $\mathbb{U}$ and $h(0) = 1$.

If we now assume that $f \in \mathcal{MSK}_{p,k,\nu}^{\alpha+1,e_1}(q, s, \xi; w)$ and make use of (1.8), we get

$$\frac{-h(z)(p-\xi) + (2p + \alpha - \xi)}{\alpha + p} = \frac{(1-\nu)\mathcal{Q}_{p,q,s}^{k,\alpha+1}(e_1)f(z) + \nu z\left(\mathcal{Q}_{p,q,s}^{k,\alpha+1}(e_1)f\right)'(z)}{(1-\nu)\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)f(z) + \nu z\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)f\right)'(z)}.$$

Upon logarithmically differentiating both sides of this last equation, we find that

$$h(z) + \frac{zh'(z)}{-(p-\xi)h(z) + (\alpha + 2p - \xi)} \tag{2.3}$$

$$= -\frac{1}{p-\xi}\left(\frac{z\left(\mathcal{Q}_{p,q,s}^{k,\alpha+1}(e_1)f\right)'(z)+\nu z^2\left(\mathcal{Q}_{p,q,s}^{k,\alpha+1}(e_1)f\right)''(z)}{(1-\nu)\left(\mathcal{Q}_{p,q,s}^{k,\alpha+1}(e_1)f\right)+\nu z\left(\mathcal{Q}_{p,q,s}^{k,\alpha+1}(e_1)f\right)'(z)}+\xi\right).$$

Thus, by the above hypothesis, we have

$$\Re\{-(p-\xi)w(z)+(\alpha+2p-\xi)\}>0 \qquad (z\in\mathbb{U}).$$

Finally, by applying Lemma 1.6 to (2.3), it follows that $h(z)\prec w(z)$, that is, that $f\in\mathcal{MSK}_{p,k,\nu}^{\alpha,e_1}(q,s,\xi,w)$. Using the same lines of arguments as above, we can similarly prove the second inclusion relation asserted by Theorem 2.1. The proof of Theorem 2.1 is now completed. □

Theorem 2.2. *For $w,\phi\in\mathcal{P}$, if*

$$\max_{z\in\mathbb{U}}\Re\{w(z)\}<\min_{z\in\mathbb{U}}\left\{\frac{\alpha+2p-\xi}{p-\xi},\frac{e_1+p-\xi}{p-\xi}\right\} \qquad (0\leqq\xi<p),$$

then each of the following inclusion relations holds true:

$$\begin{aligned}&\mathcal{MC}_{p,k,\nu}^{\alpha+1,e_1}(q,s,\eta,\xi,w,\phi)\subset\mathcal{MC}_{p,k,\nu}^{\alpha,e_1}(q,s,\eta,\xi,w,\phi)\\ &\subset\mathcal{MC}_{p,k,\nu}^{\alpha,e_1+1}(q,s,\eta,\xi,w,\phi).\end{aligned}\tag{2.4}$$

Proof. In order to prove the first part of Theorem 2.2, let

$$f\in\mathcal{MC}_{p,k,\nu}^{\alpha+1,e_1}(q,s,\eta,\xi,w,\phi).$$

Then, from Definition 1.4 of the meromorphically multivalent (p-valent) function class $\mathcal{MC}_{p,k,\nu}^{\alpha+1,e_1}(q,s,\eta,\xi,w,\phi)$, there exists a function

$$g\in\mathcal{MSK}_{p,k,\nu}^{\alpha+1,e_1}(q,s,\eta;w)$$

such that

$$-\frac{1}{p-\eta}\left(\frac{z\left(\mathcal{Q}_{p,q,s}^{k,\alpha+1}(e_1)f\right)'(z)+\nu z^2\left(\mathcal{Q}_{p,q,s}^{k,\alpha+1}(e_1)f\right)''(z)}{(1-\nu)(\mathcal{Q}_{p,q,s}^{k,\alpha+1}(e_1)g)(z)+\nu z\left(\mathcal{Q}_{p,q,s}^{k,\alpha+1}(e_1)g\right)'(z)}+\eta\right)\prec\phi(z)$$

$$(z\in\mathbb{U}).$$

We now introduce a function $h(z)$ given by

$$h(z)=-\frac{1}{p-\eta}\left(\frac{z\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)f\right)'(z)+\nu z^2\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)f\right)''(z)}{(1-\nu)(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)g)(z)+\nu z\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)g\right)'(z)}+\eta\right).\tag{2.5}$$

The function $h(z)$ is analytic in $\mathbb{U}$ with $h(0)=1$. So, upon using the identity (1.7) in (2.5), we get

$$\begin{aligned}&[(p-\eta)h(z)+\eta]\cdot[(p-\xi)r(z)+\xi]=(\alpha+2p)[(p-\eta)h(z)+\eta]\\ &+(\alpha+p)\left(\frac{z\left(\mathcal{Q}_{p,q,s}^{k,\alpha+1}(e_1)f\right)'(z)+\nu z^2\left(\mathcal{Q}_{p,q,s}^{k,\alpha+1}(e_1)f\right)''(z)}{(1-\nu)(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)g)(z)+\nu z\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)g\right)'(z)}\right),\end{aligned}\tag{2.6}$$

which, on further simplification, yields

$$\frac{-(p-\eta)zh'(z)}{-(p-\xi)r(z)+(\alpha+2p-\xi)}+[-(p-\eta)h(z)-\eta]=(\alpha+p)$$

(2.7)

$$\cdot\left(\frac{z\left(\mathcal{Q}_{p,q,s}^{k,\alpha+1}(e_1)f\right)'(z)+\nu z^2\left(\mathcal{Q}_{p,q,s}^{k,\alpha+1}(e_1)f\right)''(z)}{\left[(1-\nu)(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)g)(z)+\nu z\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)g\right)'(z)\right]\cdot[-(p-\xi)r(z)+(\alpha+2p-\xi)]}\right),$$

where, for convenience,

$$r(z)=-\frac{1}{p-\xi}\left(\frac{z\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)g\right)'(z)+\nu z^2\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)g\right)''(z)}{(1-\nu)(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)g)(z)+\nu z\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)g\right)'(z)}+\xi\right). \tag{2.8}$$

Thus, by using the identity (1.7) in (2.8), we obtain

$$(\alpha+p)\left(\frac{(1-\nu)(\mathcal{Q}_{p,q,s}^{k,\alpha+1}(e_1)g)(z)+\nu z\left(\mathcal{Q}_{p,q,s}^{k,\alpha+1}(e_1)g\right)'(z)}{-(p-\xi)r(z)+(\alpha+2p-\xi)}\right)$$

$$=(1-\nu)(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)g)(z)+\nu z\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)g\right)'(z). \tag{2.9}$$

Also, if we use (2.9) in (2.7), we get

$$h(z)+\frac{zh'(z)}{-(p-\xi)r(z)+(\alpha+2p-\xi)}$$

$$=-\frac{1}{p-\eta}\left(\frac{z\left(\mathcal{Q}_{p,q,s}^{k,\alpha+1}(e_1)f\right)'(z)+\nu z^2\left(\mathcal{Q}_{p,q,s}^{k,\alpha+1}(e_1)f\right)''(z)}{(1-\nu)\left(\mathcal{Q}_{p,q,s}^{k,\alpha+1}(e_1)g\right)(z)+\nu z\left(\mathcal{Q}_{p,q,s}^{k,\alpha+1}(e_1)g\right)'(z)}+\eta\right) \tag{2.10}$$

$$(z\in\mathbb{U}).$$

Therefore, by the above hypothesis, we have

$$\Re\{-(p-\xi)w(z)+(\alpha+2p-\xi)\}>0 \qquad (z\in\mathbb{U}).$$

Lastly, by applying Lemma 1.6 to (2.10), it follows that $h(z)\prec w(z)$, that is, that $f\in\mathcal{MC}_{p,k,\nu}^{\alpha,e_1}(q,s,\eta,\xi,w,\phi)$. The same lines of arguments can be used to prove the second inclusion relation asserted by Theorem 2.2. This evidently completes the proof of Theorem 2.2. □

3. Applications of the integral operator F_μ

In this section, we consider the integral operator F_μ defined by (see, for example, [12])

$$F_\mu(f)(z):=\frac{\mu-p+1}{z^{\mu+1}}\int_0^z t^\mu\, f(t)\,\mathrm{d}t \quad (\mu>0), \tag{3.1}$$

which readily yields

$$\begin{aligned} z\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)\ F_\mu f\right)'(z) &= (\mu-p+1)\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)f(z) \\ &\quad -(\mu+1)\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)F_\mu f(z). \end{aligned} \tag{3.2}$$

Theorem 3.1. *Let $w \in \mathcal{P}$ and*

$$\max_{z\in\mathbb{U}} \Re\{w(z)\} < \frac{\mu+1-\xi}{p-\xi} \qquad (0 \leqq \xi < p).$$

If $f \in \mathcal{MSK}_{p,k,\nu}^{\alpha,e_1}(q,s,\xi,w)$, then $F_\mu(f) \in \mathcal{MSK}_{p,k,\nu}^{\alpha,e_1}(q,s,\xi,w)$.

Proof. Let $f \in \mathcal{MSK}_{p,k,\nu}^{\alpha,e_1}(q,s,\xi,w)$ and suppose that

$$-\frac{1}{p-\xi}\left(\frac{z\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)f\right)'(z)+\nu z^2\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)f\right)''(z)}{(1-\nu)\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)f(z)+\nu z\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)f\right)'(z)}+\xi\right) =: \ell(z) \prec w(z). \tag{3.3}$$

Choosing the function $\mathfrak{q}(z)$ as follows:

$$\mathfrak{q}(z) = -\frac{1}{p-\xi}\left(\frac{z\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)\ F_\mu f\right)'(z)+\nu z^2\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)\ F_\mu f\right)''(z)}{(1-\nu)(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)\ F_\mu f)(z)+\nu z\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)\ F_\mu f\right)'(z)}+\xi\right), \tag{3.4}$$

we see that $\mathfrak{q}(z)$ is analytic in $\mathbb{U}$ and $\mathfrak{q}(0) = 1$. So, by using the identity (3.2) in (3.4) and carrying out the same procedure as in our proof of Theorem 2.1, we get

$$\ell(z) = \frac{z\mathfrak{q}'(z)}{-(p-\xi)\mathfrak{q}(z)+\mu-\xi+1} + \mathfrak{q}(z). \tag{3.5}$$

In view the above hypothesis, if we apply Lemma 1.6 to (3.5), it follows that $q \prec w$, that is, that $F_\mu(f) \in \mathcal{MSK}_{p,k,\nu}^{\alpha,e_1}(q,s,\xi,w)$. The proof theorem 3.1 is thus completed. □

Theorem 3.2. *Let $w, \phi \in \mathcal{P}$ and suppose that*

$$\max_{z\in\mathbb{U}} \Re\{w(z)\} < \frac{\mu+1-\xi}{p-\xi} \qquad (0 \leqq \xi < p).$$

If $f \in \mathcal{MC}_{p,k,\nu}^{\alpha,e_1}(q,s,\eta,\xi,w,\phi)$, then $F_\mu(f) \in \mathcal{MC}_{p,k,\nu}^{\alpha,e_1}(q,s,\eta,\xi,w,\phi)$.

Proof. To prove Theorem 3.2, we first let $f \in \mathcal{MC}_{p,k,\nu}^{\alpha,e_1}(q,s,\eta,\xi,w,\phi)$. Then, from Definition 1.4 of the meromorphically multivalent (p-valent) function class $\mathcal{MC}_{p,k,\nu}^{\alpha,e_1}(q,s,\eta,\xi,w,\phi)$, there exists a function $g \in \mathcal{MS}_{p,k,\nu}^{\alpha,e_1}(q,s,\xi,w)$ such that

$$l(z) := -\frac{1}{p-\eta}\left(\frac{z\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)f\right)'(z)+\nu z^2\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)f\right)''(z)}{(1-\nu)(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)g)(z)+\nu z\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)g\right)'(z)}+\eta\right) \prec \phi(z)$$

$$(z \in \mathbb{U})$$

We now set

$$r(z)=-\frac{1}{p-\eta}\left(\frac{z\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)\ F_\mu f\right)'(z)+\nu z^2\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)\ F_\mu f\right)''(z)}{(1-\nu)(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)\ F_\mu g)(z)+\nu z\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)F_\mu g\right)'(z)}+\eta\right), \tag{3.6}$$

where $r(z)$ is analytic in $\mathbb{U}$ with $r(0)=1$. Also let the function $\psi(z)$ be given by

$$\psi(z)=-\frac{1}{p-\xi}\left(\frac{z\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)\ F_\mu g\right)'(z)+\nu z^2\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)\ F_\mu g\right)''(z)}{(1-\nu)(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)\ F_\mu g)(z)+\nu z\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)\ F_\mu g\right)'(z)}+\xi\right). \tag{3.7}$$

The function $\psi(z)$ is analytic in $\mathbb{U}$ with $\psi(0)=1$. So, by using (1.7) in (3.7), we get

$$\begin{aligned}(\mu-p+1)&\left(\frac{z\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)g\right)'(z)+\nu z^2\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)g\right)''(z)}{(1-\nu)(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)\ F_\mu g)(z)+\nu z\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)\ F_\mu g\right)'(z)}\right)\\ &=-(p-\xi)\psi(z)+\mu+1-\xi.\end{aligned} \tag{3.8}$$

By making use of (1.7) in (3.6), we obtain

$$\begin{aligned}&\frac{(p-\eta)zr'(z)}{(p-\eta)r(z)+\eta}+\{-(p-\xi)\psi(z)+\mu+1-\xi\}\\ &\quad=(\mu-p+1)\left(\frac{z\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)f\right)'(z)+\nu z^2\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)f\right)''(z)}{(1-\nu)(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)\ F_\mu g)(z)+\nu z\left(\mathcal{Q}_{p,q,s}^{k,\alpha}(e_1)\ F_\mu g\right)'(z)}\right)\\ &\qquad\cdot\left(-\frac{1}{(p-\eta)r(z)+\eta}\right),\end{aligned} \tag{3.9}$$

which, in view of (3.8), yields

$$l(z)=r(z)+\frac{zr'(z)}{-(p-\xi)\psi(z)+\mu+1-\xi}\qquad(z\in\mathbb{U}). \tag{3.10}$$

Finally, by the above hypothesis, we can find that

$$\Re\{-(p-\xi)\phi(z)+(\mu+1-\xi)\}>0\qquad(z\in\mathbb{U}),$$

so that, by applying Lemma 1.6, it follows that $r(z)\prec\phi(z)$, that is, that $F_\mu(f)\in\mathcal{MC}_{p,k,\nu}^{\alpha,e_1}(q,s,\eta,g,\phi)$. Hence we have completed the proof of Theorem 3.2. □

4. Special cases and consequences

This section is devoted to the presentation of some special cases and consequences of our main results (Theorem 2.1, Theorem 2.2, Theorem 3.1 and Theorem 3.2).

First of all, if we put

$$w(z)=\frac{1+Az}{1+Bz}\qquad(-1\leqq B<A\leqq 1;\ z\in\mathbb{U})$$

in Theorem 2.1, we are led to the following corollary.

Corollary 4.1. *If*

$$\frac{1+A}{1+B} \leqq \min_{z\in\mathbb{U}} \left\{ \frac{\alpha+2p-\xi}{p-\xi}, \frac{e_1+p-\xi}{p-\xi} \right\},$$

then each of the following inclusion relations holds true:

$$\mathcal{MSK}_{p,k,\nu}^{\alpha+1,e_1}(q,s,\xi;A,B) \subset \mathcal{MSK}_{p,k,\nu}^{\alpha,e_1}(q,s,\xi;A,B)$$
$$\subset \mathcal{MSK}_{p,k,\nu}^{\alpha,e_1+1}(q,s,\xi;A,B).$$

Remark 4.2. Theorem-2.1 and Theorem-2.2 generalize and extend several known results in earlier investigations which are being recalled below.

1. Under the same hypothesis as before, if we set $\nu = 0$ in Theorem-2.1, we get a result obtained by Patel and Palit [20, Theorem 3.1].

2. Upon taking $p = 1$, if we further set $\nu = 0$ and $\nu = 1$ in Theorem-2.1, respectively, we get the known results derived earlier by Cho and Kim [2, Theorem 1 and Theorem 2].

3. Choosing $p = 1$ and $q = s = 1$ in Theorem-2.1, we get a result of Aghalary [1, Theorem 1].

4. In the case when $p = 1$, if we put $\nu = 0$ and $\nu = 1$ in Corollary-4.1, we get both results established by Cho and Kim [2, Corollary 1].

5. Upon taking $p = 1$ and $\nu = 0$ in Theorem 2.2, we get another result of Cho and Kim [2, Theorem 2].

Remark 4.3. By putting

$$w(z) = \frac{1+Az}{1+Bz} \qquad (-1 \leqq B < A \leqq 1;\ z \in \mathbb{U})$$

in Theorem 3.1, we are led to the following corollary.

Corollary 4.4. *Let*

$$\frac{1+A}{1+B} < \frac{\mu+1-\xi}{p-\xi} \qquad (\mu > 0;\ -1 < B < A \leqq 1;\ 0 \leqq \xi < 1).$$

Then $f \in \mathcal{MSK}_{p,k,\nu}^{\alpha,e_1}(q,s,\eta,g,A,B)$.

Remark 4.5. By assigning suitable particular values to p and ν in Theorem 3.1, Theorem 3.2 and Corollary 4.4, we can derive a number of known results. Some of these special cases are recorded below.

1. Upon taking $p = 1$, if we set $\nu = 0$ and $\nu = 1$ in Theorem 3.1, respectively, we get the results of Cho and Kim [2, Theorem 4 and Theorem 5].

2. Under the same hypothesis as before, if we set $\nu = 0$ in Theorem-3.1, we get a result obtained by Patel and Palit [20, Theorem 3.8].

3. Upon setting $p = 1$, if we take $\nu = 0$ and $\nu = 1$ in Corollary 4.4, respectively, we get both results of Cho and Kim [2, Corollary 2].

4. Upon taking $p = 1$ and $\nu = 0$ in Theorem 3.2, we get a result of Cho and Kim [2, Theorem 6], which, in turn, generalizes the results obtained earlier by Goel and Sohi [9].

5. If we take $q = 2$, $s = 1$, $\alpha + p = \mu$, $e_1 = \beta$, $e_2 = 1$ and $d_1 = \alpha + \beta$ in our main results (Theorem 2.1, Theorem 2.2, Theorem 3.1 and Theorem 3.2), then we get all of the results obtained by Mostafa [18].

5. Concluding remarks and observations

In recent years, several authors used hypergeometric functions to define and investigate many different subclasses of meromorphic functions in Geometric Function Theory of Complex Analysis. Motivated by these earlier works, we have introduced and studied the following two general subclasses of meromorphic multivalent (p-valent) functions:

$$\mathcal{MSK}_{p,k,\nu}^{\alpha.e_1}(q,s,\xi:w) \qquad \text{and} \qquad \mathcal{MC}_{p,k,\nu}^{\alpha,e_1}(q,s,\eta,\xi,w,\phi)$$

in the punctured unit disk $\mathbb{U}^*$. Each of these subclasses is defined as an analogue of the Choi-Saigo-Srivastava operator for meromorphic functions. Here, in this article, we have established inclusion properties and other results for each of these subclasses, which are associated with an integral operator F_μ. For functions f belonging to the aforementioned subclasses, we have derived some sufficient conditions for $F_\mu f$ to be member of the subclasses:

$$\mathcal{MSK}_{p,k,\nu}^{\alpha.e_1}(q,s,\xi:w) \qquad \text{and} \qquad \mathcal{MC}_{p,k,\nu}^{\alpha,e_1}(q,s,\eta,\xi,w,\phi).$$

We have also briefly considered relevant connections of the developments reported here with those in some earlier works on the subject.

We conclude our present investigation by drawing the attention of the interested readers toward the potential for further researches developing similar or analogous results based upon such other meromorphic function classes as those that were studied in (for example) several recent works (see, for details, [10], [14], [15], [24], [26], [27] and [28]; see also [29]).

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