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REDUCTIONS OF ANTI-SELF-DUAL YANG-MILLS EQUATION OVER HEISENBERG GROUP

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Abstract. In this paper, we investigate the reduction of anti-self-dual (ASD) Yang-Mills equation over Heisenberg group. We reduce the ASD equation to a system of equations in four dimensions by an action of one dimensional group generated by the Reeb vector. We also give the the complex Bogomolny equation over Heisenberg group by an action of two dimensional group.

1. Introduction

In the classical flat case, by complexifying the real space $\mathbb{R}^4$ as $\mathbb{C}^4$ by

$$(y_1, y_2, y_3, y_4) \hookrightarrow \begin{pmatrix} y_1 + \mathbf{i}y_2 & -y_3 - \mathbf{i}y_4 \\ y_3 - \mathbf{i}y_4 & y_1 - \mathbf{i}y_2 \end{pmatrix} \tag{1.1}$$

for $(y_1, y_2, y_3, y_4) \in \mathbb{R}^4$, ones can construct twistor space by using complex geometry method (cf. [4] [11] [14]). If we denote a point of $\mathbb{C}^4$ by $(y_{AA'})$ with 2-component spinor indices $A = 0, 1$, $A' = 0', 1'$, an *α-plane* in $\mathbb{C}^4$ is the set of all $(y_{AA'})$ satisfying

$$\begin{pmatrix} y_{00'} & y_{01'} \\ y_{10'} & y_{11'} \end{pmatrix} \begin{pmatrix} \pi_{0'} \\ \pi_{1'} \end{pmatrix} = \begin{pmatrix} \omega_0 \\ \omega_1 \end{pmatrix} \tag{1.2}$$

for fixed $0 \neq (\pi_{0'}, \pi_{1'}) \in \mathbb{C}^2$ and w_0, $w_1 \in \mathbb{C}$, which is the integral surfaces of the integrable distribution $V_A := \pi_{0'}\frac{\partial}{\partial y_{A0'}} - \pi_{1'}\frac{\partial}{\partial y_{A1'}}$ $(A = 0, 1)$. A connection ∇ is called *anti-self-dual* (briefly ASD) if it is flat over any α-plane, which is equivalent to $F(V_0, V_1) = 0$, where F is the curvature of ∇. Then the connection form $\mathbf{A} := A_{00'}\mathrm{d}y_{00'} + A_{01'}\mathrm{d}y_{01'} + A_{10'}\mathrm{d}y_{10'} + A_{11'}\mathrm{d}y_{11'}$ of ∇ is ASD if and only if

$$\begin{cases} \frac{\partial A_{10'}}{\partial y_{00'}} - \frac{\partial A_{00'}}{\partial y_{10'}} + [A_{00'}, A_{10'}] = 0, \\ \frac{\partial A_{10'}}{\partial y_{01'}} - \frac{\partial A_{01'}}{\partial y_{10'}} + [A_{01'}, A_{10'}] + \frac{\partial A_{11'}}{\partial y_{00'}} - \frac{\partial A_{00'}}{\partial y_{11'}} + [A_{00'}, A_{11'}] = 0, \\ \frac{\partial A_{11'}}{\partial y_{01'}} - \frac{\partial A_{00'}}{\partial y_{11'}} + [A_{01'}, A_{11'}] = 0, \end{cases} \tag{1.3}$$

where $A_{AA'} = \mathbf{A}(\frac{\partial}{\partial y_{AA'}})$. The reduction of ASD Yang-Mills equation (1.3) means reducing the number of variables to construct its resolution. Ward [15] proposed the programme of reducing the ASD Yang-Mills equation. And Mason [11] defined the associated Higgs fields and gives the classification of reductions of the ASD Yang-Mills equation.

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The 5D real Heisenberg group $\mathscr{H}^{\mathbb{R}}$ is the space $\mathbb{R}^5$ with multiplication given by

$$(\mathbf{y},s)\circ(\mathbf{y}',s')=\left(\mathbf{y}+\mathbf{y}',s+s'+2\left(y_1y_2'-y_2y_1'-y_3y_4'+y_4y_3'\right)\right), \tag{1.4}$$

where $\mathbf{y},\mathbf{y}'\in\mathbb{R}^4$ and $s,s'\in\mathbb{R}$. Similarly to (1.1), by the real imbedding $\mathbb{R}^5\longrightarrow\mathbb{C}^5$ given by

$$\begin{bmatrix} y_{00'} & y_{01'} \\ y_{10'} & y_{11'} \end{bmatrix} := \begin{bmatrix} y_1+\mathbf{i}y_2 & -y_3+\mathbf{i}y_4 \\ y_3+\mathbf{i}y_4 & y_1-\mathbf{i}y_2 \end{bmatrix}, \quad t=-\mathbf{i}s, \tag{1.5}$$

the real Heisenberg group $\mathscr{H}^{\mathbb{R}}$ can be imbedded into the 5D complex *Heisenberg group* $\mathscr{H}$, which is the complex space $\mathbb{C}^5:=\{(\mathbf{y},t)|\mathbf{y}\in\mathbb{C}^4,t\in\mathbb{C}\}$ with the multiplication given by

$$(\mathbf{y},t)\circ(\mathbf{y}',t')=\left(\mathbf{y}+\mathbf{y}',t+t'+B(\mathbf{y},\mathbf{y}')\right), \tag{1.6}$$

where $B(\mathbf{y},\mathbf{y}')=y_{00'}y'_{11'}-y_{01'}y'_{10'}+y_{10'}y'_{01'}-y_{11'}y'_{00'}$. There are two reasons to choose the imbedding. On the one hand, by an isomorphism $\mathrm{SO}(6,\mathbb{C})/P\cong\mathscr{H}$, Ren and Wang [12] used the method in [1] and gave the twistor transform over Heisenberg group. Here P is a parabolic subgroup of $\mathrm{SO}(6,\mathbb{C})$. On the other hand, with the advantage of two-component station, they could use the twistor method to study the ASD equations over $\mathscr{H}$ and $\mathscr{H}^{\mathbb{R}}$. Moreover, the anti-self-dual (ASD) equation over $\mathscr{H}^{\mathbb{R}}$ is a model of horizontal ASD equation over 5D contact manifold (cf. [2] [3] [6] [7] [8] [9] [16]). In this paper, we consider the reduction of ASD equations over 5D *Heisenberg group* $\mathscr{H}$.

We have left invariant vector fields on $\mathscr{H}$:

$$\begin{aligned} V_{00'} &:= \frac{\partial}{\partial y_{00'}}-y_{11'}T, \qquad V_{01'} := \frac{\partial}{\partial y_{01'}}+y_{10'}T, \\ V_{10'} &:= \frac{\partial}{\partial y_{10'}}-y_{01'}T, \qquad V_{11'} := \frac{\partial}{\partial y_{11'}}+y_{00'}T, \qquad T:=\frac{\partial}{\partial t}. \end{aligned} \tag{1.7}$$

It is easy to see that

$$[V_{00'},V_{11'}]=[V_{10'},V_{01'}]=2T, \tag{1.8}$$

and all other brackets vanish. Consequently, for nonhomogeneous coordinates ζ of $\mathbb{C}P^1$, if denote

$$V_A:=\zeta V_{A0'}-V_{A1'},\quad A=0,1, \tag{1.9}$$

we have

$$[V_0,V_1]=0.$$

Namely, *span*$\{V_0,V_1\}$ is an abelian Lie subalgebra and an integrable distribution for fixed $0\neq(\pi_{0'},\pi_{1'})\in\mathbb{C}^2$. Their integral surfaces are hyperplanes, which we also call *α-planes*.

A connection is called *anti-self-dual* (briefly ASD) if it is flat over any α-plane. Let $\Phi=\Phi_{00'}\theta^{00'}+\Phi_{10'}\theta^{10'}+\Phi_{01'}\theta^{01'}+\Phi_{11'}\theta^{11'}+\Phi_T\theta$ be a $\mathfrak{g}$-valued connection form on $\mathscr{H}$, where $\{\theta^{AA'},\theta\}$ are 1-forms dual to $\{V_{AA'},T\}$. Φ is ASD if and only if it satisfies the *ASD Yang-Mills equation*

$$\begin{cases} V_{00'}(\Phi_{10'}) - V_{10'}(\Phi_{00'}) + [\Phi_{00'}, \Phi_{10'}] = 0, \\ V_{01'}(\Phi_{10'}) + V_{00'}(\Phi_{11'}) - V_{10'}(\Phi_{01'}) - V_{11'}(\Phi_{00'}) + [\Phi_{00'}, \Phi_{11'}] + [\Phi_{01'}, \Phi_{10'}] = 0, \\ V_{01'}(\Phi_{11'}) - V_{11'}(\Phi_{01'}) + [\Phi_{01'}, \Phi_{11'}] = 0. \end{cases} \tag{1.10}$$

This paper is organized as follows. In Section 2, we investigate the invariance of connections and Higgs fields on manifold [11]. In Section 3, we recall the ASD equation over Heisenberg group and give its more concrete form. In Section 4, we investigate the reduction of ASD equation by Reeb vector. And we also give the the complex Bogomolny equation over Heisenberg group by the method of reduction.

2. Higgs field

In [11], Mason gave classification of the integrable systems that arise as symmetry reductions of the anti-self-dual Yang-Mills (ASDYM) equation. And he also introduced the associated Higgs field and gave the process of the reduction. In this section, we supplement some calculation details and proofs of some propositions about reduction in [11].

Let $\pi : E \to U$ be a vector bundle. Let $(e_1, e_2, \ldots, e_n)$ be the frame field of E. Then a section of E can be written as $(e_1, e_2, \ldots, e_n) \cdot \mathbf{s} \in \Gamma(U, E)$, where

$$\mathbf{s} = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix} \in \mathbb{C}^n.$$

Suppose that H is a Lie group acting on the manifold U as $m \to \rho.m$ for any $\rho \in H$ and $m \in U$. A *lift* of action of H on U to E is that, for any $\rho \in H$, the action $m \to \rho.m$ induces a linear map $\rho_* : E_m \to E_{\rho.m}$ defined by

$$\rho_*(\mathbf{s}, m) = (\Psi(\rho, m)\mathbf{s}, \rho.m)\,, \tag{2.1}$$

where $\Psi(\rho, m)$ is the $GL(E)$-valued function over $H \times M$.

Remark 2.1. Here the linear map $\rho_* : E \to E$ must satisfy the compatibility conditions:
(1) $\pi(\rho_*(\mathbf{s}, m)) = \rho(\pi((\mathbf{s}, m)))$, i.e. the following diagram

$$\begin{tikzcd} (\mathbf{s}, m) \arrow[r, "\rho."] \arrow[d, "\pi"] & (\Psi(\rho, m)\mathbf{s}, \rho.m) \arrow[d, "\pi"] \\ m \arrow[r, "\rho."] & \rho.m \end{tikzcd}$$

is commutative.
(2) $\rho \to \rho_*$ is group homomorphism, i.e., for any $\rho_1,\ \rho_2 \in H$, we have $(\rho_1\rho_2)_* = \rho_{1*}\rho_{2*}$.

Note that, $(\rho_1\rho_2)_*(\mathbf{s}, m) = (\Psi(\rho_1\rho_2, m)\mathbf{s}, \rho_1\rho_2.m)$ and

$$\rho_{1*}\rho_{2*}(\mathbf{s}, m) = \rho_{1*}(\Psi(\rho_2, m)\mathbf{s}, \rho_2.m) = (\Psi(\rho_1, \rho_2.m)\Psi(\rho_2, m)\mathbf{s}, \rho_1.\rho_2.m)\,.$$

Then the condition (2) is equivalent to

$$\Psi(\rho_1\rho_2, m) = \Psi(\rho_1, \rho_2.m)\Psi(\rho_2, m), \tag{2.2}$$

which means that Ψ is an automorphic function.

Suppose the lift of action of H on U to E satisfy the above condition. Then we can define a *pull-back* action of H on section $\mathbf{s} \in \Gamma(U, E)$ by

$$\rho^*\mathbf{s}(m) = \rho_*^{-1}\left(\mathbf{s}(\rho.m)\right).$$

Let ∇ be a connection on E. Locally,

$$\nabla\left((e_1, e_2, \ldots, e_n)\mathbf{s}\right) = (e_1, e_2, \ldots, e_n)\left(\mathrm{d}\mathbf{s} + \Phi\mathbf{s}\right),$$

where Φ is a matrix-valued 1-form, called the gauge potential.

Lemma 2.2. *The gauge potential Φ satisfy the gauge transformation.*

Proof. Let $(e_1', e_2', \ldots, e_n')$ be another frame field of E. Then there exists $A \in Gl(n)$ such that

$$\left(e_1', e_2', \ldots, e_n'\right) = (e_1, e_2, \ldots, e_n)\, A. \tag{2.3}$$

Denote $\nabla\left(e_1', e_2', \ldots, e_n'\right) = \left(e_1', e_2', \ldots, e_n'\right)\Phi'$. Then we have

$$\begin{aligned}\nabla\left(e_1', e_2', \ldots, e_n'\right) &= \nabla\left((e_1, e_2, \ldots, e_n)\, A\right) = (e_1, e_2, \ldots, e_n)\left(\mathrm{d}A + \Phi A\right)\\ &= \left(e_1', e_2', \ldots, e_n'\right)\left(A^{-1}\mathrm{d}A + A^{-1}\Phi A\right).\end{aligned}$$

So the connection form of ∇ satisfy the gauge transformation

$$\Phi' = A^{-1}\mathrm{d}A + A^{-1}\Phi A.$$

□

Proposition 2.3. *The pull back of Φ by $\rho \in H$ defined by*

$$\rho^*\Phi = \Psi^{-1}\mathrm{d}\Psi + \Psi^{-1}\Phi\Psi$$

is a connection.

Proof. Since

$$\begin{aligned}\rho_*\left((e_1', e_2', \ldots, e_n')\right) &= \rho_*\left((e_1, e_2, \ldots, e_n)A\right) = (e_1, e_2, \ldots, e_n)\Psi A\\ &= (e_1', e_2', \ldots, e_n')A^{-1}\Psi A.\end{aligned}$$

Note that

$$\begin{aligned}\rho^*\Phi' &= (A^{-1}\rho A)^{-1}\mathrm{d}(A^{-1}\Psi A) + (A^{-1}\Psi A)^{-1}\Phi'(A^{-1}\Psi A)\\ &= A^{-1}\Psi^{-1}A\left(-A^{-1}\mathrm{d}AA^{-1}\Psi A + A^{-1}\mathrm{d}\Psi A + A^{-1}\Psi\mathrm{d}A\right)\\ &+ A^{-1}\Psi^{-1}A\left(A^{-1}\mathrm{d}A + A^{-1}\Phi A\right)\left(A^{-1}\Psi A\right)\\ &= A^{-1}\mathrm{d}A + A^{-1}\left(\Psi^{-1}\mathrm{d}\Psi + \Psi^{-1}\Phi\Psi\right)A\\ &= A^{-1}\mathrm{d}A + A^{-1}\rho^*\Phi A.\end{aligned}$$

So we have the commutative diagram

$$\begin{tikzcd} \Phi \arrow[r] \arrow[d] & \Phi' \arrow[d] \\ \rho^*\Phi \arrow[r] & \rho^*\Phi'. \end{tikzcd}$$

Then $\rho^*\Phi$ is a connection.

□

For each vector $X \in \mathfrak{h}$ (the Lie algebra of H), let $\{\varphi_t\}$ be the corresponding one parameter subgroup and L'_X be the Lie-derivative along X. Locally, its lift to the bundle is defined as

$$\begin{aligned} L_X \mathbf{s} :&= \lim_{t\to 0} \frac{\varphi^*_{-t}\mathbf{s}(\varphi_t(m)) - \mathbf{s}(m)}{t} \\ &= \lim_{t\to 0} \left(\frac{\varphi^*_{-t}\mathbf{s}(\varphi_t(m)) - \varphi^*_{-t}\mathbf{s}(m)}{t} + \frac{\varphi^*_{-t}\mathbf{s}(m) - \mathbf{s}(m)}{t} \right) \\ &= \lim_{t\to 0} \left(\varphi^*_{-t} \frac{\mathbf{s}(\varphi_t(m)) - \mathbf{s}(m)}{t} + \frac{\varphi^*_{-t} - Id}{t}\mathbf{s}(m) \right) \\ &= X(\mathbf{s}) + \theta_X \mathbf{s}, \end{aligned}$$

where $\theta_X := \lim_{t\to 0} \frac{\varphi^*_{-t} - Id}{t}$ is endomorphism of E, which is a matrix-valued function on U. Higgs field is defined by

$$\phi_X \mathbf{s} := \nabla_X \mathbf{s} - L_X \mathbf{s} = (\Phi_X - \theta_X)\mathbf{s}, \tag{2.4}$$

which measures the difference between the covariant derivative along X and the Lie derivative along X. The adjoint bundle of E is $adj(E_x) = E_x \otimes E^*_x$. The connection extends in a natural way to sections of $adj(E)$ by $\nabla\phi = d\phi + [\Phi, \phi]$ for $\phi \in adj(E)$ and to forms with values in adj(E) by $\nabla\Phi = d\Phi + \Phi\wedge\Omega - (-1)^p\Omega\wedge\Phi$ for $\Omega \in \Lambda^p(M) \wedge adj(E)$. Moreover, the action of the operators L_x extends to sections of $adj(E)$ by $L_X\phi = X(\phi) + [\theta_x, \phi]$.

We say that the connection is invariant if it is preserved by the action of H, that is, if $\rho^*\nabla = \nabla$ for every $\rho \in H$. At the Lie algebra level, for every $X \in \mathfrak{h}$ the condition is that

$$L_X(\nabla \mathbf{s}) = \nabla(L_X \mathbf{s}). \tag{2.5}$$

Obviously, if ∇ is invariant under the action of X , we have

$$L_X F = L'_X F + [\theta_X, F], \tag{2.6}$$

and hence that the conformal group preserves ASD eauqtion. Then we have

Lemma 2.4. *[11] If the connection is invariant with respect to H, for $X, Y \in \mathfrak{h}$, we have*
(1) $L'_X\Phi + [\theta_X, \Phi] = \mathrm{d}\theta_X$;
(2) $X(\phi_Y) + [\theta_X, \phi_Y] = \phi_{[X,Y]}$;
(3) $X\lrcorner d\Phi = -[\theta_X, \Phi] - \mathrm{d}\phi_X$.

Proof. (1) Locally,

$$\begin{aligned}\nabla(L_X\mathbf{s}) &= \nabla\left(X(\mathbf{s})+\theta_X\mathbf{s}\right) = \mathrm{d}\left(X\mathbf{s}+\theta_X\mathbf{s}\right)+\Phi\left(X(\mathbf{s})+\theta_X\mathbf{s}\right)\\ &= \mathrm{d}X(\mathbf{s})+\mathrm{d}\theta_X\cdot\mathbf{s}+\theta_X\mathrm{d}\mathbf{s}+\Phi X(\mathbf{s})+\Phi\theta_X\mathbf{s},\\ L_X(\nabla\mathbf{s}) &= L_X\left(\mathrm{d}\mathbf{s}+\Phi\mathbf{s}\right) = L'_X(d\mathbf{s}+\Phi\mathbf{s})+\theta_X(d\mathbf{s}+\Phi\mathbf{s})\\ &= \mathrm{d}X(\mathbf{s})+L'_X\Phi\cdot\mathbf{s}+\Phi X(\mathbf{s})+\theta_X\mathrm{d}\mathbf{s}+\theta_X\Phi\mathbf{s}.\end{aligned}$$

By (2.5), we have $L'_X\Phi+[\theta_X,\Phi]=\mathrm{d}\theta_X$.
(2) For $X,Y\in\mathfrak{h}$, since $L_X(\nabla_Y\mathbf{s})=\nabla_{[X,Y]}\mathbf{s}+\nabla_Y(L_X\mathbf{s})$, we have $\nabla_{[X,Y]}\mathbf{s}=L_X(\nabla_Y\mathbf{s})-\nabla_Y(L_X\mathbf{s})$. Moreover, $L_{[X,Y]}\mathbf{s}=[L_X,L_Y]\mathbf{s}=L_XL_Y\mathbf{s}-L_YL_X\mathbf{s}$. So we have

$$\begin{aligned}\phi_{[X,Y]}\mathbf{s} &= \nabla_{[X,Y]}s-L_{[X,Y]}\mathbf{s}=(L_X(\nabla_Y\mathbf{s})-\nabla_Y(L_X\mathbf{s}))-(L_XL_Ys-L_YL_X\mathbf{s})\\ &= L_X(\nabla_Y-L_Y)\mathbf{s}-(\nabla_Y-L_Y)(L_X\mathbf{s})=L_X(\phi_Y\mathbf{s})-\phi_Y(L_X\mathbf{s})\\ &= (L_X\phi_Y)\mathbf{s}=(X(\phi_Y)+[\theta_X,\phi_Y])\mathbf{s}.\end{aligned}$$

(3) Since $L_X=\mathrm{d}\circ X\lrcorner+X\lrcorner\circ\mathrm{d}$, we have

$$X\lrcorner d\Phi=L'_X\Phi-\mathrm{d}(X\lrcorner\Phi)=\mathrm{d}\theta_X-[\theta_X,\Phi]-\mathrm{d}(X\lrcorner\Phi)=-[\theta_X,\Phi]-\mathrm{d}\phi_X,$$

by using (1). □

Let U be an elementary open set in $\mathscr{H}$. We can pick out a submanifold $S\subseteq U$ that intersects each orbit transversely at a single point and we can identify S with $S=U/H=\{xH|x\in U\}$. Take $H_x=\{x\cdot h|h\in H\}$ as the orbit for $x\in S$, and its Lie algebra is

$$\mathfrak{h}_x:=\{X\in\mathfrak{h}|X(x)=0\}.$$

By the transversality condition, we have

$$T_xU=T_xS\oplus\mathfrak{h}_x.$$

And denote $\nabla'=\nabla|_S$, $F'=F|_S$.

Theorem 2.5. *[11] For $x\in S$, we have that*
(1) if $X,Y\in T_xS$, we have $F(X,Y)=F'(X,Y)$;
(2) if $X\in T_xS$, $Y\in\mathfrak{h}_x$, we have $F(X,Y)=\nabla'_X\phi_Y$;
(3) if $X,Y\in\mathfrak{h}_x$, we have $F(X,Y)=\phi_{[X,Y]}+[\phi_X,\phi_Y]$.

Proof. (1) It holds obviously by the definition of F'.
(2) By the third identity in Lemma 2.4, for $Y\in\mathfrak{h}_x$, we have $\mathrm{d}\Phi(Y,X)=-[\theta_Y,\Phi_X]-X(\phi_Y)$ with $X\in T_xS$. Then we have

$$\begin{aligned}F(X,Y) &= d\Phi(X,Y)+[\Phi_X,\Phi_Y]=X(\phi_Y)-[\Phi_X,\theta_Y]+[\Phi_X,\Phi_Y]\\ &= X(\phi_Y)+[\Phi_X,\phi_Y]=\nabla'_X\phi_Y.\end{aligned}$$

(3) By (2) and the second identity in Lemma 2.4, we have

$$\begin{aligned}F(X,Y) &= \nabla_X\phi_Y=X(\phi_Y)+[\Phi_X,\phi_Y]=\phi_{[X,Y]}-[\theta_X,\phi_Y]+[\Phi_X,\phi_Y]\\ &= \phi_{[X,Y]}+[\phi_X,\phi_Y].\end{aligned}$$

□

3. ANTI-SELF-DUAL YANG-MILLS EQUATION

Let

$$\Phi = \Phi_{00'}\theta^{00'} + \Phi_{10'}\theta^{10'} + \Phi_{01'}\theta^{01'} + \Phi_{11'}\theta^{11'} + \Phi_T\theta \tag{3.1}$$

be a matrix-valued connection form on $\mathscr{H}$, where $\{\theta^{AA'}, \theta\}$ are 1-forms dual to $\{V_{AA'}, T\}$ and $\Phi_{AA'} = \Phi(V_{AA'})$ and $\Phi_T = \Phi(T)$. Locally,

$$\theta^{AA'} = \mathrm{d}y_{AA'},\ \theta = \mathrm{d}t + y_{11'}\mathrm{d}y_{00'} + y_{01'}\mathrm{d}y_{10'} - y_{10'}\mathrm{d}y_{01'} - y_{00'}\mathrm{d}y_{11'}.$$

Define the connection associated to the connection form Φ

$$\nabla_A = \nabla_{A1'} - \zeta\nabla_{A0'} := (V_{A1'} + \Phi_{A1'}) - \zeta(V_{A0'} + \Phi_{A0'}), \tag{3.2}$$

$A = 0, 1$, for fixed $\zeta \in \mathbb{C}$. A connection on $\mathscr{H}$ called *anti-self-dual* (briefly ASD) if it is flat over any α-plane, i.e.

$$F(V_0, V_1) = 0. \tag{3.3}$$

The ASD condition (3.3) is equivalent to

$$\zeta^2 F(V_{00'}, V_{10'}) - \zeta(F(V_{00'}, V_{11'}) + F(V_{01'}, V_{10'})) + F(V_{01'}, V_{11'}) = 0.$$

Comparing the coefficients of ζ^2, ζ^1 and ζ^0, we get

$$F(V_{00'}, V_{10'}) = 0, \quad F(V_{00'}, V_{11'}) + F(V_{01'}, V_{10'}) = 0, \quad F(V_{01'}, V_{11'}) = 0, \tag{3.4}$$

which is equivalent to (1.10).

Remark 3.1. However, if we choose the connection form with respect to 1-forms $\mathrm{d}y_{00'}$, $\mathrm{d}y_{01'}$, $\mathrm{d}y_{10'}$, $\mathrm{d}y_{11'}$, $\mathrm{d}t$, the connection form (3.1) can be written as

$$\Phi = A_{00'}\mathrm{d}y_{00'} + A_{10'}\mathrm{d}y_{10'} + A_{01'}\mathrm{d}y_{01'} + A_{11'}\mathrm{d}y_{11'} + A_t\mathrm{d}t \tag{3.5}$$

where $A_{BB'} = \Phi(\frac{\partial}{\partial y_{BB'}})$ and $A_t = \Phi(\frac{\partial}{\partial t})$. Then the ASD equations (1.10) can be rewritten as

$$\begin{cases} \frac{\partial A_{10'}}{\partial y_{00'}} - \frac{\partial A_{00'}}{\partial y_{10'}} + [A_{00'}, A_{10'}] - y_{01'}\left(\frac{\partial A_t}{\partial y_{00'}} - \frac{\partial A_{00'}}{\partial t} + [A_{00'}, A_t]\right) \\ +y_{11'}\left(\frac{\partial A_t}{\partial y_{10'}} - \frac{\partial A_{10'}}{\partial t} + [A_{10'}, A_t]\right) = 0, \\ \frac{\partial A_{11'}}{\partial y_{00'}} - \frac{\partial A_{00'}}{\partial y_{11'}} + [A_{00'}, A_{11'}] + \frac{\partial A_{10'}}{\partial y_{01'}} - \frac{\partial A_{01'}}{\partial y_{10'}} + [A_{01'}, A_{10'}] \\ -y_{01'}\left(\frac{\partial A_t}{\partial y_{01'}} - \frac{\partial A_{01'}}{\partial t} + [A_{01'}, A_t]\right) - y_{10'}\left(\frac{\partial A_t}{\partial y_{10'}} - \frac{\partial A_{10'}}{\partial t} + [A_{10'}, A_t]\right) \\ +y_{00'}\left(\frac{\partial A_t}{\partial y_{00'}} - \frac{\partial A_{00'}}{\partial t} + [A_{00'}, A_t]\right) + y_{11'}\left(\frac{\partial A_t}{\partial y_{11'}} - \frac{\partial A_{11'}}{\partial t} + [A_{11'}, A_t]\right) = 0, \\ \frac{\partial A_{11'}}{\partial y_{01'}} - \frac{\partial A_{01'}}{\partial y_{11'}} + [A_{01'}, A_{11'}] + y_{00'}\left(\frac{\partial A_t}{\partial y_{01'}} - \frac{\partial A_{01'}}{\partial t} + [A_{01'}, A_t]\right) \\ -y_{10'}\left(\frac{\partial A_t}{\partial y_{11'}} - \frac{\partial A_{11'}}{\partial t} + [A_{11'}, A_t]\right) = 0. \end{cases} \tag{3.6}$$

Moreover, if $F(T, \cdot) = 0$, the ASD equation (3.6) could reduce to classical case (1.10).

4. REDUCTION

In this section, we consider the reduction of ASD equation generated by the right action of Lie subgroups of Heisenberg group, which preserves α-surface.

4.1. **Reduction by Reeb vector.** $\Gamma := \{(0,0,0,0,s)|s \in \mathbb{R}\}$ is an Abelian subgroup of $\mathscr{H}$. It is generated by T. Consider the transversal hyperplane $S = \mathbb{C}^4 = \{y_{00'}, y_{10'}, y_{01'}, y_{11'}\}$, whose coordinates are constant along T. Let $\nabla' := \nabla|_S$ be the connection restricted to $\mathbb{C}^4$.

As $\frac{\partial}{\partial y_{AA'}} \in TS$ and $T \in T\Gamma$, by Theorem 2.5, the ASD equations (3.4) reduce to

$$\begin{cases} \dfrac{\partial A_{10'}}{\partial y_{00'}} - \dfrac{\partial A_{00'}}{\partial y_{10'}} + [A_{00'}, A_{10'}] - y_{01'}\nabla'_{\frac{\partial}{\partial y_{00'}}}\phi_T + y_{11'}\nabla'_{\frac{\partial}{\partial y_{10'}}}\phi_T = 0, \\ \dfrac{\partial A_{11'}}{\partial y_{00'}} - \dfrac{\partial A_{00'}}{\partial y_{11'}} + [A_{00'}, A_{11'}] + \dfrac{\partial A_{10'}}{\partial y_{01'}} - \dfrac{\partial A_{01'}}{\partial y_{10'}} + [A_{01'}, A_{10'}] \\ + y_{00'}\nabla'_{\frac{\partial}{\partial y_{00'}}}\phi_T + y_{11'}\nabla'_{\frac{\partial}{\partial y_{11'}}}\phi_T - y_{01'}\nabla'_{\frac{\partial}{\partial y_{01'}}}\phi_T - y_{10'}\nabla'_{\frac{\partial}{\partial y_{10'}}}\phi_T = 0, \\ \dfrac{\partial A_{11'}}{\partial y_{01'}} - \dfrac{\partial A_{01'}}{\partial y_{11'}} + [A_{01'}, A_{11'}] - y_{10'}\nabla'_{\frac{\partial}{\partial y_{11'}}}\phi_T + y_{00'}\nabla'_{\frac{\partial}{\partial y_{01'}}}\phi_T = 0, \end{cases}$$

where $A_{BB'} = \Phi(\frac{\partial}{\partial y_{BB'}})$ and ϕ_T is the Higgs field along T.

4.2. **The complex Bogomolny equation.** Consider the Lie subgroup $\Gamma = \{H_{(0,-s_1,s_1,0,s_2)|s_1,s_2\in\mathbb{R}}\}$. Its right action on $H_{(y_{00'},y_{10'},y_{01'},y_{11'},t)}$ is

$$H_{(y_{00'},y_{10'},y_{01'},y_{11'},t)} \cdot H_{(0,-s_1,s_1,0,s_2)} = H_{(y_{00'},y_{10'}-s_1,y_{01'}+s_1,y_{11'},t+s_2+y_{10'}s_1+y_{01'}s_1)}.$$

So the corresponding vector fields X_1 and X_2 are

$$\begin{aligned} X_1 &= \frac{\partial}{\partial y_{01'}} - \frac{\partial}{\partial y_{10'}} + (y_{01'} + y_{10'})T = V_{01'} - V_{10'}, \\ X_2 &= T. \end{aligned} \tag{4.1}$$

We can choose a transversal S to the orbits by

$$S = \{y_{00'}, x := y_{01'} + y_{10'}, y_{11'}, t = 0\}.$$

Since $[V_{01'}, V_{11'}] = 0$, we can choose the invariant gauge such that $\Phi_{01'} = \Phi_{11'} = 0$. In this case, the reduced Lax pair is

$$L = \frac{\partial}{\partial x} - \zeta\left(\frac{\partial}{\partial y_{00'}} + \Phi_{00'}\right), \quad M = \frac{\partial}{\partial y_{11'}} - \zeta\left(\frac{\partial}{\partial x} + \phi_{10'}\right), \tag{4.2}$$

where $\phi_{10'}$ is the higgs filed with respect to $V_{10'}$. So the ASD equations reduce to

$$\begin{cases} \frac{\partial \phi_{10'}}{\partial y_{00'}} - \frac{\partial \Phi_{00'}}{\partial x} + [\Phi_{00'}, \phi_{10'}] = 0, \\ \frac{\partial \phi_{10'}}{\partial x} - \frac{\partial \Phi_{00'}}{\partial y_{11'}} = 0, \end{cases} \tag{4.3}$$

which is the generalized complex Bogomolny equation over Heisenberg group.

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