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# EQUIVALENCE BETWEEN GENERALIZED COMPLEMENTARITY PROBLEM AND GENERALIZED VARIATIONAL INEQUALITY PROBLEM INVOLVING XOR-OPERATION WITH EXISTENCE OF SOLUTION

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Abstract. In this paper, we study a generalized complementarity problem and a variational inequality problem involving XOR-operation. Equivalence between both the problems is shown by using the technique of Karamardian [16, 17]. An iterative algorithm is defined for solving generalized variational inequality problem involving XOR-operation. An existence and convergence result is proved. We provide an example in support of some of the concepts used in our main result.

## 1. Introduction

The techniques of variational inequalities are powerful tools for solving many problems related to mechanics, optimization, transportation, economics, elasticity, etc., see for example [2, 6]. Due to their applications, variational inequalities were generalized and extended in various directions. Equally important is the area of operations research known as complementarity theory, which has received much recognition in recent past. If the convex set involved in a variational inequality problem and a complementarity problem is a convex cone, then both the problems are equivalent, see Karamardian [16, 17]. Indeed, variational inequality problems are more general than complementarity problems and include them as special cases. For more details on variational inequalities, complementarity problems and their applications, we refer to [4, 5, 7, 8, 10, 12, 13, 18–20, 27–30].

A Boolean logic operation called "exclusive or", or XOR-operation is widely used in cryptography as well as in generating parity bits for error checking and fault tolerance. XOR compares two input bits and generate one output bit. The logic is simple, if the bits are same, the result is zero. Suppose a system receiving a continuous stream of data in some fixed packet size. The parity check can help to know whether the data is received, correctly received or has been corrupted. We mention one more application of XOR-operation, that is, when a file is being transferred from server $A$ to server $B$, it is cut into blocks, sent over piece by piece, then reattached at its destination. To ensure each block is not corrupted, a checksum function is run on it, generating a unique checksum. All the symbols are then intercepted in base 2 (1's and 0's), and each chunk is XOR-ed together. One can find application of XOR-terminology in digital electronics, combination

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and sequential circuits, multibit inverter, etc.. For some suitable work related to variational inclusions involving XOR-operation, we refer to [1, 21–23].

Merging all the concepts discussed above, in this paper, we study a generalized complementarity problem and a generalized variational inequality problem involving XOR-operation. An equivalence between them is establish by taking convex set to be convex cone involved in both the problems. Finally, an iterative algorithm is defined to obtain the solution of generalized variational inequality problem involving XOR-operation. Convergence criteria is also discussed. An example is provided.

## 2. Preliminaries

We assume $E$ to be real ordered positive Hilbert space with its norm $\|.\|$ and inner product $\langle\cdot,\cdot\rangle$, $d$ is the metric induced by the norm $\|\cdot\|$, $\mathcal{CB}(E)$ is the family of nonempty, closed and bounded subsets of $E$, and $D(.,.)$ is Hausdörff metric on $\mathcal{CB}(E)$.

The following notations and definitions are required to prove our result, which can be traced in [24, 25].

**Definition 2.1.** *A nonempty closed convex subset $C$ of $E$ is said to be a cone, if*

(*i*) *for any $x \in C$ and $\lambda > 0$, $\lambda x \in C$,*
(*ii*) *for any $x \in C$ and $-x \in C$, then $x = 0$.*

**Definition 2.2.** *For arbitrary elements $x, y \in E$, $x \leq y$ (or $y \leq x$) holds, then $x$ and $y$ are said to be comparable to each other (denoted by $x \propto y$).*

**Definition 2.3.** *For arbitrary elements $x, y \in E$, $lub\{x, y\}$ and $glb\{x, y\}$ means the least upper bound and the greatest upper bound of the set $\{x, y\}$. Suppose $lub\{x, y\}$ and $glb\{x, y\}$ exist, then some binary operations are defined as:*

(*i*) $x \vee y = lub\{x, y\}$,
(*ii*) $x \wedge y = glb\{x, y\}$,
(*iii*) $x \oplus y = (x - y) \vee (y - x)$,
(*iv*) $x \odot y = (x - y) \wedge (y - x)$.

*The operations $\vee, \wedge$, $\oplus$ and $\odot$ are called OR, AND, XOR and XNOR operations, respectively.*

**Proposition 2.4.** *Let $\oplus$ be an XOR-operation and $\odot$ be an XNOR-operation. Then the following relations hold:*

(*i*) $x \odot x = 0$, $x \odot y = y \odot x$,, $x \odot y = -(x \oplus y)$,
(*ii*) *if* $x \propto 0$, *then* $-x \oplus 0 \leq x \leq x \oplus 0$,
(*iii*) $0 \leq x \oplus y$, *if* $x \propto y$,
(*iv*) *if* $x \propto y$, *then* $x \oplus y = 0$ *if and only if* $x = y$,
(*v*) $x \oplus y = y \oplus x$,
(*vi*) $x \oplus x = 0$,
(*vii*) $0 \leq x \oplus 0$,
(*viii*) $(\lambda x) \oplus (\lambda y) = |\lambda|(x \oplus y)$,
(*ix*) *if* $x \propto y$, *then* $(x \oplus 0) \oplus (y \oplus 0) \leq (x \oplus y) \oplus 0 = x \oplus y$, *for all* $x, y \in E$ *and* $\lambda \in \mathbb{R}$.

**Proposition 2.5.** *Let $C$ be a cone in $E$. Then for each $x, y \in E$, the following relations hold:*

(*i*) $\|0 \oplus 0\| = \|0\| = 0$,
(*ii*) $\|x \oplus y\| \leq \|x - y\|$,
(*iii*) *if $x \propto y$, then $\|x \oplus y\| = \|x - y\|$.*

**Definition 2.6.** *Let $\psi : E \to \mathbb{R} \cup \{+\infty\}$ be a proper convex functional. A vector $\omega \in E$ is called subgradient of $\psi$ at $x \in dom\psi$, if*

$$\langle \omega, y - x \rangle \leq \psi(y) - \psi(x), \text{ for all } y \in E.$$

*The set of all subgradients of $\psi$ at $x$ is denoted by $\partial\psi(x)$ . The mapping $\partial\psi : E \to 2^E$ defined by*

$$\partial\psi(x) = \Big\{\omega \in E : \langle \omega, y - x \rangle \leq \psi(y) - \psi(x), \text{ for all } y \in E\Big\}$$

*is called subdifferential of $\psi$.*

**Definition 2.7.** *Let $\psi : E \times E \to \mathbb{R} \cup \{+\infty\}$ be a proper convex functional. The resolvent operator $\mathcal{J}_\rho^{\partial\psi}$ is defined as*

$$\mathcal{J}_\rho^{\partial\psi(.,x)}(x) = [I + \rho\partial\psi(.,x)]^{-1}(x), \text{ for all } x \in E,$$

*where $\rho > 0$ is a constant and $I$ is the identity operator.*

**Definition 2.8.** *A mapping $\psi : E \times E \to \mathbb{R} \cup \{+\infty\}$ is said to be positive homogeneous in the first argument, if for all $\alpha > 0$ and $x \in E$, $\psi(\alpha x, y) = \alpha\psi(x, y)$, for all $y \in E$.*

**Definition 2.9.** *A multi-valued mapping $T : E \to \mathcal{CB}(E)$ is said to be D-Lipschitz continuous, if there exists a constant $\lambda_{D_T} > 0$ such that*

$$D(T(x), T(y)) \leq \lambda_{D_T}\|x - y\|, \text{ for all } x, y \in E.$$

**Definition 2.10.** *A multi-valued mapping $T : E \to 2^E$ is said to be relaxed Lipschitz continuous, if there exists a constant $k > 0$ such that*

$$\langle w_1 - w_2, x - y \rangle \leq -k\|x - y\|^2, \text{ for all } x, y \in E \text{ and } w_1 \in T(x), w_2 \in T(y).$$

## 3. Formulation of the problem

Let $E$ to be a real ordered positive Hilbert space and $C \subset E$ be a closed convex pointed cone. Let $T : C \to \mathcal{CB}(E) \setminus \{\emptyset\}$ be a multi-valued mapping with nonempty values, $\psi : C \times C \to \mathbb{R} \cup \{+\infty\}$ be a proper convex functional and $A : E \times C \to \mathbb{R}$ be a mapping. We consider the following generalized complementarity problem involving XOR-operation.

Find $x \in C$, $t \in T(x)$ such that

$$\langle A(t, x), x \rangle \oplus \psi(x, x) = 0 \text{ and } \langle A(t, x), y \rangle \oplus \psi(y, x) \geq 0, \ \forall\, y \in C. \tag{3.1}$$

Below we mention some special cases of problem (3.1).

($i$) If $E$ is a real Banach space, $A(t,x) = t, \psi(x,x) = \psi(x), \psi(y,x) = \psi(y), \mathcal{CB}(E) \setminus \{\emptyset\} = 2^E \setminus \{\emptyset\}$ and replacing $\oplus$ by $+$, we obtain the following generalized $\psi$-complementarity problem introduced and studied by Huang and Regan [11].

Find $x \in C$ and $t \in T(x)$ such that

$$\langle t, x\rangle + \psi(x) = 0 \text{ and } \langle t, y\rangle + \psi(y) \geq 0, \ \forall\, y \in C. \tag{3.2}$$

($ii$) If $E$ is a real Banach space, $\langle A(t,x), x\rangle = A(t,x), \langle A(t,x), y\rangle = A(t,y)$, $\psi(x,x) = 0 = \psi(y,x), \mathcal{CB}(E) \setminus \{\emptyset\} = 2^E \setminus \{\emptyset\}$ and replacing $\oplus$ by $+$, we can obtain the following generalized complementarity problem introduced and studied by Farajzadeh and Harandi [9].

Find $x \in C$ and $t \in T(x)$ such that

$$A(t,x) = 0 \text{ and } A(t,y) \geq 0, \ \forall\, y \in C. \tag{3.3}$$

It is worth to mention that for suitable choices of operators involved in the formulation of problem (3.1), one can obtain the problems studied by Yin and Xu [31], Bazan and Lopez [3] and Isac [14, 15], etc..

In connection with generalized complementarity problem involving XOR-operation (3.1), we study the following generalized variational inequality problem involving XOR-operation.

Find $x \in C$, $t \in T(x)$ such that

$$\langle A(t,x), y - x\rangle \oplus (\psi(y,x) - \psi(x,x)) \geq 0, \ \forall\, y \in C. \tag{3.4}$$

Problem (3.4) includes many known variational inequalities problems available in the literature.

We establish an equivalence result between generalized complementarity problem involving XOR-operation (3.1) and generalized variational inequality problem involving XOR-operation (3.4).

**Theorem 3.1.** *Let $T : C \to \mathcal{CB}(E) \setminus \{\emptyset\}$ be a multi-valued mapping with nonempty values, $\psi : C \times C \to \mathbb{R} \cup \{+\infty\}$ be a proper convex functional such that $\psi$ is positive homogeneous in first argument and $A : E \times C \to \mathbb{R}$ be a mapping. If $\langle A(t,x), x\rangle \propto \psi(x,x)$, for all $x \in C, t \in T(x)$, then generalized complementarity problem involving XOR-operation* (3.1) *and generalized variational inequality problem involving XOR-operation* (3.4) *are equivalent.*

*Proof.* Let the generalized complementarity problem involving XOR-operation (3.1) holds. That is, $x \in C$, $t \in T(x)$ such that

$$\langle A(t,x), x\rangle \oplus \psi(x,x) = 0 \text{ and } \langle A(t,x), y\rangle \oplus \psi(y,x) \geq 0, \ \forall\, y \in C.$$

Since $\langle A(t,x), x\rangle \propto \psi(x,x)$, using (iv) of Proposition 2.4, we have

$$\langle A(t,x), x\rangle = \psi(x,x). \tag{3.5}$$

As $\langle A(t,x), y\rangle \oplus \psi(y,x) \geq 0$, we have

$$\langle A(t,x), y\rangle \oplus (\psi(y,x) \oplus \psi(y,x)) \geq \psi(y,x).$$

By (vi) and (ix) of Proposition 2.4, we obtain

$$\langle A(t,x), y\rangle \oplus 0 \quad \geq \quad 0 \oplus \psi(y,x),$$

$$\langle A(t,x), y\rangle \geq \psi(y,x). \tag{3.6}$$

Because

$$\langle A(t,x), y-x\rangle = \langle A(t,x), y\rangle - \langle A(t,x), x\rangle. \tag{3.7}$$

Using (3.5) and (3.6), (3.7) becomes

$$\begin{aligned}
\langle A(t,x), y-x\rangle &\geq \psi(y,x)-\psi(x,x),\\
\langle A(t,x), y-x\rangle \oplus (\psi(y,x)-\psi(x,x)) &\geq (\psi(y,x)-\psi(x,x)) \oplus (\psi(y,x)-\psi(x,x)),\\
\langle A(t,x), y-x\rangle \oplus (\psi(y,x)-\psi(x,x)) &\geq 0.
\end{aligned}$$

Thus, the generalized variational inequality problem involving XOR-operation (3.4) holds.

Conversely, suppose that generalized variational inequality problem involving XOR-operation (3.4) holds. That is, $x \in C, t \in T(x)$ such that

$$\langle A(t,x), y-x\rangle \oplus (\psi(y,x)-\psi(x,x)) \geq 0, \ \forall\ y \in C.$$

Since $C$ is a closed convex pointed cone in $E$, clearly $y = 2x \in C$ and $y = \frac{1}{2}x \in C$. Putting $y = 2x$ and $y = \frac{1}{2}x$ in generalized variational inequality problem involving XOR-operation (3.4), respectively and using positive homogeneity of $\psi$ in the first argument, we obtain

$$\begin{aligned}
\langle A(t,x), 2x-x\rangle \oplus (\psi(2x,x)-\psi(x,x)) &\geq 0,\\
\langle A(t,x), x\rangle \oplus \psi(x,x) &\geq 0. \qquad (3.8)
\end{aligned}$$

Since

$$\begin{aligned}
\langle A(t,x), y-x\rangle \oplus (\psi(y,x)-\psi(x,x)) &\geq 0,\\
\langle A(t,x), y-x\rangle &\geq (\psi(y,x)-\psi(x,x)),\\
\langle A(t,x), \frac{1}{2}x-x\rangle &\geq \psi\Big(\frac{1}{2}x,x\Big)-\psi(x,x),\\
\langle A(t,x), -\frac{1}{2}x\rangle &\geq -\frac{1}{2}\psi(x,x),\\
\text{or } \langle A(t,x), x\rangle &\leq \psi(x,x).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\langle A(t,x), x\rangle \oplus \psi(x,x) &\leq \psi(x,x)\oplus\psi(x,x) = 0,\\
\langle A(t,x), x\rangle \oplus \psi(x,x) &\leq 0. \qquad (3.9)
\end{aligned}$$

Combining (3.8) and (3.9), we have

$$\langle A(t,x), x\rangle \oplus \psi(x,x) = 0. \tag{3.10}$$

Applying (ix) of Proposition 2.4 and (3.5), from (3.4), we have

$$\begin{aligned}
\langle A(t,x), y-x\rangle \oplus (\psi(y,x)-\psi(x,x)) &\geq 0,\\
\langle A(t,x), y-x\rangle &\geq (\psi(y,x)-\psi(x,x)),\\
\langle A(t,x), y\rangle - \langle A(t,x), x\rangle &\geq (\psi(y,x)-\psi(x,x)),\\
\langle A(t,x), y\rangle - \psi(x,x) &\geq \psi(y,x)-\psi(x,x).
\end{aligned}$$

That is,

$$\begin{aligned}\langle A(t,x), y\rangle &\geq \psi(y,x),\\ \langle A(t,x), y\rangle \oplus \psi(y,x) &\geq \psi(y,x) \oplus \psi(y,x) = 0,\\ \langle A(t,x), y\rangle \oplus \psi(y,x) &\geq 0. \qquad (3.11)\end{aligned}$$

Equation (3.10) and inequality (3.11) constitute the required generalized complementarity problem involving XOR-operation (3.1). □

## 4. Existence and Convergence Result

We provide a fixed point formulation of generalized variational inequality problem involving XOR-operation (3.4). Based on this fixed point formulation, we define an iterative algorithm to obtain solution of generalized variational inequality problem involving XOR-operation (3.4).

**Lemma 4.1.** *The generalized variational inequality problem involving XOR-operation (3.4) have a solution $x \in C$, $t \in T(x)$, if and only if the following equation is satisfied:*

$$x = \mathcal{J}_\rho^{\partial\psi(.,x)}[x + \rho A(t,x)\ ], \qquad (4.1)$$

*where $\mathcal{J}_\rho^{\partial\psi(.,x)} = [I + \rho\partial\psi(.,x)]^{-1}$ is the resolvent operator, $\rho > 0$ is a constant and $I$ is the identity operator.*

*Proof.* Assume that $x \in C$, $t \in T(x)$ satisfy the relation (4.1). Using the definition of resolvent operator $\mathcal{J}_\rho^{\partial\psi(.,x)}$, we have

$$\begin{aligned} x &= \mathcal{J}_\rho^{\partial\psi(.,x)}[x + \rho A(t,x)\ ]\\ &= [I + \rho\partial\psi(.,x)]^{-1}[x + \rho A(t,x)\ ],\\ x + \rho\partial\psi(x,x) &= [x + \rho A(t,x)\ ]. \end{aligned}$$

The above relation holds if and only if

$$A(t,x) \in \partial\psi(x,x).$$

By the definition of subdifferential of $\psi$, we have

$$\psi(y,x) - \psi(x,x) \geq \langle A(t,x), y - x\rangle.$$

Using $(vi)$ of Proposition 2.4, we have

$$\begin{aligned}\langle A(t,x), y-x\rangle \oplus (\psi(y,x) - \psi(x,x)) &\geq \langle A(t,x), y-x\rangle \oplus \langle A(t,x), y-x\rangle\\ \langle A(t,x), y-x\rangle \oplus (\psi(y,x) - \psi(x,x)) &\geq 0,\ \forall\, y \in C.\end{aligned}$$

It follows that $x \in C$, $t \in T(x)$ is a solution of generalized variational inequality problem involving XOR-operation (3.4). □

Based on Lemma 4.1, we construct the following iterative algorithm for solving generalized variational inequality problem involving XOR-operation (3.4).

**Algorithm 5.** *Let $C \subset E$ be a closed convex pointed cone and $t_n \propto t_{n-1}$, for $n = 0, 1, 2, \cdots$ . For $x_0 \in C$, $t_0 \in T(x_0)$ and $\alpha \in [0, 1]$, let*

$$x_1 = (1-\alpha)x_0 + \alpha \mathcal{J}_\rho^{\partial\psi(.,x_0)}[x_0 + \rho A(t_0, x_0)].$$

*Since $t_0 \in T(x_0) \in \mathcal{CB}(E)$, by Nadler [26], there exists $t_1 \in T(x_1)$ and suppose that $t_0 \propto t_1$, using $(iii)$ of Proposition 2.5, we have*

$$\|t_0 \oplus t_1\| = \|t_0 - t_1\| \leq (1+1)D(T(x_0), T(x_1)).$$

*Continuing the above procedure inductively, we compute the sequences $\{x_n\}$ and $\{t_n\}$ for $x_n \in C, t_n \in T(x_n)$ by the following scheme:*

$$x_{n+1} = (1-\alpha)x_n + \alpha \mathcal{J}_\rho^{\partial\psi(.,x_n)}[x_n + \rho A(t_n, x_n)], \tag{5.1}$$

$$\|t_n \oplus t_{n+1}\| = \|t_n - t_{n+1}\| \leq \left(1 + \frac{1}{n+1}\right) D(T(x_n), T(x_{n+1})), \tag{5.2}$$

*for $n = 0, 1, 2, \cdots$*

We prove existence and convergence result for generalized variational inequality problem involving XOR-operation (3.4).

**Theorem 5.1.** *Let $E$ be a real ordered positive Hilbert space and $C$ be a closed convex pointed cone in $E$ with partial ordering " $\leq$ ", $x_n \propto x_{n-1}, t_n \propto t_{n-1}, n = 1, 2, \cdots$ . Let $T : C \to \mathcal{CB}(E) \setminus \{\emptyset\}$ is a multi-valued mapping with nonempty values such that $T$ is $D$-Lipschitz continuous with constant $\lambda_{D_T}$. Let $\psi : C \times C \to \mathbb{R} \cup \{+\infty\}$ be a proper convex functional such that the resolvent operators associated with $\psi(., x_n)$ and $\psi(., x_{n-1})$ are comparable, that is $\mathcal{J}_\rho^{\partial\psi(.,x_n)} \propto \mathcal{J}_\rho^{\partial\psi(.,x_{n-1})}$.*

*Let $A : E \times C \to \mathbb{R}$ be a mapping such that $A$ is Lipschitz continuous and relaxed Lipschitz continuous in both the arguments with constants $\lambda_{A_1}, \lambda_{A_2}, \lambda_{C_1}$ and $\lambda_{C_2}$, respectively. If the following conditions are satisfied:*

$$\left\|\mathcal{J}_\rho^{\partial\psi(.,x_n)}(z) - \mathcal{J}_\rho^{\partial\psi(.,x_{n-1})}(z)\right\| \leq \mu\|x_n - x_{n-1}\|, \forall\ z \in C, \tag{5.3}$$

*and*

$$\left|\rho - \frac{\lambda_{A_1}\lambda_{D_T}^2 + \lambda_{C_2}}{(\lambda_{A_1}\lambda_{D_T} + \lambda_{A_2})^2}\right| < \sqrt{\frac{(\lambda_{C_1}\lambda_{D_T}^2 + \lambda_{C_2})^2 - (\lambda_{A_1}\lambda_{D_T} + \lambda_{A_2})^2(2\mu - \mu^2)}{(\lambda_{A_1}\lambda_{D_T} + \lambda_{A_2})^4}}, \tag{5.4}$$

*where all the constants involved in* (5.3) *and* (5.4) *are positive, then the sequences $\{x_n\}$ and $\{t_n\}$ generated by Algorithm 5 strongly converge to the solution $x$ and $t$ of generalized variational inequality problem involving XOR-operation* (3.4)*, respectively.*

*Proof.* Since $x_{n+1} \propto x_n$, $n = 1, 2, \cdots$ , using $(iii)$ of Proposition 2.4 and (5.1) of Algorithm 5, we have

$$\begin{aligned} 0 \leq x_{n+1} \oplus x_n &= \Big[(1-\alpha)x_n + \alpha \mathcal{J}_\rho^{\partial\psi(.,x_n)}[x_n + \rho A(t_n, x_n)]\Big] \\ &\quad \oplus \Big[(1-\alpha)x_{n-1} + \alpha \mathcal{J}_\rho^{\partial\psi(.,x_{n-1})}[x_{n-1} + \rho A(t_{n-1}, x_{n-1})]\Big] \\ &= (1-\alpha)(x_n \oplus x_{n-1}) + \alpha\Big[\mathcal{J}_\rho^{\partial\psi(.,x_n)}[x_n + \rho A(t_n, x_n)] \end{aligned}$$

$$\oplus \mathcal{J}_{\rho}^{\partial\psi(.,x_{n-1})}[x_{n-1} + \rho A(t_{n-1}, x_{n-1})]\Big]. \tag{5.5}$$

From (5.5), it follows that

$$\begin{aligned}
\|x_{n+1} \oplus x_n\| &= \Big\|(1-\alpha)(x_n \oplus x_{n-1}) + \alpha\Big[\mathcal{J}_{\rho}^{\partial\psi(.,x_n)}[x_n + \rho A(t_n, x_n)] \\
&\quad \oplus \mathcal{J}_{\rho}^{\partial\psi(.,x_{n-1})}[x_{n-1} + \rho A(t_{n-1}, x_{n-1})]\Big]\Big\| \\
&\leq (1-\alpha)\|x_n \oplus x_{n-1}\| + \alpha\Big\|\mathcal{J}_{\rho}^{\partial\psi(.,x_n)}[x_n + \rho A(t_n, x_n)] \\
&\quad \oplus \mathcal{J}_{\rho}^{\partial\psi(.,x_{n-1})}[x_{n-1} + \rho A(t_{n-1}, x_{n-1})]\Big\|. \qquad (5.6)
\end{aligned}$$

As $x_n \propto x_{n-1}$, $\mathcal{J}_{\rho}^{\partial\psi(.,x_n)} \propto \mathcal{J}_{\rho}^{\partial\psi(.,x_{n-1})}$, $n = 0, 1, 2, \cdots$, using (iii) of Proposition 2.5, condition (5.3) and nonexpensiveness of the resolvent operator $\mathcal{J}_{\rho}^{\partial\psi(.,x)}$, (5.6) becomes

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq (1-\alpha)\|x_n - x_{n-1}\| + \alpha\Big\|\mathcal{J}_{\rho}^{\partial\psi(.,x_n)}[x_n + \rho A(t_n, x_n)] \\
&\quad -\mathcal{J}_{\rho}^{\partial\psi(.,x_{n-1})}[x_{n-1} + \rho A(t_{n-1}, x_{n-1})]\Big\| \\
&\leq (1-\alpha)\|x_n - x_{n-1}\| + \alpha\Big\|[\mathcal{J}_{\rho}^{\partial\psi(.,x_n)}[x_n + \rho A(t_n, x_n)] \\
&\quad -\mathcal{J}_{\rho}^{\partial\psi(.,x_{n-1})}[x_n + \rho A(t_n, x_n)] + \mathcal{J}_{\rho}^{\partial\psi(.,x_{n-1})}[x_n + \rho A(t_n, x_n)] \\
&\quad -\mathcal{J}_{\rho}^{\partial\psi(.,x_{n-1})}[x_{n-1} + \rho A(t_{n-1}, x_{n-1})]\Big\| \\
&\leq (1-\alpha)\|x_n - x_{n-1}\| + \alpha\Big\|[\mathcal{J}_{\rho}^{\partial\psi(.,x_n)}[x_n + \rho A(t_n, x_n)] \\
&\quad -\mathcal{J}_{\rho}^{\partial\psi(.,x_{n-1})}[x_n + \rho A(t_n, x_n)]\Big\| + \alpha\Big\|\mathcal{J}_{\rho}^{\partial\psi(.,x_{n-1})}[x_n + \rho A(t_n, x_n)] \\
&\quad -\mathcal{J}_{\rho}^{\partial\psi(.,x_{n-1})}[x_{n-1} + \rho A(t_{n-1}, x_{n-1})]\Big\| \\
&\leq (1-\alpha)\|x_n - x_{n-1}\| + \alpha\mu\|x_n - x_{n-1}\| \\
&\quad +\|[x_n + \rho A(t_n, x_n)] - [x_{n-1} + \rho A(t_{n-1}, x_{n-1})]\| \\
&= (1-\alpha)\|x_n - x_{n-1}\| + \alpha\mu\|x_n - x_{n-1}\| \\
&\quad +\|(x_n - x_{n-1}) + \rho\big(A(t_n, x_n) - A(t_{n-1}, x_{n-1})\big)\|. \qquad (5.7)
\end{aligned}$$

Since $T$ is $D$-Lipschitz continuous with constant $\lambda_{D_T}$, $A$ is relaxed Lipschitz continuous in both the arguments with constants $\lambda_{C_1}$ and $\lambda_{C_2}$, respectively, using (5.2) of Algorithm 5, we have

$$\begin{aligned}
&\|(x_n - x_{n-1}) + \rho(A(t_n, x_n) - A(t_{n-1}, x_{n-1})\|^2 \\
= &\|x_n - x_{n-1}\|^2 + 2\rho\langle A(t_n, x_n) - A(t_{n-1}, x_{n-1}), x_n - x_{n-1}\rangle \\
&+\rho^2\|A(t_n, x_n) - A(t_{n-1}, x_{n-1})\|^2 \\
= &\|x_n - x_{n-1}\|^2 + 2\rho\langle A(t_n, x_n) - A(t_{n-1}, x_n), x_n - x_{n-1}\rangle \\
&+2\rho\langle A(t_{n-1}, x_n) - A(t_{n-1}, x_{n-1}), x_n - x_{n-1}\rangle \\
&+\rho^2\|A(t_n, x_n) - A(t_{n-1}, x_{n-1})\|^2 \\
\leq &\|x_n - x_{n-1}\|^2 - 2\rho\lambda_{C_1}\|t_n - t_{n-1}\|^2 - 2\rho\lambda_{C_2}\|x_n - x_{n-1}\|^2
\end{aligned}$$

$$
\begin{aligned}
& +\rho^2\|A(t_n,x_n)-A(t_{n-1},x_{n-1})\|^2.\\
\leq\ & \|x_n-x_{n-1}\|^2-2\rho\lambda_{C_1}\lambda_{D_T}^2\|x_n-x_{n-1}\|^2-2\rho\lambda_{C_2}\|x_n-x_{n-1}\|^2\\
& +\rho^2\|A(t_n,x_n)-A(t_{n-1},x_{n-1})\|^2. \qquad (5.8)
\end{aligned}
$$

Since $A$ is Lipschitz continuous in both the arguments with constants $\lambda_{A_1}$ and $\lambda_{A_2}$, respectively, and $T$ is $D$-Lipschitz continuous with constant $\lambda_{D_T}$, we have

$$
\begin{aligned}
& \|A(t_n,x_n)-A(t_{n-1},x_{n-1})\|\\
=\ & \|A(t_n,x_n)-A(t_{n-1},x_n)+A(t_{n-1},x_n)-A(t_{n-1},x_{n-1})\|\\
\leq\ & \|A(t_n,x_n)-A(t_{n-1},x_n)\|+\|A(t_{n-1},x_n)-A(t_{n-1},x_{n-1})\|\\
\leq\ & \lambda_{A_1}\|t_n-t_{n-1}\|+\lambda_{A_2}\|x_n-x_{n-1}\|\\
\leq\ & \lambda_{A_1}D(T(x_n),T(x_{n-1}))+\lambda_{A_2}\|x_n-x_{n-1}\|\\
\leq\ & \lambda_{A_1}\lambda_{D_T}\|x_n-x_{n-1}\|+\lambda_{A_2}\|x_n-x_{n-1}\|\\
\leq\ & (\lambda_{A_1}\lambda_{D_T}+\lambda_{A_2})\|x_n-x_{n-1}\|. \qquad (5.9)
\end{aligned}
$$

Using (5.9), (5.8) becomes

$$
\begin{aligned}
& \|(x_n-x_{n-1})+\rho(A(t_n,x_n)-A(t_{n-1},x_{n-1})\|^2\\
\leq\ & \|x_n-x_{n-1}\|^2-2\rho\lambda_{D_T}^2\lambda_{C_1}\|x_n-x_{n-1}\|^2-2\rho\lambda_{C_2}\|x_n-x_{n-1}\|^2\\
& +\rho^2(\lambda_{A_1}\lambda_{D_T}+\lambda_{A_2})^2\|x_n-x_{n-1}\|^2\\
\leq\ & \|x_n-x_{n-1}\|^2-2\rho\lambda_{D_T}^2\lambda_{C_1}\|x_n-x_{n-1}\|^2-2\rho\lambda_{C_2}\|x_n-x_{n-1}\|^2\\
& +\rho^2(\lambda_{A_1}\lambda_{D_T}+\lambda_{A_2})^2\|x_n-x_{n-1}\|^2\\
\leq\ & \left[1-2\rho\lambda_{D_T}^2\lambda_{C_1}-2\rho\lambda_{C_2}+\rho^2(\lambda_{A_1}\lambda_{D_T}+\lambda_{A_2})^2\right]\|x_n-x_{n-1}\|^2. \qquad (5.10)
\end{aligned}
$$

Thus, we have

$$\|(x_n-x_{n-1})+\rho(A(t_n,x_n)-A(t_{n-1},x_{n-1})\|\leq\Theta\|x_n-x_{n-1}\|, \qquad (5.11)$$

where

$$\Theta=\sqrt{1-2\rho\lambda_{D_T}^2\lambda_{C_1}-2\rho\lambda_{C_2}+\rho^2(\lambda_{A_1}\lambda_{D_T}+\lambda_{A_2})^2}.$$

Using (5.11), (5.7) becomes

$$
\begin{aligned}
\|x_{n+1}-x_n\| &\leq (1-\alpha)\|x_n-x_{n-1}\|+\alpha\mu\|x_n-x_{n-1}\|+\alpha\Theta\|x_n-x_{n-1}\|\\
&\leq [(1-\alpha)+\alpha\mu+\alpha\Theta]\|x_n-x_{n-1}\|\\
&= \xi\|x_n-x_{n-1}\|, \qquad (5.12)
\end{aligned}
$$

where $\xi=[(1-\alpha)+\alpha\mu+\alpha\Theta]$ and $\Theta=\sqrt{1-2\rho\lambda_{D_T}^2\lambda_{C_1}-2\rho\lambda_{C_2}+\rho^2(\lambda_{A_1}\lambda_{D_T}+\lambda_{A_2})^2}$. By condition (5.4), we have $\xi<1$ and consequently $\{x_n\}$ is a Cauchy sequence in $E$. Let $x_n\to x\in E$, as $n\to\infty$. From (5.2) of Algorithm5 and as $T$ is $D$-Lipschitz continuous, we have

$$
\begin{aligned}
\|t_n\oplus t_{n+1}\|=\|t_n-t_{n+1}\| &\leq \left(1+\frac{1}{n+1}\right)D(T(x_n),T(x_{n+1})\\
&\leq \left(1+\frac{1}{n+1}\right)\lambda_{D_T}\|x_n-x_{n+1}\|. \qquad (5.13)
\end{aligned}
$$

It follows that $\{t_n\}$ is also a Cauchy sequence and consequently $t_n \to t \in E$. Lastly, we prove that $t \in T(x)$. Infact, $t_n \in T(x_n)$ and

$$\begin{aligned} d(t_n, T(x)) &\leq max\left\{ d(t_n, T(x)), \underset{u \in T(x)}{sup}\, d(T(x_n), u) \right\} \\ &\leq max\left\{ \underset{v \in T(x_n)}{sup}\, d(v, T(x)), \underset{u \in T(x)}{sup}\, d(T(x_n), u) \right\} \\ &= D(T(x_n), T(x)). \end{aligned}$$

we have

$$\begin{aligned} d(t, T(x)) &\leq \|t - t_n\| + d(t_n, T(x)) \\ &\leq \|t - t_n\| + D(T(x_n), T(x)) \\ &\leq \|t - t_n\| + \lambda_{D_T} \|x_n - x\| \to 0, \ as\ n \to \infty. \end{aligned}$$

Thus, we have $d(t, T(x)) = 0$ and so $T(x) \in CB(E)$ and $t \in T(x)$. By Lemma 4.1, the result follows.

□

**Remark 5.2.** *By combining Theorem 3.1 and Theorem 5.1, we emphasize that the solution of generalized variational inequality involving XOR-operation* (3.4) *is also a solution of generalized complementarity problem involving XOR-operation* (3.1).

## 6. Numerical Example

We provide a numerical example in support of some of the concepts used in our main result.

**Example 6.1.** *Let $E = \mathbb{R}, C = [0, \infty) \subset E$.*

**(i)** *We define the mapping $A : E \times C \to \mathbb{R}$ by*

$$A(x, y) = -\frac{x}{2} - \frac{y}{2}.$$

*Now,*

$$\begin{aligned} \|A(x, w) - A(y, w)\| &= \left\| \left(-\frac{x}{2} - \frac{w}{2}\right) - \left(-\frac{y}{2} - \frac{w}{2}\right) \right\| \\ &= \left\| \frac{x}{2} - \frac{y}{2} \right\| \\ &= \frac{1}{2}\|x - y\| \\ &\leq \frac{3}{4}\|x - y\|, \end{aligned}$$

*that is, $A$ is Lipschitz continuous in first argument with constant $\lambda_{A_1} = \frac{3}{4}$.*

$$\begin{aligned} \|A(w, x) - A(w, y)\| &= \left\| \left(-\frac{w}{2} - \frac{x}{2}\right) - \left(-\frac{w}{2} - \frac{y}{2}\right) \right\| \\ &= \left\| \frac{x}{2} - \frac{y}{2} \right\| \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\|x-y\| \\
&\leq \frac{3}{2}\|x-y\|,
\end{aligned}
$$

*that is, $A$ is Lipschitz continuous in second argument with constant $\lambda_{A_2} = \frac{3}{2}$.*

$$
\begin{aligned}
\langle A(x,w) - A(y,w), x-y \rangle &= \left\langle \left(-\frac{x}{2} - \frac{w}{2}\right) - \left(-\frac{y}{2} - \frac{w}{2}\right), x-y \right\rangle \\
&= \left\langle -\frac{x}{2} + \frac{y}{2}, x-y \right\rangle \\
&= -\frac{1}{2}\|x-y\| \\
&\leq -\frac{1}{4}\|x-y\|,
\end{aligned}
$$

*that is, $A$ is relaxed Lipschitz continuous in first argument with constant $\lambda_{C_1} = \frac{1}{4}$.*

$$
\begin{aligned}
\langle A(w,x) - A(w,y), x-y \rangle &= \left\langle \left(-\frac{w}{2} - \frac{x}{2}\right) - \left(-\frac{w}{2} - \frac{y}{2}\right), x-y \right\rangle \\
&= \left\langle -\frac{x}{2} + \frac{y}{2}, x-y \right\rangle \\
&= -\frac{1}{2}\|x-y\| \\
&\leq -\frac{1}{3}\|x-y\|,
\end{aligned}
$$

*that is, $A$ is relaxed Lipschitz continuous in second argument with constant $\lambda_{C_2} = \frac{1}{3}$.*

**(ii)** *We define the mapping $T : C \to \mathcal{CB}(E)$ by*

$$T(x) = \left\{\frac{x}{2}\right\}.$$

*Since*

$$
\begin{aligned}
D(T(x), T(y)) &= \max\left\{ \sup_{x \in T(x)} d(x, T(y)), \sup_{y \in T(y)} d(T(x), y) \right\} \\
&= \max\left\{ \sup_{x \in T(x)} |x - T(y)|, \sup_{y \in T(y)} |T(x) - y| \right\} \\
&= \max\left\{ \sup\left|\frac{x}{4} - \frac{y}{4}\right|, \sup\left|\frac{x}{4} - \frac{y}{4}\right| \right\} \\
&\leq \frac{1}{2}|x-y|,
\end{aligned}
$$

*that is, $T$ is $D$-Lipschitz continuous with constant $\lambda_{D_T} = \frac{1}{2}$.*

**(iii)** *We define the mapping $\psi : C \times C \to \mathbb{R} \cup \{+\infty\}$ by*

$$\psi(x,y) = x^2$$

*The subdifferential of $\psi$ is given by*

$$\partial\psi(x,y) = 2x$$

*and the resolvent operator*

$$\begin{aligned} J_\rho^{\partial\psi(\cdot,x)}(x) &= [I+\rho\partial\psi(\cdot,x)]^{-1}(x) \\ &= [x+1\times 2x]^{-1} \\ &= (3x)^{-1} \\ &= \frac{x}{3}. \end{aligned}$$

*Clearly, if* $x_1 \propto x_2$ *so that* $J_\rho^{\partial\psi(\cdot,x)}(x_1) \propto J_\rho^{\partial\psi(\cdot,x)}(x_2)$.

**(iv)** *Additionally, the condition* (5.4) *is satisfied for the constants computed above* $\lambda_{A_1} = \frac{3}{4}, \lambda_{A_2} = \frac{3}{2}, \lambda_{C_1} = \frac{1}{4}, \lambda_{C_2} = \frac{1}{3}, \lambda_{D_T} = \frac{1}{2}, \rho = 1$ *and* $\mu = 3$. *That is,*

$$\left|\rho - \frac{\lambda_{A_1}\lambda_{D_T}^2 + \lambda_{C_2}}{(\lambda_{A_1}\lambda_{D_T} + \lambda_{A_2})^2}\right| = 0.8518$$

*and*

$$\sqrt{\frac{(\lambda_{C_1}\lambda_{D_T}^2 + \lambda_{C_2})^2 - (\lambda_{A_1}\lambda_{D_T} + \lambda_{A_2})^2(2\mu-\mu^2)}{(\lambda_{A_1}\lambda_{D_T} + \lambda_{A_2})^4}} = 0.9229$$

*Hence,*

$$\left|\rho - \frac{\lambda_{A_1}\lambda_{D_T}^2 + \lambda_{C_2}}{(\lambda_{A_1}\lambda_{D_T} + \lambda_{A_2})^2}\right| < \sqrt{\frac{(\lambda_{C_1}\lambda_{D_T}^2 + \lambda_{C_2})^2 - (\lambda_{A_1}\lambda_{D_T} + \lambda_{A_2})^2(2\mu-\mu^2)}{(\lambda_{A_1}\lambda_{D_T} + \lambda_{A_2})^4}}.$$

## References

[1] I. Ahmad, C. T. Pang, R. Ahmad and M. Ishtyak, *System of Yosida inclusions involving XOR-operation*, J. Nonlinear Convex Anal. **18** (2017), 831–845.

[2] C. Baiocchi and A. Capelo, *Variational and Quasi Variational Inequalities*, Wiley, New York, 1984.

[3] F. F. Bazán and R. López, *Asymptotic analysis, existence and sensitivity results for a class of multivalued complementarity problems*, ESAIM Control Optim. Calc. Var. **12** (2006), 271–293.

[4] Y. J. Cho, J. Li and N. J. Huang, *Solvability of implicit complementarity problems*, Math. Comput. Modelling **45** (2007), 1001–1009.

[5] R. W. Cottle and G. B. Dantzig, *Complementarity pivot theory of mathematical programing*, Linear Algebra Appl., **1**(1968), 103-125.

[6] J. Crank, *Free and Moving Boundary Problems*, Oxford Univ. Press (Clarendon), London, 1984.

[7] R. W. Cottle, *Complementarity and variational problems*, Sympos. Math. **19** (1976), 177–208.

[8] X. P. Ding and F. Q. Xia, *A new class of completely generalized quasivariational inclusions in Banach spaces*, J. Copmut. Appl. Math. **147** (2002), 369-316.

[9] A. P. Farajzadeh and A. Amini-Harandi, *Generalized Complementarity Problems in Banach Spaces*, Albanian J. Math. **3** (2009), 35–42.

[10] F. Facchinei and J. S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*, Springer-Verlag, New York, 2003.

[11] N. J. Huang, J. Li and D. O'Regan, *Generalized $f$-complementarity problems in Banach Spaces*, Nonlinear Anal. **68** (2008), 3828–3840.

[12] A. Hassouni and A. Moudafi, *A Perturbed algorithm for variational inclusions*, J. Math. Anal. Appl. **185** (1994), 706–712.

[13] P. T. Harker and J. S. Pang, *Finite-dimensional variational inequalities and nonlinear complementarity problems, A survey of theory, algorithms and applications*, Math. Program. **48** (1990), 161–220.

[14] G. Isac, *Complementarity Problems, in: Lecture Notes in Mathematics*, Springer-Verlag, Berlin, **1528**(1992).
[15] G. Isac, *Topological Methods in Complementarity Theory*, Kluwer Academic publishers, Dordrecht, Boston, London, 2000.
[16] S. Karamardian, *Generalized complementarity problem*, J. Optim. Theory Appl. **8** (1971), 161–168.
[17] S. Karamardian, *The complementarity problem*, Math. Program. **2** (1972), 107–129.
[18] I. V. Konnov, *A scalarization approach for vector variational inequalities with applications*, J. Global Optim. **32** (2005), 517–527.
[19] J. L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, New York/Berlin, 1971.
[20] J. L. Lions and G. Stampacchia, *Variational inequalities*, Comm. Pure Appl. Math. **20** (1967), 493–519.
[21] H. G. Li, *Nonlinear inclusion problems for ordered RME set-valued mappings in ordered Hilbert spaces*, Nonlinear Funct. Anal. Appl. **16** (2001), 1–8.
[22] H. G. Li, *A nonlinear inclusion problem involving* $(\alpha, \lambda)$*-NODM set-valued mappings in ordered Hilbert space*, Appl. Math. Lett. **25** (2012), 1384–1388.
[23] H. G. Li, X. B. Pan, Z. Y. Deng and C. Y. Wang, *Solving frameworks involving* $(\gamma G, \lambda)$*-weak-GRD set-valued mappings in positive Hilbert spaces*, Fixed Point Theory Appl. **2014** (2014): Article No. 146.
[24] H. G. Li, D. Qiu and Y. Zou, *Characterizations of weak-ANODD set-valued mappings with applications to approximate solution of GNMOQV inclusions involving* $\oplus$ *operator in ordered Banach spaces*, Fixed Point Theory Appl. **2013** (2013): Article No. 241.
[25] H. G. Li, L. P. Li and M. M. Jin, *A class of nonlinear mixed ordered inclusion problems for ordered* $(\alpha_A, \lambda)$*-ANODM set valued mappings with strong comparison mapping* $A$, Fixed Point Theory Appl. **2014** (2014): Article No. 79.
[26] S. B. Nadler Jr., *Multivalued Contraction Mapping*, Pacific J. Math. **30** (1969), 475–488.
[27] J. S. Pang, *The implicit complementarity problem*, in: Nonlinear Programming 4, O.L. Mangasarian, R. Meyer, and S. M. Robinson (eds.), Academic press, New York, 1981.
[28] J. S. Pang, *On the convergence of a basic iterative method for the implicit complementarity problems*, J. Optim. Theory Appl. **37** (1982), 149–162.
[29] J. S. Pang and D. Chan, *Iterative methods for variational and complementarity problems*, Math. Program. **24** (1982), 284–313.
[30] J. C. Yao, *On the generalized complementarity problem*, J. Aust. Math. Soc. Ser. B **35** (1994), 420–428.
[31] H. Yin and C. Xu, *Vector variational inequality and implicit vector complementarity problems*, in: Vector Variational Inequalities and Vector Equilibria, F. Giannessi (Ed.), vol. 38 of Nonconvex Optimization and Its Applications, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000, pp. 491–505.

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