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INERTIAL-BASED ITERATIVE ALGORITHMS FOR SOLVING GENERALIZED SPLIT COMMON NULL POINT PROBLEMS IN REAL HILBERT SPACES

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Abstract. Based on the recent important result of S. Reich and T. M. Tuyen [iterative methods for solving the generalized split common null point problem in real Hilbert spaces, Optimization, DOI:10.1080/023 31934 2019.1655562], we propose and study an inertial-based algorithm without prior knowledge of the operator norm for solving generalized split common null point problems in real Hilbert spaces. We compared our algorithm with that of Reich and Tuyen with a numerical example and it is seen that our algorithm out performs that of Reich and Tuyen.

1. Introduction

Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $T : H_1 \to H_2$ be a bounded linear operator with adjoint $T^* : H_2 \to H_1$. The *split feasibility problem* (SFP) is formulated as follows:

$$\text{Find an element } x^* \in \Gamma := C \cap T^{-1}(Q). \tag{1.1}$$

The SFP was first introduced by Censor and Elfving [?] for modelling certain inverse problems. It is also known to play an important role in medical image reconstruction and signal processing (see [?]). Consequently, the problem has motivated the research of many mathematicians. See for example, Byrne [?, ?], Censor *et al.* [?, ?], Masad and Reich [?], Moudafi [?], Schopfer *et al.* [?], Shehu*et al.* [?], Sahu *et a.l* [?] Reich and Tuyen [?], Xu [?, ?].

A popular algorithm for solving the SFP is the CQ method of Byrne [?, ?]: for any starting point $x \in H_1$, the sequence $\{x_n\}$ is defined by

$$x_{n+1} = P_C^{H_1}\big(x_n - \gamma T^*(I^{H_1} - P_Q^{H_2})Tx_n\big) \ \ \forall\, n \geq 0, \tag{1.2}$$

where $P_C^{H_1}$ and $P_Q^{H_2}$ are the metric projections from H_1 onto C and from H_2 onto Q respectively and $\gamma \in (0, 2/||T||^2)$.
Since every closed and convex subset C of a Hilbert space H is the null point set of the maximal monotone operator$A = \partial i_C$, where i_C is the indicator function of C, the SFP becomes a special case of the split common null point problem (SCNPP) defined as follows; find a point $x^* \in H_1$ such that

$$0 \in A_1(x^*) \text{ and } \ 0 \in A_2(Tx^*), \tag{1.3}$$

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where $A_i : H_i \to 2^{H_i},\ i = 1, 2$ are maximal monotone operators and $T : H_1 \to H_2$ is a bounded linear operator. This problem has also been investigated by many researchers (See for example, Takahashi et al. [26, 27] and Tuyen [28] and the references therein). Other related problems to the aforementioned are; the split common fixed point problem (SCFPP) [9], the split variational inequality problem (SVIP) [13].

We remark that the SFP, SCFPP, SCNPP and the SVIP as well as several other related problems can be reformulated as the following generalized split problem. Let V and W be two Hilbert or Banach spaces and let $T : V \to W$ be a mapping from V to W. Suppose that (P_1) and (P_2) are two given problems in V and W respectively. Then the problem is to find an element x^* in V such that x^* is a solution to (P_1) and $T(x^*)$ is a solution to (P_2). We denote this problem by (P). A more general form of problem (P) is defined as follows: Let $V_1, V_2, \ldots, V_N$ be Hilbert or Banach spaces and let $T_i : V_i \to V_{i+1}, i = 1, 2, \ldots, N-1$ be mappings from V_i to $V_{i+1.}$ Suppose that $(P_i), i = 1, 2, \ldots, N$ are N problems on V_i, respectively. Then the more general form of problem (P) is to find an element $x^* \in V_1$ such that x^* is a solution to (P_1), $T_1(x^*)$ is a solution to $(P_2), \ldots,$ and $T_{N-1}(T_{N-2}(\ldots T_2(T_1(x^*))))$ is a solution to (P_N). We denote this problem by (GP). There are practical problems which can be modelled in the form of problem (GP). For instance, the production line balancing problem, where the quantity of semi-finished products from the previous process has to be equal to that intended for the next process [22]. Since the methods for solving the SCNPP can be applied to related problems such as the SFP, the SCFPP and the SVIP. In [22], Reich and Tuyen considered the following generalized split common null point problem (GSCNPP): Let $H_i, i = 1, 2, \ldots, N$ be real Hilbert spaces and let $A_i : H_i \to 2^{H_i}, i = 1, 2, \ldots, N$ be maximal monotone operators on H_i, respectively. Let $T_i : H_i \to H_{i+1}, i = 1, 2, \ldots, N-1$ be bounded linear operators such that $T_i \neq 0$ and

$$S := A_1^{-1}(0) \cap T_1^{-1}(A_2^{-1}(0)) \cap \cdots \cap T_1^{-1}(T_2^{-1} \ldots (T_{N-1}^{-1}(A_N^{-1}(0)))) \neq \emptyset.$$

Consider the following problem:

$$\text{Find an element } \quad x^* \in S, \tag{1.4}$$

that is, a point $x^* \in H_1$ such that

$$0 \in A_1(x^*), 0 \in A_2(T_1 x^*), \ldots, 0 \in A_N(T_{N-1}(T_{N-2}, \ldots, T_1(x^*))).$$

Since this new problem is much more general than the split feasibility problem (SFP), it turns out to have many more applications. In Reich and Tuyen [22], the authors in order to solve Problem 1.4, proposed and studied different modifications of the CQ method and established several strong convergence theorems for their algorithms. Furthermore, they presented several applications of their results and exhibited a numerical example to illustrate the performance of their algorithms. We remark that the step size γ in the algorithms studied by Reich and Tuyen [22] depends on the operator norm of the bounded linear operators, $T_i, i = 1, 2, \ldots, N-1$. This is restrictive since the norms of the bounded linear operators, $T_i, i = 1, 2, \ldots, N-1$ are not known precisely. In fact, it is known that computation of the norm of bounded linear operators is very difficult in general and in some cases impossible.

Although the results obtained in Reich and Tuyen [22] are novel and important in application, their algorithms have a draw back arising from the restriction on γ. It is therefore of interest to obtain those results without the restrictive condition on the step size, γ.

Recently, inertial type algorithms for solving optimization problems have become of great interest to numerous researchers. Since Polyak [21] studied an inertial extrapolation process for solving smooth convex minimization problems, there have been growing interests in the design and study of iterative methods with inertial term. For example, inertial forward-backward splitting methods, Attouch et al. [1], Cholamjiak et al. [10], inertial ADMM, Bot and Csetnek [3], and inertial forward backward- forward method, Lorenz and Pock [14]. The inertial term is based upon a discrete analogue of a second order dissipative dynamical system, (see Attouch et al. [1]) and is known for its efficiency in improving the convergence rate of iterative methods. The inertial type algorithms have been tested in the solution of certian number of problems (for example, imaging and data analysis problems, motion of a body in a potential field) and the tests show that they actually give remarkable speed-up when compared with corresponding algorithms without inertial term (see for example, Attouch *et al.* [1], Beck and Teboulle [2], Bot and Csetnek [3], [23], [12], Shehu *et al.* [25] and the references therein).

Inspired by the above mentioned results, the goals of this paper are; to construct inertia based algorithms such that the step size is independent of prior knowledge of the operator norms of the associated bounded linear operators, to prove strong convergence of the algorithms to solution of Problem 1.4, to test the performance of the obtained result using numerical example and finally to campare the performance of our algorithm with that of Reich and Tuyen [22].

The rest of the paper is organised as follows; Section 2 contains definition of terms and needed Lemmas, in Section 3, we present the major contributions of the paper, in Section 4 , we give applications of our results and in Section 5, we give a numerical example to test the performance of our algorithm and compare it with the performance of the algorithm studied by Reich and Tuyen [22].

2. Preliminaries

In this section, we present some definitions and known results needed for our convergence analysis.

Let C be a nonempty, closed and convex subset of H. It is known that for each $x \in H$, there is a unique point $P_C^H x \in C$ such that

$$||x - P_C^H x|| = \inf_{v \in C} ||x - v|| \tag{2.1}$$

The mapping $P_C^H : H \to C$ defined by (2.1) is called the metric projection from H onto C. P_C^H is known to satisfy the following inequality:

$$\langle x - P_C^H x, y - x \rangle \leq 0 \; \forall x \in H, \; \forall \; y \in C. \tag{2.2}$$

A mapping $T : C \to C$ is said to be nonexpansive if $||Tx - Ty|| \leq ||x - y|| \; \forall \; x, y \in C$. We denote by $F(T)$ the set of fixed points of a mapping T, that is, $F(T) = \{x \in C : Tx = x\}$.

Definition 2.1. A set-valued mapping $T : H \to 2^H$ is said to be monotone if for any $x, y \in H$,

$$\langle x - y, f - g\rangle \geq 0,$$

where $f \in Tx$ and $g \in Ty$. The Graph of T is defined by

$$Gr(T) := \{(x, f) \in H \times H : f \in Tx\}.$$

When $Gr(T)$ is not properly contained in the graph of any other monotone mapping, we say that T is maximal. If A is monotone, then we can define, for each $\lambda > 0$, a nonexpansive single-valued mapping $J^A_\lambda : R(I^H + \lambda A) \to D(A)$ by

$$J^A_\lambda := (I^H + \lambda A)^{-1}.$$

It is known that J^A_λ is a single valued nonexpansive mapping.

Lemma 2.2 ([17]). *Let $\{a_n\}$ and $\{c_n\}$ be sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \delta_n)a_n + b_n + c_n \ \forall\ n \geq 1,$$

where $\{\delta_n\}$ is a sequence in $(0, 1)$ and $\{b_n\}$ is a real sequence. Assume $\sum_{n=1}^{\infty} c_n < \infty$. Then the following results hold:

(i) *If $b_n \leq \delta_n M$ for some $M \geq 0$, then $\{a_n\}$ is a bounded sequence.*
(ii) *If $\sum_{n=1}^{\infty} \delta_n = \infty$ and $\limsup_{n\to\infty} \frac{b_n}{\delta_n} \leq 0$, then $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.3. *([16, 22]) Let $A : D(A) \subset H \to 2^H$ be a monotone operator. Then the following four statements hold true.*

(i) *For $r \geq s > 0$, we have*

$$||x - J^A_s x|| \leq 2||x - J^A_r x|| \ \ \text{for all } x \in R(I^H + rA) \cap R(I^H + sA).$$

(ii) *For all $r > 0$ and for all $x, y \in R(I^H + rA)$, we have*

$$\langle x - y, J^A_r x - J^A_r y\rangle \geq ||J^A_r x - J^A_r y||^2.$$

(iii) *For all $r > 0$ and for all $x, y \in R(I^H + rA)$, we have*

$$\langle (I^A - J^A_r)x - (I^A - J^A_r)y, x - y\rangle \geq ||(I^A - J^A_r)x - (I^A - J^A_r)y||^2.$$

(iv) *If $S = A^{-1}(0) \neq \emptyset$, then for all points $\xi^* \in S$ and $x \in R(I^H + rA)$, we have*

$$||J^A_r x - \xi^*||^2 \leq ||x - \xi^*||^2 - ||x - J^A_r x||^2.$$

Lemma 2.4 ([11]). *Assume that T is a nonexpansive mapping of a closed and convex subset C of a Hilbert space H into H. Then the mapping $I^H - T$ is demiclosed on C, that is, whenever $\{x_n\}$ is a sequence in C which weakly converges to some point $x \in C$ and the sequence $(I^H - T)(x_n)$ strongly converges to some point y, it follows that $(I^H - T)(x) = y$.*

Lemma 2.5 ([18]). *Let $\{s_n\}$ be a real sequence which does not decrease at infinity in the sense that there exists a subsequence $\{s_{n_k}\}$ such that $s_{n_k} \leq s_{n_k+1}$ $\forall k \geq 0$. Define an integer sequence $\{\tau(n)\}$, where $n > n_0$, by $\tau(n) := max\{n_0 \leq k \leq n : s_k < s_{k+1}\}$. Then $\tau(n) \to \infty$ as $n \to \infty$ and for all $n > n_0$, we have $max\{s_{\tau(n)}, s_n\} \leq s_{\tau(n)+1}$.*

Lemma 2.6 ([31]). *Let $\{s_n\}$ be a sequence of nonnegative numbers,$\{\alpha_n\}$ be a sequence in $(0,1)$ and let $\{c_n\}$ be a sequence of real numbers satisfying the following two conditions: (i) $s_{n+1} \leq (1-\alpha_n)s_n+\alpha_n c_n$; (ii) $\sum_{n=0}^{\infty}\alpha_n = \infty, \limsup_{n\to\infty} c_n \leq 0$. Then $lim_{n\to\infty} s_n = 0$.*

3. Main contributions

In this section, we present our major contributions of the paper.

Lemma 3.1. *Let $H_i,\ i = 1,2,3,\dots,N$ be real Hilbert spaces, $T_i : H_i \to H_{i+1},\ 1,2,3,\dots,N-1$ be bounded linear operators and let $A : H_1 \to 2^{H_1}$ be a maximal monotone operator on H_1. Then, for any $r > 0$ and $\gamma \in \big(\epsilon, \frac{2||(I-J_r^A)T_{N-1}T_{N-2}\dots T_1x-(I-J_r^A)T_{N-1}T_{N-2}\dots T_1y||^2}{||T_1^*T_2^*\dots T_{N-1}^*(I-J_r^A)T_{N-1}T_{N-2}\dots T_1x-T_1^*T_2^*\dots T_{N-1}^*(I-J_r^A)T_{N-1}T_{N-2}\dots T_1y||^2} - \epsilon\big)$ if $T_1^*T_2^*\dots T_{N-1}^*(I-J_r^A)T_{N-1}T_{N-2}\dots T_1x-T_1^*T_2^*\dots T_{N-1}^*(I-J_r^A)T_{N-1}T_{N-2}\dots T_1y \neq 0$, else $\gamma = k$ where k is a nonnegative constant , then the operator*

$$\mathcal{F}(x) := x - \gamma T_1^*T_2^*\dots T_{N-1}^*(I-J_r^A)T_{N-1}T_{N-2}\dots T_1x$$

is a nonexpansive self-mapping of H_1.

Proof. For any $x, y \in H_1$, using Lemma 2.3 (iii) and the condition on γ, we get

$$\begin{aligned}
||\mathcal{F}(x)-\mathcal{F}(y)||^2 &= ||x - \gamma T_1^*T_2^*\dots T_{N-1}^*(I-J_r^A)T_{N-1}T_{N-2}\dots T_1x \\
&\quad - y - \gamma T_1^*T_2^*\dots T_{N-1}^*(I-J_r^A)T_{N-1}T_{N-2}\dots T_1y||^2 \\
&= ||x - y - \gamma\big(T_1^*T_2^*\dots T_{N-1}^*(I-J_r^A)T_{N-1}T_{N-2}\dots T_1x \\
&\quad - T_1^*T_2^*\dots T_{N-1}^*(I-J_r^A)T_{N-1}T_{N-2}\dots T_1y\big)||^2 \\
&= ||x-y||^2 - 2\gamma\big\langle T_{N-1}T_{N-2}\dots T_1x - T_{N-1}T_{N-2} \\
&\dots T_1y, (I-J_r^A)T_{N-1}T_{N-2}\dots T_1x \\
&\quad - (I-J_r^A)T_{N-1}T_{N-2}\dots T_1y\big\rangle \\
&\quad + \gamma^2||T_1^*T_2^*\dots T_{N-1}^*(I-J_r^A)T_{N-1}T_{N-2}\dots T_1x \\
&\quad - T_1^*T_2^*\dots T_{N-1}^*(I-J_r^A)T_{N-1}T_{N-2}\dots T_1y||^2 \\
&\leq ||x-y||^2 - 2\gamma||(I-J_r^A)T_{N-2}\dots T_1x - (I-J_r^A)T_{N-2}\dots T_1y||^2 \\
&\quad + \gamma^2||T_1^*T_2^*\dots T_{N-1}^*(I-J_r^A)T_{N-1}T_{N-2}\dots T_1x \\
&\quad - T_1^*T_2^*\dots T_{N-1}^*(I-J_r^A)T_{N-1}T_{N-2}\dots T_1y||^2 \\
&\leq ||x-y||^2 - \gamma\big[2||(I-J_r^A)T_{N-2}\dots T_1x - (I-J_r^A)T_{N-2}\dots T_1y||^2 \\
&\quad - \gamma||T_1^*T_2^*\dots T_{N-1}^*(I-J_r^A)T_{N-1}T_{N-2}\dots T_1x \\
&\quad - T_1^*T_2^*\dots T_{N-1}^*(I-J_r^A)T_{N-1}T_{N-2}\dots T_1y||^2\big] \\
&\leq ||x-y||^2. \qquad (3.1)
\end{aligned}$$

Hence, $\mathcal{F}$ is nonexpansive. □

We consider Problem 1.4 for the case $N = 3$. The general case will be studied at the end of this section. For any $x_0, x_{-1}, u \in H_1$, let $\{x_n\}$ be the sequence generated by

$$\begin{aligned} w_n &= x_n + \theta_n(x_n - x_{n-1}) \\ y_{1,n} &= w_n - \gamma_{1,n} T_1^* T_2^* (I^{H_3} - J^{A_3}_{\beta_{3,n}}) T_2 T_1 w_n \\ y_{2,n} &= y_{1,n} - \gamma_{2,n} T_1^* (I^{H_2} - J^{A_2}_{\beta_{2,n}}) T_1 y_{1,n} \\ y_{3,n} &= J^{A_1}_{\beta_{1,n}} y_{2,n} \\ x_{n+1} &= \alpha_n u + (1-\alpha_n) y_{3,n}, \ n \geq 0. \end{aligned} \tag{3.2}$$

where $\{\beta_{i,n}\}, i = 1, 2, 3$, are sequences of positive numbers and$\{\alpha_n\}$ is a sequence in $(0, 1)$. We shall prove strong convergence of the sequence $\{x_n\}$ under the following conditions:

(1) $\gamma_{1,n} \in \big(\epsilon, \frac{2\|(I^{H_3} - J^{A_3}_{\beta_{3,n}})T_2T_1w_n\|^2}{\|T_1^*T_2^*(I^{H_3} - J^{A_3}_{\beta_{3,n}})T_2T_1w_n\|^2} - \epsilon\big)$ if $T_1^*T_2^*(I^{H_3} - J^{A_3}_{\beta_{3,n}})T_2T_1w_n \neq 0$,

else $\gamma_{1,n} = k_1$ where k_1 is a nonnegative constant .

(2) $\gamma_{2,n} \in \big(\epsilon, \frac{2\|(I^{H_2} - J^{A_2}_{\beta_{2,n}})T_1y_{1,n}\|^2}{\|T_1^*(I^{H_2} - J^{A_2}_{\beta_{2,n}})T_1y_{1,n}\|^2} - \epsilon\big)$ if $T_1^*(I^{H_2} - J^{A_2}_{\beta_{2,n}})T_1y_{1,n} \neq 0$,

else $\gamma_{2,n} = k_2$, where k_2 is a nonnegative constant .

(3) $\min\{\inf\{\beta_{1,n}\}, \inf\{\beta_{2,n}\}, \inf\{\beta_{3,n}\}\} \geq \beta > 0$. (4) $\lim \alpha_n = 0$, $\sum \alpha_n = \infty$, $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$.

Theorem 3.2. *If conditions* (1), (2), (3) *and* (4) *hold, then the sequences* $\{x_n\}$ *and* $\{w_n\}$ *generated by* (3.2) *converge strongly to* $P_S^{H_1}u$ *as* $n \to \infty$.

Proof. The proof is divided into two steps.

Step 1: The sequences $\{x_n\}$ and $\{y_{i,n}\}$ $i = 1, 2, 3$ are bounded. Let $q \in S$ be fixed, then

$$\begin{aligned} \|x_{n+1} - q\| &= \|\alpha_n u + (1-\alpha_n)y_{3,n} - q\| \\ &\leq \alpha_n\|u - q\| + (1-\alpha_n)\|y_{3,n} - q\| \end{aligned} \tag{3.3}$$

Using Lemma2.3 (iv) and the fact that $q \in S$, we get

$$\|y_{3,n} - q\|^2 \leq \|y_{2,n} - q\|^2 - \|y_{2,n} - J^{A_1}_{\beta_{1,n}}y_{2,n}\|^2 \tag{3.4}$$

Now, from $(I^{H_2} - J^{A_2}_{\beta_{2,n}})T_1q = 0$ and Lemma 2.3(iii), we have

$$\begin{aligned} \|y_{2,n} - q\|^2 &= \|y_{1,n} - \gamma_{2,n}T_1^*(I^{H_2} - J^{A_2}_{\beta_{2,n}})T_1y_{1,n} - q\|^2 \\ &= \|y_{1,n} - q\|^2 - 2\gamma_{2,n}\langle y_{1,n} - q, T_1^*(I^{H_2} - J^{A_2}_{\beta_{2,n}})T_1y_{1,n}\rangle \\ &+ \gamma_{2,n}^2\|T_1^*(I^{H_2} - J^{A_2}_{\beta_{2,n}})T_1y_{1,n}\|^2 \\ &\leq \|y_{1,n} - q\|^2 - 2\gamma_{2,n}\|(I^{H_2} - J^{A_2}_{\beta_{2,n}})T_1y_{1,n}\|^2 \\ &+ \gamma_{2,n}^2\|T_1^*(I^{H_2} - J^{A_2}_{\beta_{2,n}})T_1y_{1,n}\|^2 \\ &\leq \|y_{1,n} - q\|^2 - \gamma_{2,n}\big(2\|(I^{H_2} - J^{A_2}_{\beta_{2,n}})T_1y_{1,n}\|^2 \\ &- \gamma_{2,n}\|T_1^*(I^{H_2} - J^{A_2}_{\beta_{2,n}})T_1y_{1,n}\|^2\big). \end{aligned} \tag{3.5}$$

From $(I^{H_3} - J^{A_3}_{\beta_{2,n}})T_2T_1q = 0$ and Lemma 2.3(iiii), we have

$$\begin{aligned}
||y_{1,n}-q||^2 &= ||w_n-\gamma_{1,n}T_1^*T_2^*(I^{H_3}-J^{A_3}_{\beta_{3,n}})T_2T_1w_n-q||^2\\
&= ||w_n-q||^2-2\gamma_{1,n}\langle w_n-q,T_1^*T_2^*(I^{H_3}-J^{A_3}_{\beta_{3,n}})T_2T_1w_n\rangle\\
&+ \gamma_{1,n}^2||T_1^*T_2^*(I^{H_3}-J^{A_3}_{\beta_{3,n}})T_2T_1w_n||^2\\
&\le ||w_n-q||^2-2\gamma_{1,n}||(I^{H_3}-J^{A_3}_{\beta_{3,n}})T_2T_1w_n||^2\\
&+ \gamma_{1,n}^2||T_1^*T_2^*(I^{H_3}-J^{A_3}_{\beta_{3,n}})T_2T_1w_n||^2\\
&= ||w_n-q||^2-\gamma_{1,n}\big(2||(I^{H_3}-J^{A_3}_{\beta_{3,n}})T_2T_1w_n||^2\\
&- \gamma_{1,n}||T_1^*T_2^*(I^{H_3}-J^{A_3}_{\beta_{3,n}})T_2T_1w_n||^2\big). \qquad (3.6)
\end{aligned}$$

Using (3.3) – (3.6) and condition (1),, we obtain

$$\begin{aligned}
||x_{n+1}-q|| &\le \alpha_n||u-q||+(1-\alpha_n)||w_n-q||\\
&=(1-\alpha_n)||x_n-q+\theta_n(x_n-x_{n-1})||+\alpha_n||u-q||\\
&\le(1-\alpha_n)||x_n-q||+(1-\alpha_n)\theta_n||x_n-x_{n-1}||+\alpha_n||u-q||\\
&\le(1-\alpha_n)||x_n-q||+\alpha_n\big(||u-q||+\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||\big) \qquad (3.7)
\end{aligned}$$

Applying condition (4) and Lemma 2.2 (i) in (3.7), we have that $\{||x_n-q||\}$ is bounded and so $\{x_n\}$ is bounded. Consequently, $\{w_n\}$ is bounded . Moreover, it follows from (3.4) – (3.6) and condition (2) that the sequences $\{y_{i,n}\}$ $i=1,2,3$ are also bounded.

Step 2 : $x_n \to P_S^{H_1}u$

Let $\xi = P_S^{H_1}u$. By convexity of $||\cdot||^2$, we have

$$\begin{aligned}
||x_{n+1}-\xi||^2 &\le \alpha_n||u-\xi||^2+(1-\alpha_n)||y_{3,n}-\xi||^2\\
&\le \alpha_n||u-\xi||^2+||y_{3,n}-\xi||^2 \qquad (3.8)
\end{aligned}$$

Using (3.4) – (3.6), we obtain

$$\begin{aligned}
||x_{n+1}-\xi||^2 &\le \alpha_n||u-\xi||^2+||y_{3,n}-\xi||^2\\
&\le \alpha_n||u-\xi||^2+||y_{2,n}-\xi||^2-||y_{2,n}-J^{A_1}_{\beta_{1,n}}y_{2,n}||^2\\
&\le \alpha_n||u-\xi||^2+||y_{1,n}-\xi||^2-\gamma_{2,n}\big(2||(I^{H_2}-J^{A_2}_{\beta_{2,n}})T_1y_{1,n}||^2\\
&\quad -\gamma_{2,n}||T_1^*(I^{H_2}-J^{A_2}_{\beta_{2,n}})T_1y_{1,n}||^2\big)-||y_{2,n}-J^{A_1}_{\beta_{1,n}}y_{2,n}||^2\\
&\le \alpha_n||u-\xi||^2+||w_n-\xi||^2\\
&\quad -\gamma_{1,n}\big(2||(I^{H_3}-J^{A_3}_{\beta_{1,n}})T_2T_1w_n||^2-\gamma_{1,n}||T_1^*T_2^*(I^{H_3}-J^{A_3}_{\beta_{3,n}})T_2T_1w_n||^2\big)\\
&\quad -\gamma_{2,n}\big(2||(I^{H_2}-J^{A_2}_{\beta_{2,n}})T_1y_{1,n}||^2-\gamma_{2,n}||T_1^*(I^{H_2}-J^{A_2}_{\beta_{2,n}})T_1y_{1,n}||^2\big)\\
&\quad -||y_{2,n}-J^{A_1}_{\beta_{1,n}}y_{2,n}||^2 \qquad (3.9)
\end{aligned}$$

$$\begin{aligned}||w_n-\xi||^2 &= ||x_n-\xi+\theta_n(x_n-x_{n-1})||^2\\ &\leq ||x_n-\xi||^2+2||x_n-\xi||\theta_n||x_n-x_{n-1}||+\theta_n^2||x_n-x_{n-1}||^2\\ &= ||x_n-\xi||^2+\theta_n[2||x_n-\xi||+\theta_n||x_n-x_{n-1}||]||x_n-x_{n-1}|| \qquad (3.10)\end{aligned}$$

Now using (3.10) in (3.9), we get

$$\begin{aligned}||x_{n+1}-\xi||^2 &\leq \alpha_n||u-\xi||^2+||x_n-\xi||^2\\ &+ \theta_n||x_n-x_{n-1}||[2||x_n-\xi||+\theta_n||x_n-x_{n-1}||]\\ &- \gamma_{1,n}\big(2||(I^{H_3}-J^{A_3}_{\beta_{3,n}})T_2T_1w_n||^2\\ &- \gamma_{1,n}||T_1^*T_2^*(I^{H_3}-J^{A_3}_{\beta_{3,n}})T_2T_1w_n||^2\big)\\ &- \gamma_{2,n}\big(2||(I^{H_2}-J^{A_2}_{\beta_{2,n}})T_1y_{1,n}||^2\\ &- \gamma_{2,n}||T_1^*(I^{H_2}-J^{A_2}_{\beta_{2,n}})T_1y_{1,n}||^2\big)\\ &- ||y_{2,n}-J^{A_1}_{\beta_{1,n}}y_{2,n}||^2 \qquad (3.11)\end{aligned}$$

So,

$$\begin{aligned}&||y_{2,n}-J^{A_1}_{\beta_{1,n}}y_{2,n}||^2+\gamma_{1,n}\big(2||(I^{H_3}-J^{A_3}_{\beta_{3,n}})T_2T_1w_n||^2\\ -\;& \gamma_{1,n}||T_1^*T_2^*(I^{H_3}-J^{A_3}_{\beta_{3,n}})T_2T_1w_n||^2\big)\\ +\;& \gamma_{2,n}\big(2||(I^{H_2}-J^{A_2}_{\beta_{2,n}})T_1y_{1,n}||^2-\gamma_{2,n}||T_1^*(I^{H_2}-J^{A_2}_{\beta_{2,n}})T_1y_{1,n}||^2\big)\\ \leq\;& \alpha_n||u-\xi||^2+||x_n-\xi||^2-||x_{n+1}-\xi||^2\\ +\;& \theta_n||x_n-x_{n-1}||[2||x_n-\xi||+\theta_n||x_n-x_{n-1}||]\\ =\;& \alpha_n||u-\xi||^2+||x_n-\xi||^2-||x_{n+1}-\xi||^2\\ +\;& \alpha_n\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||[2||x_n-\xi||+\theta_n||x_n-x_{n-1}||] \qquad (3.12)\end{aligned}$$

Furthermore, from (3.4) – (3.6), we obtain

$$\begin{aligned}||x_{n+1}-\xi||^2 &\leq (1-\alpha_n)||y_{3,n}-\xi||^2+2\alpha_n\langle u-\xi,x_{n+1}-\xi\rangle\\ &\leq (1-\alpha_n)||w_n-\xi||^2+2\alpha_n\langle u-\xi,x_{n+1}-\xi\rangle\\ &\leq (1-\alpha_n)\big[||x_n-\xi||^2+\theta_n||x_n-x_{n-1}||[2||x_n-\xi||+\theta_n||x_n-x_{n-1}||]\big]\\ &+ 2\alpha_n\langle u-\xi,x_{n+1}-\xi\rangle\\ &= (1-\alpha_n)||x_n-\xi||^2+\alpha_n\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||[2||x_n-\xi||\\ &+ \theta_n||x_n-x_{n-1}||]\\ &+ 2\alpha_n\langle u-\xi,x_{n+1}-\xi\rangle \qquad (3.13)\end{aligned}$$

Set $\sigma_n=||x_n-\xi||^2$, $c_n=\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||[2||x_n-\xi||+\theta_n||x_n-x_{n-1}||]+2\langle u-\xi,x_{n+1}-\xi\rangle$, then (3.13) becomes

$$\sigma_{n+1} \leq (1-\alpha_n)\sigma_n+\alpha_nc_n \qquad (3.14)$$

We now show that $\sigma_n \to 0$ as $n \to \infty$ by considering two possible cases.
Case A: The sequence $\{\sigma_n\}$ eventually decreases, that is, there exists $N_0 \geq 0$ such that $\{\sigma_n\}$ decreases for $n \geq N_0$ and so $\{\sigma_n\}$ converges. From conditions (4) and (3.12), we have

$$||y_{2,n} - J_{\beta_{1,n}}^{A_1} y_{2,n}|| \to 0, \tag{3.15}$$

That is

$$||y_{3,n} - y_{2,n}|| \to 0, \tag{3.16}$$

Next, from (3.12), we obtain

$$\begin{aligned} & \gamma_{1,n}\big(2||(I^{H_3} - J_{\beta_{3,n}}^{A_3})T_2T_1w_n||^2 \\ - & \gamma_{1,n}||T_1^*T_2^*(I^{H_3} - J_{\beta_{3,n}}^{A_3})T_2T_1w_n||^2\big) \\ \leq & \alpha_n||u-\xi||^2 + ||x_n-\xi||^2 - ||x_{n+1}-\xi||^2 \\ + & \alpha_n\frac{\theta_n}{\alpha_n}||x_n - x_{n-1}||\big[2||x_n-\xi|| \\ + & \theta_n||x_n-x_{n-1}||\big] \to 0,\ n\to\infty. \end{aligned} \tag{3.17}$$

From condition (1), it follows that

$$\gamma_{1,n} < \frac{2||(I^{H_3} - J_{\beta_{3,n}}^{A_3})T_2T_1w_n||^2}{||T_1^*T_2^*(I^{H_3} - J_{\beta_{3,n}}^{A_3})T_2T_1w_n||^2} - \epsilon$$

So

$$\begin{aligned} \gamma_{1,n}||T_1^*T_2^*(I^{H_3} - J_{\beta_{3,n}}^{A_3})T_2T_1w_n||^2 &< 2||(I^{H_3} - J_{\beta_{3,n}}^{A_3})T_2T_1w_n||^2 \\ &- \epsilon||T_1^*T_2^*(I^{H_3} - J_{\beta_{3,n}}^{A_3})T_2T_1w_n||^2, \end{aligned}$$

which gives

$$\begin{aligned} \epsilon||T_1^*T_2^*(I^{H_3} - J_{\beta_{3,n}}^{A_3})T_2T_1w_n||^2 &< 2||(I^{H_3} - J_{\beta_{3,n}}^{A_3})T_2T_1w_n||^2 \\ &- \gamma_{1,n}||T_1^*T_2^*(I^{H_3} - J_{\beta_{3,n}}^{A_3})T_2T_1w_n||^2 \to 0 \end{aligned} \tag{3.18}$$

That is

$$||T_1^*T_2^*(I^{H_3} - J_{\beta_{3,n}}^{A_3})T_2T_1w_n|| \to 0$$

Furthermore, from (3.17), we have

$$\begin{aligned} 2\epsilon||(I^{H_3} - J_{\beta_{3,n}}^{A_3})T_2T_1w_n||^2 &< \gamma_{1,n}\big(2||(I^{H_3} - J_{\beta_{3,n}}^{A_3})T_2T_1w_n||^2 \\ &\leq \alpha_n||u-\xi||^2 + ||x_n-\xi||^2 \\ &- ||x_{n+1}-\xi||^2 + \alpha_n\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||\big[2||x_n-\xi|| \\ &+ \theta_n||x_n-x_{n-1}||\big] \\ &+ \gamma_{1,n}^2||T_1^*T_2^*(I^{H_3} - J_{\beta_{3,n}}^{A_3})T_2T_1w_n||^2 \to 0 \end{aligned} \tag{3.19}$$

Thus,

$$||(I^{H_3} - J^{A_3}_{\beta_{3,n}})T_2T_1w_n|| \to 0,\ n \to \infty. \tag{3.20}$$

Furthermore, using condition (2) and similar argument, we obtain

$$||(I^{H_2} - J^{A_2}_{\beta_{2,n}})T_1y_{1,n}|| \quad \to \quad 0,\ n \to \infty \tag{3.21}$$

From (3.2), (3.20) and (3.21), we obtain

$$||y_{1,n} - w_n|| \quad \to \quad 0, \quad ||y_{1,n} - y_{2,n}|| \to 0 \tag{3.22}$$

Utilizing conclusions (3.16) and (3.22), and noticing (3.2) we obtain

$$||y_{3,n} - w_n|| \quad \to \quad 0, \quad ||w_n - x_n|| \to 0 \tag{3.23}$$

It also follows from (3.2), condition (4) and boundedness of $\{y_{3,n}\}$ that

$$||x_{n+1} - y_{3,n}|| \quad = \quad \alpha_n||u - y_{3,n}|| \to 0 \tag{3.24}$$

Having in hand (3.24) and (3.23), we have

$$||x_{n+1} - x_n|| \leq ||x_{n+1} - y_{3,n}|| + ||y_{3,n} - w_n|| + ||w_n - x_n|| \to 0\ n \to \infty. \tag{3.25}$$

Next we show that $\limsup_{n\to\infty} c_n \leq 0$. Indeed, suppose that $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u - \xi, x_n - \xi\rangle = \lim_{k\to\infty}\langle u - \xi, x_{n_k} - \xi\rangle.$$

Since the subsequence $\{x_{n_k}\}$ is bounded, there exists a further subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_l}} \rightharpoonup \xi^*$. We may assume without any loss of generality that $x_{n_k} \rightharpoonup \xi^*$.

We claim that $\xi^* \in S$. From (3.23), (3.16) and (3.22) we obtain that $y_{i,n_k} \to \xi^*$ $i = 1, 2, 3$. Since T_1 and T_2 are bounded linear operators,we have $T_1y_{1,n_k} \rightharpoonup T_1\xi^*$ and $T_2T_1x_{n_k} \rightharpoonup T_2T_1\xi^*$. It follows from Lemma 2.3(i) , (3.15) , (3.21) and (3.22) that

$$||y_{2,n_k} - J^{A_1}_{\beta_{2,n}}y_{2,n_k}|| \quad \to 0, \quad ||(I^{H_2} - J^{A_2}_{\beta_{2,n}})T_1y_{1,n_k}|| \to 0,$$

$$||(I^{H_3} - J^{A_3}_{\beta_{2,n}})T_2T_1w_{n_k}||^2 \quad \to \quad 0. \tag{3.26}$$

Thus, from $y_{2,n_k} \rightharpoonup \xi^*$, $T_1y_{1,n_k} \rightharpoonup T_1\xi^*$, $T_2T_1x_{n_k} \rightharpoonup T_2T_1\xi^*$. and Lemma 2.4, we conclude that $\xi^* \in F(J^{A_1}_{\beta_{1,n}})$, $T_1\xi^* \in F(J^{A_2}_{\beta_{2,n}})$ and $T_2T_1\xi^* \in F(J^{A_3}_{\beta_{3,n}})$, that is, $\xi^* \in S$.

From $\xi = P^{H_1}_S u$ and (2.1), we deduce that

$$\limsup_{n\to\infty}\langle u - \xi, x_n - \xi\rangle = \langle u - \xi, \xi^* - \xi\rangle \leq 0,$$

which when combined with (3.25), implies that $\limsup_{n\to\infty} c_n \leq 0$, as claimed. Hence all conditions of Lemma 2.6 are satisfied. Therefore we conclude that $\sigma_n \to 0$ that is

$$x_n \to P^{H_1}_S u.$$

Case B. Suppose the sequence $\{\sigma_n\}$ is not a monotone sequence. Then, as in Lemma 2.5, we can define an integer sequence$\{\tau(n)\}$, where $n \geq n_0$ (for some n_0 large enough), by

$$\tau(n) := \max\{k \leq n : \sigma_k < \sigma_{k+1}\}.$$

Moreover, $\{\tau(n)\}$ is an increasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $\sigma_{\tau(n)} < \sigma_{\tau(n+1)}$ for all $n \geq n_0$. From (3.12), we deduce that

$$\begin{aligned} 0 < \sigma_{\tau(n+1)} - \sigma_{\tau(n)} &\leq \alpha_{\tau(n)}||u-\xi||^2 + \theta_{\tau(n)}||x_{\tau(n)} - x_{\tau(n)-1}||[2||x_{\tau(n)} - \xi|| \\ &+ \theta_{\tau(n)}||x_{\tau(n)} - x_{\tau(n)-1}||] \\ &= \alpha_{\tau(n)}||u-\xi||^2 + \alpha_{\tau(n)}\frac{\theta_{\tau(n)}}{\alpha_{\tau(n)}}||x_{\tau(n)} - x_{\tau(n)-1}||[2||x_{\tau(n)} - \xi|| \\ &+ \theta_{\tau(n)}||x_{\tau(n)} - x_{\tau(n)-1}||]. \end{aligned} \tag{3.27}$$

Since $\alpha_{\tau(n)} \to 0$ and noticing condition $C4$ we have from (3.27) we get

$$\sigma_{\tau(n+1)} - \sigma_{\tau(n)} \to 0. \tag{3.28}$$

Furthermore, we have $\sigma_{\tau(n+1)} \leq (1-\alpha_{\tau(n)})\sigma_{\tau(n)} + \alpha_{\tau(n)}c_{\tau(n)}$, where

$$\limsup_{n\to\infty} c_{\tau(n)} \leq 0.$$

Since $\sigma_{\tau(n+1)} > \sigma_{\tau(n)}$ and $\alpha_{\tau(n)} > 0$, we have $\sigma_{\tau(n)} \leq c_{\tau(n)}$. Also since $\limsup_{n\to\infty} c_{\tau(n)} \leq 0$, we have $\lim_{n\to\infty} \sigma_{\tau(n)} = 0$. This together with (3.28) implies that $\lim_{n\to\infty} \sigma_{\tau(n+1)} = 0$. Thus,

$$0 < \sigma_n \leq \max\{\sigma_{\tau(n)}, \sigma_n\} \leq \sigma_{\tau(n+1)} \to 0.$$

Consequently, $\sigma_n \to 0$, that is, $x_n \to \xi := P_S^{H_1}u$. Knowing that $||w_n - x_n|| \to 0$, we also have that $w_n \to \xi := P_S^{H_1}u$. This completes the proof.

□

Next , we study strong convergence of the sequence $\{z_n\}$ generated by $z_0, z_{-1}, u \in H_1$,

$$\begin{aligned} w_n &= z_n + \theta_n(z_n - z_{n-1}) \\ t_{1,n} &= w_n - \gamma_{1,n}T_1^*T_2^*(I^{H_3} - J_{\beta_{3,n}}^{A_3})T_2T_1w_n \\ t_{2,n} &= t_{1,n} - \gamma_{2,n}T_1^*(I^{H_2} - J_{\beta_{2,n}}^{A_2})T_1y_{1,n} \\ t_{3,n} &= J_{\beta_{1,n}}^{A_1}t_{2,n} \\ z_{n+1} &= \alpha_n f(z_n) + (1-\alpha_n)t_{3,n}, \; n \geq 0. \end{aligned} \tag{3.29}$$

where $f : H_1 \to H_1$ is a contraction with coeffiecient $\delta \in [0, 1)$.

Theorem 3.3. *If conditions* (1), (2, (3) *and* (4) *hold, then the sequence* $\{z_n\}$ *generated by* (3.29) *converges strongly to a point* $\xi^* \in S$*, which is the unique solution to the variational inequality*

$$\left\langle (I^{H_1} - f)\xi^*, y - \xi^* \right\rangle \; \forall \; y \in S.$$

Proof. $P_S^{H_1} f$ is a strict contraction, so by Banach's fixed point theorem, $P_S^{H_1} f$ has a unique fixed point ξ^* which is the unique solution to the variational inequality

$$\big\langle \big(I^{H_1} - f\big)\xi^*, y - \xi^* \big\rangle \ \forall\, y \in S.$$

Using Theorem 3.2 , with $f(\xi^*)$ replacing u in (3.2) , we see that the sequence $\{x_n\}$ converges strongly to $P_S^{H_1} f(\xi^*) = \xi^*$.

Now we assert that $||z_n - x_n|| \to 0$ as $n \to \infty$. Using nonexpansiveness of J_λ^A, $\lambda > 0$ with A maximal monotone and Lemma 3.1, we obtain

$$\begin{aligned}
||z_{n+1} - x_{n+1}|| &\leq \alpha_n ||f(z_n) - f(\xi^*)|| + (1-\alpha_n)||t_{3,n} - y_{3,n}|| \\
&\leq \alpha_n \delta ||z_n - \xi^*|| + (1-\alpha_n)||t_{2,n} - y_{2,n}|| \\
&\leq \alpha_n \delta ||z_n - \xi^*|| + (1-\alpha_n)||t_{1,n} - y_{1,n}|| \\
&\leq \alpha_n \delta ||z_n - \xi^*|| + (1-\alpha_n)||z_n - w_n|| \\
&\leq \big(1 - (1-\delta)\alpha_n\big)||z_n - w_n|| + \alpha_n \delta ||w_n - \xi^*|| \qquad (3.30)
\end{aligned}$$

Notice that

$$\begin{aligned}
||z_n - w_n|| &\leq ||z_n - x_n|| + ||x_n - w_n||, \quad \text{so} \\
||z_{n+1} - x_{n+1}|| &\leq \big(1 - (1-\delta)\alpha_n\big)||z_n - x_n|| + \big(1 - (1-\delta)\alpha_n\big)||x_n - w_n|| \\
&+ \alpha_n \delta ||w_n - \xi^*|| \\
&\leq \big(1 - (1-\delta)\alpha_n\big)||z_n - x_n|| + ||x_n - w_n|| + \alpha_n \delta ||w_n - \xi^*|| \\
&= \big(1 - (1-\delta)\alpha_n\big)||z_n - x_n|| + \theta_n ||x_n - x_{n-1}|| + \alpha_n \delta ||w_n - \xi^*|| \\
&= \big(1 - (1-\delta)\alpha_n\big)||z_n - x_n|| + \alpha_n \frac{\theta_n}{\alpha_n} ||x_n - x_{n-1}|| + \alpha_n \delta ||w_n - \xi^*|| \\
&= \big(1 - (1-\delta)\alpha_n\big)||z_n - x_n|| + \alpha_n \Big[\frac{\theta_n}{\alpha_n} ||x_n - x_{n-1}|| + \delta ||w_n - \xi^*||\Big] \qquad (3.31)
\end{aligned}$$

Using condition (4) and the fact that $w_n \to \xi^*$, we obtain

$$\frac{\theta_n}{\alpha_n} ||x_n - x_{n-1}|| + \delta ||w_n - \xi^*|| \to 0, \ n \to \infty.$$

Hence by Lemma 2.6, we conclude that $||z_n - x_n|| \to 0, \quad n \to \infty$.
Consequently,

$$||z_n - \xi^*|| \leq ||z_n - x_n|| + ||x_n - \xi^*|| \to 0, \quad n \to \infty.$$

This completes the proof. □

Remark 3.4. In Theorem 3.3, if the sequence $\{z_n\}$ is defined by $z_0, z_{-1}, u \in H_1$,

$$\begin{aligned}
w_n &= z_n + \theta_n (z_n - z_{n-1}) \\
t_{1,n} &= w_n - \gamma_{1,n} T_1^* T_2^* (I^{H_3} - J_{\beta_{3,n}}^{A_3}) T_2 T_1 w_n \\
t_{2,n} &= t_{1,n} - \gamma_{2,n} T_1^* (I^{H_2} - J_{\beta_{2,n}}^{A_2}) T_1 y_{1,n} \\
t_{3,n} &= J_{\beta_{1,n}}^{A_1} t_{2,n}
\end{aligned}$$

$$z_{n+1} = \alpha_n f(t_{3,n}) + (1-\alpha_n)t_{3,n}, \ n \geq 0. \tag{3.32}$$

where $f : H_1 \to H_1$ is a contraction with coefficient $\delta \in [0,1)$. If conditions $(1),(2),(3)$ and (4) hold, then the sequence $\{z_n\}$ generated by (3.32) converges strongly to a point $\xi^* \in S$, which is the unique solution to the variational inequality

$$\big\langle \big(I^{H_1} - f\big)\xi^*, y - \xi^* \big\rangle \ \forall \ y \in S.$$

Notice that with $\{x_n\}$ defined in (3.2) and $u = f(\xi^*)$, we have

$$||z_{n+1} - x_{n+1}|| \leq \alpha_n \delta ||t_{3,n} - \xi^*|| + (1-\alpha_n)||t_{3,n} - y_{3,n}|| \tag{3.33}$$

Since $\xi^* \in S$, we have $J^{A_1}_{\beta_{1,n}}(\xi^*) = \xi^*, J^{A_2}_{\beta_{2,n}}(T_1\xi^*) = \xi^*,$ and $J^{A_3}_{\beta_{3,n}}(T_2T_1\xi^*) = \xi^*$. Thus it follows from Lemma 3.1 that

$$||t_{3,n} - \xi^*|| \leq ||t_{2,n} - \xi^*|| \tag{3.34}$$
$$||t_{2,n} - \xi^*|| \leq ||t_{1,n} - \xi^*|| \tag{3.35}$$
$$||t_{1,n} - \xi^*|| \leq ||w_n - \xi^*||. \tag{3.36}$$

It follows from (3.30) and (3.34) – (3.6) that

$$\begin{aligned} ||z_{n+1} - x_{n+1}|| &\leq \big(1-(1-\delta)\alpha_n\big)||z_n - x_n|| + \big(1-(1-\delta)\alpha_n\big)||x_n - w_n|| \\ &+ \alpha_n \delta ||w_n - \xi^*|| \\ &\leq \big(1-(1-\delta)\alpha_n\big)||z_n - x_n|| + \alpha_n \big[\frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| + \delta||w_n - \xi^*||\big]. \qquad (3.37) \end{aligned}$$

By similar argument to the proof of Theorem 3.3, we obtain that $z_n \to \xi^*$.

Finally, we observe that by applying arguments which are similar to those used in the proofs of Theorems 3.1 and 3.2, we obtain the following theorem regarding Problem 1.4.

Theorem 3.5. *Assume that the following conditions hold:*

(1) $\gamma_{1,n} \in \big(\epsilon, \frac{2||(I^{H_3} - J^{A_3}_{\beta_{3,n}})T_{N-1}T_{n-2}...T_1w_n||^2}{||T_1^*T_2^*...T_{N-1}(I^{H_3} - J^{A_3}_{\beta_{3,n}})T_{N-1}T_{n-2}...T_1w_n||^2} - \epsilon\big)$
if $T_1^*T_2^* \dots T_{N-1}(I^{H_3} - J^{A_3}_{\beta_{3,n}})T_{N-1}T_{n-2}\dots T_1w_n \neq 0$,
else $\gamma_{1,n} = k_1$ *where* k_1 *is a nonnegative constant* ,
$\gamma_{i,n} \in \big(\epsilon, \frac{2||\big(I^{H_{N-(i-1)}} - J^{A_{N-(i-1)}}_{\beta_{N-(i-1),n}}\big)T_{N-i}...T_1y_{(i-1),n}||^2}{||T_1^*...T_{N-1}\big(I^{H_{N-(i-1)}} - J^{A_{N-(i-1)}}_{\beta_{N-(i-1),n}}\big)T_{N-i}...T_1y_{(i-1),n}||^2} - \epsilon\big)$
if $T_1^* \dots T_{N-i}\big(I^{H_{N-(i-1)}} - J^{A_{N-(i-1)}}_{\beta_{N-(i-1),n}}\big)T_{N-i}\dots T_1y_{(i-1),n} \neq 0$,
else $\gamma_{i,n} = k_2$ *where* k_2 *is a nonnegative constant* $, i = 2,3,\dots,N-1$

(2) $\min\{\inf_n\{\beta_{i,n}\}i = 1,2,\dots,N\} \geq \beta > 0.$ (3) $\lim \alpha_n = 0, \ \sum \alpha_n = \infty,$
$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| = 0.$

Then the sequence $\{x_n\}$ *generated by* $x_0, x_{-1} \in H_1$, *and*

$$\begin{aligned} w_n &= x_n + \theta_n(x_n - x_{n-1}) \\ y_{1,n} &= w_n - \gamma_{1,n}T_1^*T_2^* \dots T_{N-1}^*(I^{H_N} - J^{A_N}_{\beta_{N,n}})T_{N-1}T_{N-2}\dots T_1w_n \end{aligned}$$

$$
\begin{aligned}
y_{2,n} &= y_{1,n} - \gamma_{2,n} T_1^* T_2^* \dots T_{N-2}^* (I^{H_{N-1}} - J_{\beta_{N-1,n}}^{A_{N-1}}) T_{N-2} \dots T_1 y_{1,n} \\
&\vdots \\
y_{N-1,n} &= y_{N-2,n} - \gamma_{N-2,n} T_1^* (I^{H_2} - J_{\beta_{2,n}}^{A_2}) T_1 y_{1,n} \\
y_{N,n} &= J_{\beta_{1,n}}^{A_1} \big(y_{N-1,n} \big) \\
x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n) y_{N,n}, \ \textit{or} \\
x_{n+1} &= \alpha_n f(y_{N,n}) \\
&+ (1 - \alpha_n) y_{N,n}, \ n \geq 0,
\end{aligned} \tag{3.38}
$$

where $\{\beta_{i,n}\}, i = 1, 2, \dots, N$ *are sequences of positive numbers and* $\{\alpha_n\}$ *is a sequence in* $(0, 1)$, *converges strongly to an element* $\xi^* \in S$, *which is the unique solution to the variational inequality*

$$\big\langle \big(I^{H_1} - f\big)\xi^*, y - \xi^* \big\rangle \ \forall \ y \in S.$$

4. Applications

4.1. **Generalized split feasibility problem.** Let C be a nonempty, closed and convex subset of a real Hilbert space H. Denote by i_C the indicator function of C, that is,

$$i_C(x) = \begin{cases} 0, & \text{if } x \in C \\ \infty & \text{if } x \notin C \end{cases} \tag{4.1}$$

Then i_C is a proper, lower semicontinuous and convex function. Hence its subdifferential ∂i_C is a maximal monotone operator. It is known that

$$\partial i_C(x) = N(x, C) = \{v \in H : \langle x - y, v \rangle \geq 0 \ \forall \ y \in C\},$$

where $N(x, C)$ is the normal cone of C at x.
We denote the resolvent operator of ∂i_C by J_r, where $r > 0$. Suppose $x = J_r y$ for each $y \in H$, that is,

$$\frac{y - x}{r} \in \partial i_C(x) = N(x, C).$$

Then we have

$$\langle y - x, x - v \rangle \geq 0 \ \ \forall \ v \in C.$$

Since this inequality characterizes the metric projection, it follows that $x = P_C^H y$. Applying Theorem 3.5 yields the following result regarding an algorithm for solving the generalized split feasibility problem in Hilbert spaces.

Theorem 4.1. *Let* $H_i, i = 1, 2, \dots, N$, *be real Hilbert spaces and let* C_i, $i = 1, 2, \dots, N$ *be closed and convex subsets of* H_i, *respectively. Let* $T_i : H_i \to H_{i+1}$ $i = 1, 2, \dots, N-1$ *be bounded linear operators such that*

$$S := C_1 \cap T_1^{-1}(C_2) \cap \dots \cap T_1^{-1}(T_2^{-1}(\dots (T_{N-1}^{-1}(C_N)))) \neq \emptyset.$$

If conditions (1) *and* (3) *of Theorem 3.5 hold, then the sequence* $\{x_n\}$ *generated by* $x_0, x_{-1} \in H_1$, *and*

$$w_n = x_n + \theta_n (x_n - x_{n-1})$$

$$
\begin{aligned}
y_{1,n} &= w_n - \gamma_{1,n} T_1^* T_2^* \dots T_{N-1}^* (I^{H_N} - P_{C_N}^{H_N}) T_{N-1} T_{N-2} \dots T_1 w_n \\
y_{2,n} &= y_{1,n} - \gamma_{2,n} T_1^* T_2^* \dots T_{N-2}^* (I^{H_{N-1}} - P_{C_{N-1}}^{H_{N-1}}) T_{N-2} \dots T_1 y_{1,n} \\
&\vdots \\
y_{N-1,n} &= y_{N-2,n} - \gamma_{N-2,n} T_1^* (I^{H_2} - P_{C_2}^{H_2}) T_1 y_{N-2,n} \\
y_{N,n} &= P_{C_1}^{H_1} (y_{N-1,n}) \\
x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n) y_{N,n}, \quad or \\
x_{n+1} &= \alpha_n f(y_{N,n}) + (1-\alpha_n) y_{N,n}, \ n \geq 0 \qquad (4.2)
\end{aligned}
$$

converges strongly to an element $\xi^* \in S$, which is the unique solution to the variational inequality

$$\big\langle \big(I^{H_1} - f\big)\xi^*, y - \xi^* \big\rangle \ \forall \ y \in S.$$

Remark 4.2. Other applications to various problems of contemporary interest such as : Generalised Split common null point problem, Generalized split equilibrium problem and Generalized split varational inequality problem studied in Reich and Tuyen [22] can easiy be obtained when we use the algorithms developed and studied in this work. We do not consider them as that amounts to mere repetition.

5. Numerical Example

In this section, we adapt the numerical example, Example 5.1 of Reich and Tuyen [22] to examine the convergence of the sequence $\{x_n\}$ defined in Theorem 4.1 of this work. Furthermore, we compare the performance of the sequence $\{x_n\}$ of Theorem 4.4 of Reich and Tuyen [22] with the performance of the sequence $\{x_n\}$ of algorithm 4.2 of Theorem 4.1 of our work.

Example 5.1. Consider the following problem: find an element $\xi^* \in \mathbb{R}^4$ such that

$$\xi^* \in S := S_1 \cap T_1^{-1}(S_2) \cap T_1^{-1}(T_2^{-1}(S_3) \neq \emptyset,$$

where $S_1 = \{x \in \mathbb{R}^4 : ||x - a_1|| \leq K_1^2\}, S_2 = \{x \in \mathbb{R}^6 : ||x - a_2|| \leq K_2^2\}, S_3 = \{x \in \mathbb{R}^8 : ||x - a_3|| \leq K_3^2\}$. and $T_1 : \mathbb{R}^4 \to \mathbb{R}^6$ and $T_2 : \mathbb{R}^6 \to \mathbb{R}^8$ are bounded linear operators, the elements of the representing matrices of which are randomly generated in $[-5, 5]$. The coordinates of the centres a_1, a_2, a_3 are randomly generated in $[-1, 1]$, the radii K_1, K_2, K_3 are randomly generated in the intervals $[4, 8], [6, 12]$ and $[8, 16]$, respectively, and the coordinates of the initial point x_0 are randomly generated in $[-2, 2]$.

Table 1. Numerical results comparing our Algorithm (4.2) with Algorithm of Theorem 4.4 of [22]

Table 1: Numerical results with $\mathbf{TOL}_n < 10^{-3}$.

No. of runs	Algorithm (4.2			Reich and Tuyen		
	CPU	TOL_n	Iter	CPU	TOL_n	Iter
1	0.0147	$9.2895e^{-04}$	26	0.0370	$9.7693e^{-04}$	30
2	0.0249	$9.3055e^{-04}$	23	0.0605	$9.7872e^{-04}$	28
3	0.0356	$9.5573e^{-04}$	22	0.0356	$9.7210e^{-04}$	29
4	0.0133	$9.1936e^{-04}$	16	0.0343	$9.4517e^{-04}$	23

Table 2: Numerical results with $\mathbf{TOL}_n < 10^{-4}$.

No. of runs	Algorithm (4.2)			Reich and Tuyen		
	CPU	TOL_n	Iter	CPU	TOL_n	Iter
1	0.0188	$9.8944e^{-05}$	68	0.0422	$9.9382e^{-05}$	78
2	0.0209	$9.8308e^{-05}$	96	0.0742	$9.8560e^{-05}$	111
3	0.0545	$9.9239e^{-05}$	102	0.0810	$9.9312e^{-05}$	122
4	0.0183	$9.8528e^{-05}$	69	0.0390	$9.8775e^{-05}$	91

Remark 5.2. In Example 5.1 above, , the function TOL_n is given by

$$\begin{aligned} TOL_n &= \frac{1}{3}(||x_n - P_{S_1}^{\mathbb{R}^4}(x_n)||^2 + ||T_1(x_n) - P_{S_2}^{\mathbb{R}^6}(T_1x_n)||^2 \\ &+ ||T_2(T_1x_n) - P_{S_3}^{\mathbb{R}^8}(T_2(T_1x_n))||^2) \ \forall\ n \geq 1. \end{aligned}$$

It is clear that if at any nth step, $TOL_n = 0$, we get that x_n is a solution to the problem.

6. Conclusion

In this paper, we constructed inertia based algorithms such that the step size is independent of prior knowledge of the operator norms of the associated bounded linear operators, and proved strong convergence of the algorithms to solution of Problem 1.4. Adapting the example in [22], we compared the performance of one of our algorithms , algorithm 4.2 with algorithm of Theorem 4.4 of Reich and Tuyen [22] . From the table of values and the graphs above, it is seen that our algorithm out performs that of Reich and Tuyen [22] since our algorithm takes less CPU time to converge.

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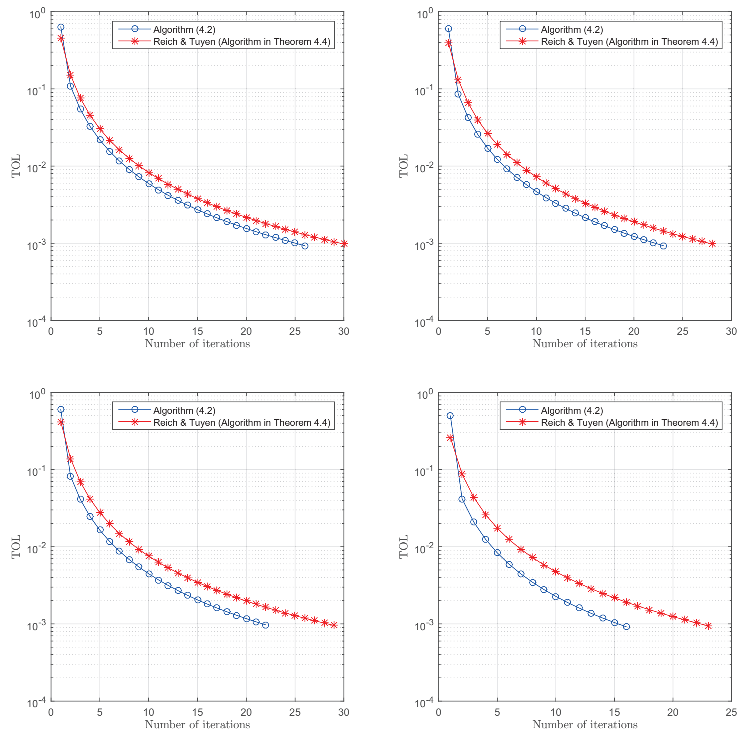
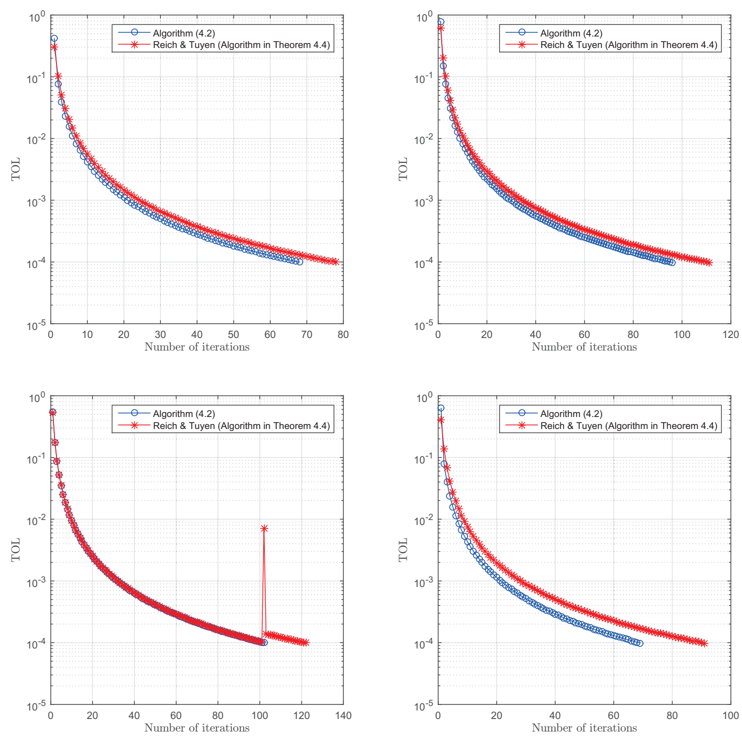

Algorithm (4.2)
Reich & Tuyen (Algorithm in Theorem 4.4)
TOL
Number of iterations


Algorithm (4.2)
Reich & Tuyen (Algorithm in Theorem 4.4)
TOL
Number of iterations

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