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VISCOSITY EXTRAGRADIENT IMPLICIT RULE FOR A SYSTEM OF VARIATIONAL INCLUSIONS

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Abstract. We consider solving a general system of variational inclusions with the variational inclusion for two accretive operators and a common fixed point problem of countably many pseudocontractive mappings as constraints in a q-uniformly smooth and uniformly convex Banach space with $q \in (1, 2]$. A viscosity extragradient implicit rule for solving it is proposed and the strong convergence of the suggested algorithm under some appropriate assumptions is established.

1. Introduction

Assume always that H is a real Hilbert space endowed with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. Given a nonempty closed convex subset $C \subset H$. Let P_C be the metric (nearest point) projection from H onto C. Given a mapping $A : C \to H$. Consider the variational inequality problem (VIP) of finding a point $z^* \in C$ s.t. $\langle Az^*, y - z^* \rangle \geq 0 \ \forall y \in C$. Here the solution set of the VIP is denoted by VI(C, A). To the most of our knowledge, Korpelevich's extragradient method [13] is now one of the most popular methods for solving the VIP. This method was first invented by Korpelevich in 1976. Here it is specified below: for any given $x_0 \in C$, the sequence $\{x_i\}$ is generated by

$$\begin{cases} y_i = P_C(x_i - \ell A x_i), \\ x_{i+1} = P_C(x_i - \ell A y_i) \quad \forall i \geq 0, \end{cases} \tag{1.1}$$

with $\ell \in (0, \frac{1}{L})$. Whenever VI$(C, A) \neq \emptyset$, the sequence $\{x_i\}$ has only weak convergence. Actually, the convergence of $\{x_i\}$ only requires that the mapping A is monotone and Lipschitz continuous. Till now, Korpelevich's extragradient method has received great attention given by many authors, who improved and modified it in various ways; see e.g., [4–10, 12, 21, 25, 28–30] and references therein.

Let the operators A and B be α-inverse-strongly monotone on H and maximal monotone on H, respectively. Consider the variational inclusion (VI) of finding a point $x^* \in H$ s.t. $0 \in (A + B)x^*$. Recently, Takahashi et al. [24] designed a

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Halpern-type iterative method, i.e., for any given $x_0, u \in H$, $\{x_i\}$ is the sequence generated by

$$x_{i+1} = \beta_i x_i + (1-\beta_i)(\alpha_i u + (1-\alpha_i)J^B_{\lambda_i}(x_i - \lambda_i A x_i)) \quad \forall i \geq 0. \tag{1.2}$$

They proved strong convergence of $\{x_i\}$ to a solution $x^* \in (A+B)^{-1}0$. Later on, Pholasa et al. [18] extended the result in [24] to the setting of Banach spaces.

In order to solve the FPP of a nonexpansive mapping $S : C \to C$ and the VI for an α-inverse-strongly monotone mapping $A : C \to H$ and a maximal monotone operator $B : D(B) \subset C \to H$, Takahashi et al. [23] suggested a Mann-type Halpern iterative method, i.e., for any given $x_1 = x \in C$, $\{x_i\}$ is the sequence generated by

$$x_{i+1} = \beta_i x_i + (1-\beta_i)S(\alpha_i x + (1-\alpha_i)J^B_{\lambda_i}(x_i - \lambda_i A x_i)) \quad \forall i \geq 1, \tag{1.3}$$

where $\{\lambda_i\} \subset (0, 2\alpha)$ and $\{\alpha_i\}, \{\beta_i\} \subset (0,1)$. They proved the strong convergence of $\{x_i\}$ to a point of $\mathrm{Fix}(S) \cap (A+B)^{-1}0$ under some mild conditions. In the practical life, many mathematical models have been formulated as the VI. Without doubt, many researchers have presented and developed a great number of iterative methods for solving the VI in several approaches; see e.g., [6–8, 14, 16, 18, 22–24] and the references therein. Thanks to the importance and interesting of the VI, many mathematicians are now interested in finding a common solution of the VI and FPP.

In 2011, Manaka and Takahashi [16] suggested an iterative process, i.e., for any given $x_0 \in C$, $\{x_i\}$ is the sequence generated by

$$x_{i+1} = \alpha_i x_i + (1-\alpha_i)SJ^B_{\lambda_i}(x_i - \lambda_i A x_i) \quad \forall i \geq 0, \tag{1.4}$$

where $\{\alpha_i\} \subset (0,1)$, $\{\lambda_i\} \subset (0,\infty)$, $A : C \to H$ is an inverse-strongly monotone mapping, $B : D(B) \subset C \to 2^H$ is a maximal monotone operator, and $S : C \to C$ is a nonexpansive mapping. They proved weak convergence of $\{x_i\}$ to a point of $\mathrm{Fix}(S) \cap (A+B)^{-1}0$ under some suitable conditions.

Furthermore, let $q \in (1,2]$ and assume that E is a uniformly convex and q-uniformly smooth Banach space with q-uniform smoothness coefficient κ_q. Let $f : E \to E$ be a ρ-contraction and $S : E \to E$ be a nonexpansive mapping. Let $A : E \to E$ be an α-inverse-strongly accretive mapping of order q and $B : E \to 2^E$ be an m-accretive operator. Very recently, in order to solve the FPP of S and the VI of finding $x^* \in E$ s.t. $0 \in (A+B)x^*$, Sunthrayuth and Cholamjiak [22] proposed a modified viscosity-type extragradient method , i.e., for any given $x_0 \in E$, $\{x_i\}$ is the sequence generated by

$$\begin{cases} y_i = J^B_{\lambda_i}(x_i - \lambda_i A x_i), \\ z_i = J^B_{\lambda_i}(x_i - \lambda_i A y_i + r_i(y_i - x_i)), \\ x_{i+1} = \alpha_i f(x_i) + \beta_i x_i + \gamma_i S z_i \quad \forall i \geq 0, \end{cases} \tag{1.5}$$

where $J^B_{\lambda_i} = (I + \lambda_i B)^{-1}$, $\{r_i\}, \{\alpha_i\}, \{\beta_i\}, \{\gamma_i\} \subset (0,1)$ and $\{\lambda_i\} \subset (0,\infty)$ are such that: (i) $\alpha_i + \beta_i + \gamma_i = 1$; (ii) $\lim_{i\to\infty} \alpha_i = 0$, $\sum_{i=1}^\infty \alpha_i = \infty$; (iii) $\{\beta_i\} \subset [a,b] \subset (0,1)$; and (iv) $0 < \lambda \leq \lambda_i < \lambda_i/r_i \leq \mu < (\alpha q/\kappa_q)^{1/(q-1)}$, $0 < r \leq r_i < 1$. They proved the strong convergence of $\{x_i\}$ to a point of $\mathrm{Fix}(S) \cap (A+B)^{-1}0$, which solves a certain VIP.

On the other hand, let $J : E \to 2^{E^*}$ be the normalized duality mapping from E into 2^{E^*} defined by $J(x) = \{\phi \in E^* : \langle x, \phi\rangle = \|x\|^2 = \|\phi\|^2\}\ \forall x \in E$, where $\langle\cdot,\cdot\rangle$ represents the generalized duality pairing between E and E^*. It is known that if E is smooth then J is single-valued. Let C be a nonempty closed convex subset of a smooth Banach space E. Let $A_1, A_2 : C \to E$ and $B_1, B_2 : C \to 2^E$ be nonlinear mappings with $B_i x \neq \emptyset\ \forall x \in C, i = 1, 2$. Consider the general system of variational inclusions (GSVI) of finding $(x^*, y^*) \in C \times C$ s.t.

$$\begin{cases} 0 \in \zeta_1(A_1 y^* + B_1 x^*) + x^* - y^*, \\ 0 \in \zeta_2(A_2 x^* + B_2 y^*) + y^* - x^*, \end{cases} \tag{1.6}$$

where ζ_i is a positive constant for $i = 1, 2$. It is known that problem (1.6) has been transformed into a fixed point problem in the following way.

Lemma 1.1 (see [9, Lemma 2]). *Let $B_1, B_2 : C \to 2^E$ be two m-accretive operators and $A_1, A_2 : C \to E$ be two operators. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of problem (1.6) if and only if $x^* \in \mathrm{Fix}(G)$, where $\mathrm{Fix}(G)$ is the fixed point set of the mapping $G := J^{B_1}_{\zeta_1}(I - \zeta_1 A_1) J^{B_2}_{\zeta_2}(I - \zeta_2 A_2)$, and $y^* = J^{B_2}_{\zeta_2}(I - \zeta_2 A_2)x^*$.*

Suppose that E is a uniformly convex and 2-uniformly smooth Banach space with 2-uniform smoothness coefficient κ_2. Let $B_1, B_2 : C \to 2^E$ be two m-accretive operators and $A_i : C \to E$ $(i = 1, 2)$ be ζ_i-inverse-strongly accretive operator. Let $f : C \to C$ be a contraction with constant $\delta \in [0, 1)$. Let $V : C \to C$ be a nonexpansive operator and $T : C \to C$ be a λ-strict pseudocontraction. Very recently, using Lemma 1.1, Ceng et al. [9] introduced a composite viscosity implicit rule for solving the GSVI (1.6) with the FPP constraint of T, i.e., for any given $x_0 \in C$, the sequence $\{x_i\}$ is generated by

$$\begin{cases} y_i = J^{B_2}_{\zeta_2}(x_i - \zeta_2 A_2 x_i), \\ x_i = \alpha_i f(x_{i-1}) + \delta_i x_{i-1} + \beta_i V x_{i-1} \\ \qquad + \gamma_i[\mu S x_i + (1-\mu) J^{B_1}_{\zeta_1}(y_i - \zeta_1 A_1 y_i)] \quad \forall i \geq 1, \end{cases} \tag{1.7}$$

where $\mu \in (0, 1)$, $S := (1 - \alpha)I + \alpha T$ with $0 < \alpha < \min\{1, \frac{2\lambda}{\kappa_2}\}$, and the sequences $\{\alpha_i\}, \{\delta_i\}, \{\beta_i\}, \{\gamma_i\} \subset (0, 1)$ are such that (i) $\alpha_i + \delta_i + \beta_i + \gamma_i = 1\ \forall i \geq 1$; (ii) $\lim_{i\to\infty} \alpha_i = 0$, $\lim_{i\to\infty} \frac{\beta_i}{\alpha_i} = 0$; (iii) $\lim_{i\to\infty} \gamma_i = 1$; (iv) $\sum_{i=0}^{\infty} \alpha_i = \infty$. They proved that $\{x_i\}$ converges strongly to a point of $\mathrm{Fix}(G) \cap \mathrm{Fix}(T)$, which solves a certain VIP.

In a q-uniformly smooth and uniformly convex Banach space with $q \in (1, 2]$, let the VI denote a variational inclusion for two accretive operators and let the CFPP indicate a common fixed point problem of countably many pseudocontractive mappings. In this paper, we introduce a viscosity extragradient implicit rule for solving the GSVI (1.6) with the VI and CFPP constraints. We then prove the strong convergence of the suggested method to a solution of the GSVI (1.6) with the VI and CFPP constraints under some approximate assumptions.

2. Preliminaries

Let E be a real Banach space with the dual E^*, and $\emptyset \neq C \subset E$ be a closed convex set. For convenience, we shall use the following symbols: $x_n \to x$ (resp.,

$x_n \rightharpoonup x$) indicates the strong (resp., weak) convergence of the sequence $\{x_n\}$ to x. Given a self-mapping T on C. We use the symbols $\mathbf{R}$ and $\mathrm{Fix}(T)$ to denote the set of all real numbers and the fixed point set of T, respectively. Recall that T is called a nonexpansive mapping if $\|Tx - Ty\| \le \|x - y\|\ \forall x, y \in C$. A mapping $f : C \to C$ is called a contraction if $\exists \delta \in [0, 1)$ s.t. $\|f(x) - f(y)\| \le \delta\|x - y\|\ \forall x, y \in C$. Also, recall that the normalized duality mapping J defined by

$$J(x) = \{\phi \in E^* : \langle x, \phi\rangle = \|x\|^2 = \|\phi\|^2\} \quad \forall x \in E. \tag{2.1}$$

is the one from E into the family of nonempty (by Hahn-Banach's theorem) weak* compact subsets of E^*, satisfying $J(\tau u) = \tau J(u)$ and $J(-u) = -J(u)$ for all $\tau > 0$ and $u \in E$.

The modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf\{1 - \frac{\|x + y\|}{2} : x, y \in E,\ \|x\| = \|y\| = 1,\ \|x - y\| \ge \epsilon\}.$$

The modulus of smoothness of E is the function $\rho_E : \mathbf{R}_+ := [0, \infty) \to \mathbf{R}_+$ defined by

$$\rho_E(\tau) = \sup\{\frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : x, y \in E,\ \|x\| = \|y\| = 1\}.$$

A Banach space E is said to be uniformly convex if $\delta_E(\epsilon) > 0\ \forall \epsilon \in (0, 2]$. It is said to be uniformly smooth if $\lim_{\tau\to0^+} \frac{\rho_E(\tau)}{\tau} = 0$. Also, it is said to be q-uniformly smooth with $q > 1$ if $\exists c > 0$ s.t. $\rho_E(t) \le ct^q\ \forall t > 0$. If E is q-uniformly smooth, then $q \le 2$ and E is also uniformly smooth and if E is uniformly convex, then E is also reflexive and strictly convex. It is known that Hilbert space H is 2-uniformly smooth. Further, sequence space ℓ_p and Lebesgue space L_p are $\min\{p, 2\}$-uniformly smooth for every $p > 1$ [26].

Let $q > 1$. The generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{\phi \in E^* : \langle x, \phi\rangle = \|x\|^q,\ \|\phi\| = \|x\|^{q-1}\}, \tag{2.2}$$

where $\langle\cdot, \cdot\rangle$ denotes the generalized duality pairing between E and E^*. In particular, if $q = 2$, then $J_2 = J$ is the normalized duality mapping of E. It is known that $J_q(x) = \|x\|^{q-2}J(x)\ \forall x \ne 0$ and that J_q is the subdifferential of the functional $\frac{1}{q}\|\cdot\|^q$. If E is uniformly smooth, the generalized duality mapping J_q is one-to-one and single-valued. Furthermore, J_q satisfies $J_q = J_p^{-1}$, where J_p is the generalized duality mapping of E^* with $\frac{1}{p} + \frac{1}{q} = 1$. Note that no Banach space is q-uniformly smooth for $q > 2$.

Let $q > 1$ and E be a real normed space with the generalized duality mapping J_q. Then the following inequality is an immediate consequence of the subdifferential inequality of the functional $\frac{1}{q}\|\cdot\|^q$:

$$\|x + y\|^q \le \|x\|^q + q\langle y, j_q(x + y)\rangle \quad \forall x, y \in E,\ j_q(x + y) \in J_q(x + y). \tag{2.3}$$

Lemma 2.1 (see [11])**.** *If $T : C \to C$ is a continuous and strong pseudocontraction mapping, then T has a unique fixed point in C.*

The following lemma can be obtained from the result in [26].

Lemma 2.2. *Let $q > 1$ and $r > 0$ be two fixed real numbers and let E be uniformly convex. Then there exist strictly increasing, continuous and convex functions $g, h : \mathbf{R}_+ \to \mathbf{R}_+$ with $g(0) = 0$ and $h(0) = 0$ such that*

(a) $\|\mu x + (1-\mu)y\|^q \le \mu\|x\|^q + (1-\mu)\|y\|^q - \mu(1-\mu)g(\|x-y\|)$ with $\mu \in [0,1]$;
(b) $h(\|x-y\|) \le \|x\|^q - q\langle x, j_q(y)\rangle + (q-1)\|y\|^q$

for all $x, y \in B_r$ and $j_q(y) \in J_q(y)$, where $B_r := \{x \in E : \|x\| \le r\}$.

The following lemma is an analogue of Lemma 2.2 (a).

Lemma 2.3. *Let $q > 1$ and $r > 0$ be two fixed real numbers and let E be uniformly convex. Then there exists a strictly increasing, continuous and convex function $g : \mathbf{R}_+ \to \mathbf{R}_+$ with $g(0) = 0$ such that $\|\lambda x + \mu y + \nu z\|^q \le \lambda\|x\|^q + \mu\|y\|^q + \nu\|z\|^q - \lambda\mu g(\|x-y\|)$ for all $x, y, z \in B_r$ and $\lambda, \mu, \nu \in [0,1]$ with $\lambda + \mu + \nu = 1$.*

Proposition 2.1 (see [2])**.** *Let $\{S_n\}_{n=0}^{\infty}$ be a sequence of self-mappings on C such that $\sum_{n=1}^{\infty} \sup_{x\in C} \|S_n x - S_{n-1}x\| < \infty$. Then for each $y \in C$, $\{S_n y\}$ converges strongly to some point of C. Moreover, let S be a self-mapping on C defined by $Sy = \lim_{n\to\infty} S_n y$ for all $y \in C$. Then $\lim_{n\to\infty} \sup_{x\in C} \|S_n x - Sx\| = 0$.*

Proposition 2.2 (see [26])**.** *Let $q \in (1,2]$ a fixed real number and let E be q-uniformly smooth. Then $\|x+y\|^q \le \|x\|^q + q\langle y, J_q(x)\rangle + \kappa_q\|y\|^q$ $\forall x, y \in E$, where κ_q is the q-uniform smoothness coefficient of E.*

Let D be a subset of C and let Π be a mapping of C into D. Then Π is said to be sunny if $\Pi[\Pi(x) + t(x - \Pi(x))] = \Pi(x)$, whenever $\Pi(x) + t(x - \Pi(x)) \in C$ for $x \in C$ and $t \ge 0$. A mapping Π of C into itself is called a retraction if $\Pi^2 = \Pi$. If a mapping Π of C into itself is a retraction, then $\Pi(z) = z$ for each $z \in R(\Pi)$, where $R(\Pi)$ is the range of Π. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D. In terms of [19], we know that if E is smooth and Π is a retraction of C onto D, then the following statements are equivalent:

(i) Π is sunny and nonexpansive;
(ii) $\|\Pi(x) - \Pi(y)\|^2 \le \langle x - y, J(\Pi(x) - \Pi(y))\rangle$ $\forall x, y \in C$;
(iii) $\langle x - \Pi(x), J(y - \Pi(x))\rangle \le 0$ $\forall x \in C, y \in D$.

Let $B : C \to 2^E$ be a set-valued operator with $Bx \ne \emptyset$ $\forall x \in C$. Let $q > 1$. An operator B is said to be accretive if for each $x, y \in C$, $\exists j_q(x-y) \in J_q(x-y)$ s.t. $\langle u - v, j_q(x-y)\rangle \ge 0$ $\forall u \in Bx, v \in By$. An accretive operator B is said to be α-inverse-strongly accretive of order q if for each $x, y \in C$, $\exists j_q(x-y) \in J_q(x-y)$ s.t. $\langle u - v, j_q(x-y)\rangle \ge \alpha\|u - v\|^q$ $\forall u \in Bx, v \in By$ for some $\alpha > 0$. If $E = H$ a Hilbert space, then B is called α-inverse-strongly monotone. An accretive operator B is said to be m-accretive if $(I + \lambda B)C = E$ for all $\lambda > 0$. For an accretive operator B, we define the mapping $J_\lambda^B : (I + \lambda B)C \to C$ by $J_\lambda^B = (I + \lambda B)^{-1}$ for each $\lambda > 0$. Such J_λ^B is called the resolvent of B for $\lambda > 0$.

Lemma 2.4 (see [14])**.** *Let $B : C \to 2^E$ be an m-accretive operator. Then the following statements hold:*

(i) *the resolvent identity:* $J_\lambda^B x = J_\mu^B(\frac{\mu}{\lambda}x + (1 - \frac{\mu}{\lambda})J_\lambda^B x)$ $\forall \lambda, \mu > 0,\ x \in E$;

(ii) *if* J_λ^B *is a resolvent of* B *for* $\lambda > 0$, *then* J_λ^B *is a firmly nonexpansive mapping with* $\mathrm{Fix}(J_\lambda^B) = B^{-1}0$, *where* $B^{-1}0 = \{x \in C : 0 \in Bx\}$;
(iii) *if* $E = H$ *a Hilbert space,* B *is maximal monotone.*

Let $A : C \to E$ be an α-inverse-strongly accretive mapping of order q and $B : C \to 2^E$ be an m-accretive operator. In the sequel, we will use the notation $T_\lambda := J_\lambda^B(I - \lambda A) = (I + \lambda B)^{-1}(I - \lambda A)\ \forall \lambda > 0$.

Proposition 2.3 (see [14]). *The following statements hold:*
(i) $\mathrm{Fix}(T_\lambda) = (A + B)^{-1}0\ \forall \lambda > 0$;
(ii) $\|y - T_\lambda y\| \le 2\|y - T_r y\|$ *for* $0 < \lambda \le r$ *and* $y \in C$.

Proposition 2.4 (see [27]). *Let* E *be uniformly smooth,* $T : C \to C$ *be a nonexpansive mapping with* $\mathrm{Fix}(T) \ne \emptyset$ *and* $f : C \to C$ *be a fixed contraction. For each* $t \in (0, 1)$, *let* $z_t \in C$ *be the unique fixed point of the contraction* $C \ni z \mapsto tf(z)+(1-t)Tz$ *on* C, *i.e.,* $z_t = tf(z_t) + (1 - t)Tz_t$. *Then* $\{z_t\}$ *converges strongly to a fixed point* $x^* \in \mathrm{Fix}(T)$, *which solves the VIP:* $\langle (I - f)x^*, J(x^* - x)\rangle \le 0\ \forall x \in \mathrm{Fix}(T)$.

Proposition 2.5 (see [14]). *Let* E *be* q*-uniformly smooth with* $q \in (1, 2]$. *Suppose that* $A : C \to E$ *is an* α*-inverse-strongly accretive mapping of order* q. *Then, for any given* $\lambda \ge 0$,

$$\|(I - \lambda A)x - (I - \lambda A)y\|^q \le \|x - y\|^q - \lambda(\alpha q - \kappa_q \lambda^{q-1})\|Ax - Ay\|^q \quad \forall x, y \in C,$$

where $\kappa_q > 0$ *is the* q*-uniform smoothness coefficient of* E. *In particular, if* $0 \le \lambda \le (\frac{\alpha q}{\kappa_q})^{\frac{1}{q-1}}$, *then* $I - \lambda A$ *is nonexpansive.*

Lemma 2.5 (see [9]). *Let* E *be* q*-uniformly smooth with* $q \in (1, 2]$. *Let* $B_1, B_2 : C \to 2^E$ *be two* m*-accretive operators and* $A_i : C \to E$ $(i = 1, 2)$ *be* σ_i*-inverse-strongly accretive mapping of order* q. *Define an operator* $G : C \to C$ *by* $G := J_{\zeta_1}^{B_1}(I - \zeta_1 A_1)J_{\zeta_2}^{B_2}(I - \zeta_2 A_2)$. *If* $0 \le \zeta_i \le (\frac{\sigma_i q}{\kappa_q})^{\frac{1}{q-1}}$ $(i = 1, 2)$, *then* G *is nonexpansive.*

Lemma 2.6 (see [1]). *Let* E *be smooth,* $A : C \to E$ *be accretive and* Π_C *be a sunny nonexpansive retraction from* E *onto* C. *Then* $\mathrm{VI}(C, A) = \mathrm{Fix}(\Pi_C(I - \lambda A))\ \forall \lambda > 0$, *where* $\mathrm{VI}(C, A)$ *is the solution set of the VIP of finding* $z \in C$ *s.t.* $\langle Az, J(z - y)\rangle \le 0\ \forall y \in C$.

Recall that if $E = H$ a Hilbert space, then the sunny nonexpansive retraction Π_C from E onto C coincides with the metric projection P_C from H onto C. Moreover, if E is uniformly smooth and T is a nonexpansive self-mapping on C with $\mathrm{Fix}(T) \ne \emptyset$, then $\mathrm{Fix}(T)$ is a sunny nonexpansive retract from E onto C [20]. By Lemma 2.6 we know that, $x^* \in \mathrm{Fix}(T)$ solves the VIP in Proposition 2.4 if and only if x^* solves the fixed point equation $x^* = \Pi_{\mathrm{Fix}(T)} f(x^*)$.

Lemma 2.7 (see [15]). *Let* $\{\Gamma_n\}$ *be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence* $\{\Gamma_{n_i}\}$ *of* $\{\Gamma_n\}$ *which satisfies* $\Gamma_{n_i} < \Gamma_{n_i+1}$ *for each integer* $i \ge 1$. *Define the sequence* $\{\tau(n)\}_{n \ge n_0}$ *of integers as follows:*

$$\tau(n) = \max\{k \le n : \Gamma_k < \Gamma_{k+1}\},$$

where integer $n_0 \ge 1$ *such that* $\{k \le n_0 : \Gamma_k < \Gamma_{k+1}\} \ne \emptyset$. *Then, the following hold:*

(i) $\tau(n_0) \le \tau(n_0+1) \le \cdots$ *and* $\tau(n) \to \infty$*;*
(ii) $\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}$ *and* $\Gamma_n \le \Gamma_{\tau(n)+1}$ $\forall n \ge n_0$.

Lemma 2.8 (see [3])**.** *Let E be strictly convex, and $\{T_n\}_{n=0}^{\infty}$ be a sequence of nonexpansive mappings on C. Suppose that $\bigcap_{n=0}^{\infty} \mathrm{Fix}(T_n)$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{n=0}^{\infty} \lambda_n = 1$. Then a mapping S on C defined by $Sx = \sum_{n=0}^{\infty} \lambda_n T_n x$ $\forall x \in C$ is defined well, nonexpansive and $\mathrm{Fix}(S) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(T_n)$ holds.*

Lemma 2.9 (see [27])**.** *Let $\{a_n\}$ be a sequence in $[0, \infty)$ such that $a_{n+1} \le (1 - s_n)a_n + s_n \nu_n$ $\forall n \ge 0$, where $\{s_n\}$ and $\{\nu_n\}$ satisfy the conditions: (i) $\{s_n\} \subset [0, 1]$, $\sum_{n=0}^{\infty} s_n = \infty$; (ii) $\limsup_{n\to\infty} \nu_n \le 0$ or $\sum_{n=0}^{\infty} |s_n \nu_n| < \infty$. Then $\lim_{n\to\infty} a_n = 0$.*

3. Main result

Throughout this paper, suppose that C is a nonempty closed convex subset of a q-uniformly smooth and uniformly convex Banach space E with $q \in (1, 2]$. Let $B_1, B_2 : C \to 2^E$ be two m-accretive operators and $A_i : C \to E$ be σ_i-inverse-strongly accretive mapping of order q for $i = 1, 2$. Let the mapping $G : C \to C$ be defined as $G := J_{\zeta_1}^{B_1}(I - \zeta_1 A_1) J_{\zeta_2}^{B_2}(I - \zeta_2 A_2)$ with constants $\zeta_1, \zeta_2 > 0$. Let $f : C \to C$ be a δ-contraction with constant $\delta \in [0, 1)$ and $\{S_n\}_{n=0}^{\infty}$ be a countable family of ℓ-uniformly Lipschitzian pseudocontractive self-mappings on C. Let $A : C \to E$ and $B : C \to 2^E$ be a σ-inverse-strongly accretive mapping of order q and an m-accretive operator, respectively. Assume that $\Omega := \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{Fix}(G) \cap (A+B)^{-1}0 \ne \emptyset$.

Algorithm 3.1. Viscosity extragradient implicit rule for the GSVI (1.6) with the VI and CFPP constraints.

Initial Step. Given $\xi \in (0, 1)$ and $x_0 \in C$ arbitrarily.

Iteration Steps. Given the current iterate x_n, calculate x_{n+1} as follows:

Step 1. Compute $w_n = s_n x_n + (1 - s_n)(\xi S_n w_n + (1 - \xi) G w_n)$;

Step 2. Compute

$$\begin{cases} v_n = J_{\zeta_2}^{B_2}(w_n - \zeta_2 A_2 w_n), \\ u_n = J_{\zeta_1}^{B_1}(v_n - \zeta_1 A_1 v_n), \\ y_n = J_{\lambda_n}^{B}(u_n - \lambda_n A u_n); \end{cases}$$

Step 3. Compute $z_n = J_{\lambda_n}^{B}(u_n - \lambda_n A y_n + r_n(y_n - u_n))$;

Step 4. Compute $x_{n+1} = \alpha_n f(u_n) + \beta_n u_n + \gamma_n G z_n$,
where $\{r_n\}, \{s_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $\{\lambda_n\} \subset (0, \infty)$.

Set $n := n + 1$ and go to Step 1.

Lemma 3.1. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. Then $\{x_n\}$ is bounded.*

Proof. Let $p \in \Omega := \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{Fix}(G) \cap (A+B)^{-1}0$. Then we observe that

$$p = Gp = S_n p = J_{\lambda_n}^B (p - \lambda_n Ap) = J_{\lambda_n}^B ((1-r_n)p + r_n(p - \frac{\lambda_n}{r_n} Ap)).$$

By Proposition 2.5 and Lemma 2.5, we know that $I - \zeta_1 A_1$, $I - \zeta_2 A_2$ and $G := J_{\zeta_1}^{B_1}(I - \zeta_1 A_1) J_{\zeta_2}^{B_2}(I - \zeta_2 A_2)$ are nonexpansive mappings. Moreover, it can be readily seen that for each $n \geq 0$, there is only an element $w_n \in C$ s.t.

$$w_n = s_n x_n + (1-s_n)(\xi S_n w_n + (1-\xi) G w_n). \tag{3.1}$$

In fact, consider the mapping $F_n x = s_n x_n + (1-s_n)(\xi S_n x + (1-\xi)Gx)\ \forall x \in C$. Note that $S_n : C \to C$ is a continuous pseudocontraction. Hence we obtain that for all $x, y \in C$,

$$\begin{aligned}
&\langle F_n x - F_n y, J(x-y) \rangle \\
&= (1-s_n)\langle (\xi S_n x + (1-\xi)Gx) - (\xi S_n y + (1-\xi)Gy), J(x-y) \rangle \\
&= (1-s_n)[\xi \langle S_n x - S_n y, J(x-y) \rangle + (1-\xi)\langle Gx - Gy, J(x-y) \rangle] \\
&\leq (1-s_n)\|x-y\|^2.
\end{aligned}$$

Also, from $\{s_n\} \subset (0,1]$, we get $0 \leq 1 - s_n < 1\ \forall n \geq 0$. Thus, F_n is a continuous and strong pseudocontractive self-mapping on C. Using Lemma 2.1, we deduce that for each $n \geq 0$, there is only an element $w_n \in C$, satisfying (3.1). Since each $S_n : C \to C$ is a pseudocontraction mapping, we get

$$\begin{aligned}
&\|w_n - p\|^2 \\
&= s_n \langle x_n - p, J(w_n - p) \rangle + (1-s_n)\langle \xi S_n w_n + (1-\xi) G w_n - p, J(w_n - p) \rangle \\
&\leq s_n \|x_n - p\| \|w_n - p\| + (1-s_n)[\xi \|w_n - p\|^2 + (1-\xi)\|w_n - p\|^2] \\
&= s_n \|x_n - p\| \|w_n - p\| + (1-s_n)\|w_n - p\|^2,
\end{aligned}$$

and hence

$$\|w_n - p\| \leq \|x_n - p\| \quad \forall n \geq 0.$$

Using $u_n = G w_n$, we deduce from the nonexpansivity of G that

$$\|u_n - p\| \leq \|w_n - p\| \leq \|x_n - p\| \quad \forall n \geq 0. \tag{3.2}$$

Using Lemma 2.4 (ii) and Proposition 2.5, we have

$$\begin{aligned}
\|y_n - p\|^q &= \|J_{\lambda_n}^B (u_n - \lambda_n A u_n) - J_{\lambda_n}^B (p - \lambda_n Ap)\|^q \\
&\leq \|(I - \lambda_n A) u_n - (I - \lambda_n A) p\|^q \\
&\leq \|u_n - p\|^q - \lambda_n (\sigma q - \kappa_q \lambda_n^{q-1}) \|A u_n - Ap\|^q
\end{aligned}, \tag{3.3}$$

which hence leads to

$$\|y_n - p\| \leq \|u_n - p\|.$$

By the convexity of $\|\cdot\|^q$ for all $q\in(1,2]$ and (3.3), we deduce that

$$\begin{aligned}
\|z_n-p\|^q &= \|J^B_{\lambda_n}((1-r_n)u_n+r_n(y_n-\frac{\lambda_n}{r_n}Ay_n)) \\
&\quad - J^B_{\lambda_n}((1-r_n)p+r_n(p-\frac{\lambda_n}{r_n}Ap))\|^q \\
&\le (1-r_n)\|u_n-p\|^q \\
&\quad + r_n\|(I-\frac{\lambda_n}{r_n}A)y_n-(I-\frac{\lambda_n}{r_n}A)p\|^q \\
&\le (1-r_n)\|u_n-p\|^q \\
&\quad + r_n[\|y_n-p\|^q-\frac{\lambda_n}{r_n}(\sigma q-\frac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}})\|Ay_n-Ap\|^q] \qquad (3.4)\\
&\le (1-r_n)\|u_n-p\|^q \\
&\quad + r_n[\|u_n-p\|^q-\lambda_n(\sigma q-\kappa_q\lambda_n^{q-1})\|Au_n-Ap\|^q \\
&\quad - \frac{\lambda_n}{r_n}(\sigma q-\frac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}})\|Ay_n-Ap\|^q] \\
&= \|u_n-p\|^q-r_n\lambda_n(\sigma q-\kappa_q\lambda_n^{q-1})\|Au_n-Ap\|^q-\lambda_n(\sigma q \\
&\quad -\frac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}})\|Ay_n-Ap\|^q.
\end{aligned}$$

This ensures that

$$\|z_n-p\|\le\|u_n-p\|.$$

So it follows from (3.2) that

$$\begin{aligned}
\|x_{n+1}-p\| &= \|\alpha_n(f(u_n)-p)+\beta_n(u_n-p)+\gamma_n(Gz_n-p)\| \\
&\le \alpha_n\|f(u_n)-p\|+\beta_n\|u_n-p\|+\gamma_n\|Gz_n-p\| \\
&\le \alpha_n(\|f(u_n)-f(p)\|+\|f(p)-p\|)+\beta_n\|u_n-p\|+\gamma_n\|Gz_n-p\| \\
&\le \alpha_n(\delta\|x_n-p\|+\|f(p)-p\|)+\beta_n\|x_n-p\|+\gamma_n\|x_n-p\| \\
&= (1-\alpha_n(1-\delta))\|x_n-p\|+\alpha_n\|f(p)-p\| \\
&\le \max\{\|x_n-p\|,\frac{\|f(p)-p\|}{1-\delta}\}.
\end{aligned}$$

By induction, we have $\|x_n-p\|\le\max\{\|x_0-p\|,\frac{\|f(p)-p\|}{1-\delta}\}\ \forall n\ge0$. Therefore, $\{x_n\}$ is bounded, and so are $\{u_n\},\{w_n\},\{y_n\},\{z_n\},\{Gz_n\},\{Au_n\},\{Ay_n\}$. This completes the proof. □

Now we state and prove the main result of this paper.

Theorem 3.2. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. Suppose that the following conditions hold:*

(C1) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=0}^\infty\alpha_n=\infty$;
(C2) $0<a\le\beta_n\le b<1$ *and* $0<c\le s_n\le d<1$;
(C3) $0<r\le r_n<1$ *and* $0<\lambda\le\lambda_n<\frac{\lambda_n}{r_n}\le\mu<(\frac{\sigma q}{\kappa_q})^{\frac{1}{q-1}}$;

(C4) $0<\zeta_i<(\frac{\sigma_i q}{\kappa_q})^{\frac{1}{q-1}}$ *for* $i=1,2$.

Assume that $\sum_{n=0}^{\infty}\sup_{x\in D}\|S_{n+1}x-S_nx\|<\infty$ *for any bounded subset* D *of* C. *Let* $S:C\to C$ *be a mapping defined by* $Sx=\lim_{n\to\infty}S_nx\ \forall x\in C$, *and suppose that* $\mathrm{Fix}(S)=\bigcap_{n=0}^{\infty}\mathrm{Fix}(S_n)$. *Then* $x_n\to x^*\in\Omega$, *which is the unique solution to the VIP:* $\langle(I-f)x^*,J(x^*-p)\rangle\le 0\ \forall p\in\Omega$, *i.e., the fixed point equation* $x^*=\Pi_\Omega f(x^*)$.

Proof. First of all, let $x^*\in\Omega$ and $y^*=J_{\zeta_2}^{B_2}(x^*-\zeta_2A_2x^*)$. Since $v_n=J_{\zeta_2}^{B_2}(I-\zeta_2A_2)w_n$ and $u_n=J_{\zeta_1}^{B_1}(I-\zeta_1A_1)v_n$, we have $u_n=Gw_n$. Using Proposition 2.5 we have

$$\begin{aligned}\|v_n-y^*\|^q&=\|J_{\zeta_2}^{B_2}(w_n-\zeta_2A_2w_n)-J_{\zeta_2}^{B_2}(x^*-\zeta_2A_2x^*)\|^q\\&\le\|w_n-x^*\|^q-\zeta_2(\sigma_2q-\kappa_q\zeta_2^{q-1})\|A_2w_n-A_2x^*\|^q,\end{aligned}$$

and

$$\begin{aligned}\|u_n-x^*\|^q&=\|J_{\zeta_1}^{B_1}(v_n-\zeta_1A_1v_n)-J_{\zeta_1}^{B_1}(y^*-\zeta_1A_1y^*)\|^q\\&\le\|v_n-y^*\|^q-\zeta_1(\sigma_1q-\kappa_q\zeta_1^{q-1})\|A_1v_n-A_1y^*\|^q.\end{aligned}$$

Combining the last two inequalities, we have

$$\begin{aligned}\|u_n-x^*\|^q\le{}&\|w_n-x^*\|^q-\zeta_2(\sigma_2q-\kappa_q\zeta_2^{q-1})\|A_2w_n-A_2x^*\|^q\\&-\zeta_1(\sigma_1q-\kappa_q\zeta_1^{q-1})\|A_1v_n-A_1y^*\|^q.\end{aligned}$$

Using Lemma 2.3, from (2.3), (3.2) and (3.4) we obtain that

$$\begin{aligned}\|x_{n+1}-x^*\|^q\le{}&\|\alpha_n(f(u_n)-f(x^*))+\beta_n(u_n-x^*)\\&+\gamma_n(Gz_n-x^*)\|^q+q\alpha_n\langle f(x^*)-x^*,J_q(x_{n+1}-x^*)\rangle\\\le{}&\alpha_n\|f(u_n)-f(x^*)\|^q+\beta_n\|u_n-x^*\|^q\\&+\gamma_n\|Gz_n-x^*\|^q-\beta_n\gamma_ng(\|u_n-Gz_n\|)\\&+q\alpha_n\langle(f-I)x^*,J_q(x_{n+1}-x^*)\rangle\\\le{}&\alpha_n\delta\|u_n-x^*\|^q+\beta_n\|u_n-x^*\|^q\\&+\gamma_n[\|u_n-x^*\|^q-r_n\lambda_n(\sigma q-\kappa_q\lambda_n^{q-1})\|Au_n-Ax^*\|^q\\&-\lambda_n(\sigma q-\frac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}})\|Ay_n-Ax^*\|^q]-\beta_n\gamma_ng(\|u_n-Gz_n\|)\\&+q\alpha_n\langle(f-I)x^*,J_q(x_{n+1}-x^*)\rangle\\\le{}&\alpha_n\delta\|x_n-x^*\|^q+\beta_n\|x_n-x^*\|^q\\&+\gamma_n[\|x_n-x^*\|^q-\zeta_2(\sigma_2q-\kappa_q\zeta_2^{q-1})\|A_2w_n-A_2x^*\|^q\\&-\zeta_1(\sigma_1q-\kappa_q\zeta_1^{q-1})\|A_1v_n-A_1y^*\|^q\\&-r_n\lambda_n(\sigma q-\kappa_q\lambda_n^{q-1})\|Au_n-Ax^*\|^q\\&-\lambda_n(\sigma q-\frac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}})\|Ay_n-Ax^*\|^q]-\beta_n\gamma_ng(\|u_n-Gz_n\|)\\&+q\alpha_n\langle(f-I)x^*,J_q(x_{n+1}-x^*)\rangle\\={}&(1-\alpha_n(1-\delta))\|x_n-x^*\|^q-\gamma_n[\zeta_2(\sigma_2q-\kappa_q\zeta_2^{q-1})\|A_2w_n-A_2x^*\|^q\end{aligned}\tag{3.5}$$

$$+\zeta_1(\sigma_1 q-\kappa_q\zeta_1^{q-1})\|A_1v_n-A_1y^*\|^q$$
$$+r_n\lambda_n(\sigma q-\kappa_q\lambda_n^{q-1})\|Au_n-Ax^*\|^q$$
$$+\lambda_n(\sigma q-\frac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}})\|Ay_n-Ax^*\|^q]-\beta_n\gamma_n g(\|u_n-Gz_n\|)$$
$$+q\alpha_n\langle(f-I)x^*,J_q(x_{n+1}-x^*)\rangle.$$

For each $n\geq 0$, we set

$$\Gamma_n=\|x_n-x^*\|^q,$$
$$\varepsilon_n=\alpha_n(1-\delta),$$
$$\eta_n=\gamma_n[\zeta_2(\sigma_2 q-\kappa_q\zeta_2^{q-1})\|A_2w_n-A_2x^*\|^q+\zeta_1(\sigma_1 q-\kappa_q\zeta_1^{q-1})\|A_1v_n-A_1y^*\|^q$$
$$+r_n\lambda_n(\sigma q-\kappa_q\lambda_n^{q-1})\|Au_n-Ax^*\|^q+\lambda_n(\sigma q-\frac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}})\|Ay_n-Ax^*\|^q]$$
$$+\beta_n\gamma_n g(\|u_n-Gz_n\|)$$
$$\delta_n=q\alpha_n\langle(f-I)x^*,J_q(x_{n+1}-x^*)\rangle.$$

Then (3.5) can be rewritten as the following formula:

$$\Gamma_{n+1}\leq(1-\varepsilon_n)\Gamma_n-\eta_n+\delta_n\quad\forall n\geq 0,\tag{3.6}$$

and hence

$$\Gamma_{n+1}\leq(1-\varepsilon_n)\Gamma_n+\delta_n\quad\forall n\geq 0.\tag{3.7}$$

We next show the strong convergence of $\{\Gamma_n\}$ by the following two cases:

Case 1. Suppose that there exists an integer $n_0\geq 1$ such that $\{\Gamma_n\}$ is non-increasing. Then

$$\Gamma_n-\Gamma_{n+1}\to 0.$$

From (3.6), we get

$$0\leq\eta_n\leq\Gamma_n-\Gamma_{n+1}+\delta_n-\varepsilon_n\Gamma_n.$$

Note that combining $\varepsilon_n\to 0$ and $\delta_n\to 0$ guarantees $\eta_n\to 0$. So it follows that $\lim_{n\to\infty}g(\|u_n-Gz_n\|)=0$,

$$\lim_{n\to\infty}\|A_2w_n-A_2x^*\|=\lim_{n\to\infty}\|A_1v_n-A_1y^*\|=0\tag{3.8}$$

and

$$\lim_{n\to\infty}\|Au_n-Ax^*\|=\lim_{n\to\infty}\|Ay_n-Ax^*\|=0.\tag{3.9}$$

Since g is a strictly increasing, continuous and convex function with $g(0)=0$, we deduce that

$$\lim_{n\to\infty}\|u_n-Gz_n\|=0.\tag{3.10}$$

On the other hand, using Lemma 2.2 (b) and Lemma 2.4 (ii), we get

$$\|v_n-y^*\|^q=\|J_{\zeta_2}^{B_2}(w_n-\zeta_2A_2w_n)-J_{\zeta_2}^{B_2}(x^*-\zeta_2A_2x^*)\|^q$$
$$\leq\langle w_n-\zeta_2A_2w_n-(x^*-\zeta_2A_2x^*),J_q(v_n-y^*)\rangle$$
$$=\langle w_n-x^*,J_q(v_n-y^*)\rangle+\zeta_2\langle A_2x^*-A_2w_n,J_q(v_n-y^*)\rangle$$

$$\leq \frac{1}{q}[\|w_n - x^*\|^q + (q-1)\|v_n - y^*\|^q - \tilde{h}_1(\|w_n - x^* - v_n + y^*\|)] + \zeta_2\langle A_2x^* - A_2w_n, J_q(v_n - y^*)\rangle,$$

which hence attains

$$\|v_n - y^*\|^q \leq \|w_n - x^*\|^q - \tilde{h}_1(\|w_n - v_n - x^* + y^*\|) + q\zeta_2\|A_2x^* - A_2w_n\|\|v_n - y^*\|^{q-1}.$$

In a similar way, we get

$$\begin{aligned}\|u_n - x^*\|^q &= \|J_{\zeta_1}^{B_1}(v_n - \zeta_1A_1v_n) - J_{\zeta_1}^{B_1}(y^* - \zeta_1A_1y^*)\|^q \\ &\leq \langle v_n - \zeta_1A_1v_n - (y^* - \zeta_1A_1y^*), J_q(u_n - x^*)\rangle \\ &= \langle v_n - y^*, J_q(u_n - x^*)\rangle + \zeta_1\langle A_1y^* - A_1v_n, J_q(u_n - x^*)\rangle \\ &\leq \frac{1}{q}[\|v_n - y^*\|^q + (q-1)\|u_n - x^*\|^q - \tilde{h}_2(\|v_n - y^* - u_n + x^*\|)] \\ &\quad + \zeta_1\langle A_1y^* - A_1v_n, J_q(u_n - x^*)\rangle,\end{aligned}$$

which hence attains

$$\begin{aligned}\|u_n - x^*\|^q &\leq \|v_n - y^*\|^q - \tilde{h}_2(\|v_n - y^* - u_n + x^*\|) \\ &\quad + q\zeta_1\|A_1y^* - A_1v_n\|\|u_n - x^*\|^{q-1} \\ &\leq \|x_n - x^*\|^q - \tilde{h}_1(\|w_n - v_n - x^* + y^*\|) \\ &\quad + q\zeta_2\|A_2x^* - A_2w_n\|\|v_n - y^*\|^{q-1} \\ &\quad - \tilde{h}_2(\|v_n - u_n + x^* - y^*\|) + q\zeta_1\|A_1y^* - A_1v_n\|\|u_n - x^*\|^{q-1}.\end{aligned} \tag{3.11}$$

Using Lemma 2.2 (b) and Lemma 2.4 (ii) again, we get

$$\begin{aligned}\|y_n - x^*\|^q &= \|J_{\lambda_n}^B(u_n - \lambda_nAu_n) - J_{\lambda_n}^B(x^* - \lambda_nAx^*)\|^q \\ &\leq \langle (u_n - \lambda_nAu_n) - (x^* - \lambda_nAx^*), J_q(y_n - x^*)\rangle \\ &\leq \frac{1}{q}[\|(u_n - \lambda_nAu_n) - (x^* - \lambda_nAx^*)\|^q + (q-1)\|y_n - x^*\|^q \\ &\quad - h_1(\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|)],\end{aligned}$$

which together with (3.3), implies that

$$\begin{aligned}\|y_n - x^*\|^q &\leq \|(u_n - \lambda_nAu_n) - (x^* - \lambda_nAx^*)\|^q - h_1(\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|) \\ &\leq \|u_n - x^*\|^q - h_1(\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|).\end{aligned}$$

This together with (3.4) and (3.11), implies that

$$\begin{aligned}\|x_{n+1} - x^*\|^q &\leq \alpha_n\|f(u_n) - x^*\|^q + \beta_n\|u_n - x^*\|^q \\ &\quad + \gamma_n\|Gz_n - x^*\|^q \\ &\leq \alpha_n\|f(u_n) - x^*\|^q + \beta_n\|u_n - x^*\|^q \\ &\quad + \gamma_n[(1-r_n)\|u_n - x^*\|^q + r_n\|y_n - x^*\|^q] \\ &\leq \alpha_n\|f(u_n) - x^*\|^q + \beta_n\|u_n - x^*\|^q \\ &\quad + \gamma_n\{(1-r_n)\|u_n - x^*\|^q + r_n[\|u_n - x^*\|^q \\ &\quad - h_1(\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|)]\}\end{aligned}$$

$$
\begin{aligned}
&= \alpha_n\|f(u_n) - x^*\|^q + \beta_n\|u_n - x^*\|^q \\
&\quad + \gamma_n\{\|u_n - x^*\|^q - r_n h_1(\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|)\} \\
&\le \alpha_n\|f(u_n) - x^*\|^q + \beta_n\|x_n - x^*\|^q \\
&\quad + \gamma_n\{\|x_n - x^*\|^q - \tilde{h}_1(\|w_n - v_n - x^* + y^*\|) \\
&\quad - \tilde{h}_2(\|v_n - u_n + x^* - y^*\|) \\
&\quad + q\zeta_1\|A_1y^* - A_1v_n\|\|u_n - x^*\|^{q-1} \\
&\quad + q\zeta_2\|A_2x^* - A_2w_n\|\|v_n - y^*\|^{q-1} \\
&\quad - r_n h_1(\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|)\} \\
&\le \alpha_n\|f(u_n) - x^*\|^q + \|x_n - x^*\|^q \\
&\quad - \gamma_n\{\tilde{h}_1(\|w_n - v_n - x^* + y^*\|) \\
&\quad + \tilde{h}_2(\|v_n - u_n + x^* - y^*\|) \\
&\quad + r_n h_1(\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|)\} \\
&\quad + q\zeta_1\|A_1y^* - A_1v_n\|\|u_n - x^*\|^{q-1} \\
&\quad + q\zeta_2\|A_2x^* - A_2w_n\|\|v_n - y^*\|^{q-1},
\end{aligned}
$$

which immediately yields

$$
\begin{aligned}
&\gamma_n\{\tilde{h}_1(\|w_n - v_n - x^* + y^*\|) + \tilde{h}_2(\|v_n - u_n + x^* - y^*\|) \\
&\quad + r_n h_1(\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|)\} \\
&\qquad \le \alpha_n\|f(u_n) - x^*\|^q + \Gamma_n - \Gamma_{n+1} + q\zeta_1\|A_1y^* - A_1v_n\|\|u_n - x^*\|^{q-1} \\
&\qquad\quad + q\zeta_2\|A_2x^* - A_2w_n\|\|v_n - y^*\|^{q-1}.
\end{aligned}
$$

Note that $\tilde{h}_1, \tilde{h}_2$ and h_1 are strictly increasing, continuous and convex functions with $\tilde{h}_1(0) = \tilde{h}_2(0) = h_1(0) = 0$. So it follows from (3.8) and (3.9) that $\|w_n - v_n - x^* + y^*\| \to 0$, $\|v_n - u_n + x^* - y^*\| \to 0$ and $\|u_n - y_n\| \to 0$ as $n \to \infty$. This immediately implies that

$$
\lim_{n\to\infty}\|w_n - u_n\| = \lim_{n\to\infty}\|u_n - y_n\| = 0. \tag{3.12}
$$

Furthermore, we put $p_n := \xi S_n w_n + (1-\xi)Gw_n$ for all $n \ge 0$. Then we obtain that

$$
\begin{aligned}
\|w_n - x^*\|^q &= \langle s_n x_n + (1 - s_n)(\xi S_n w_n + (1-\xi)Gw_n) - x^*, J_q(w_n - x^*)\rangle \\
&\le s_n\langle x_n - x^*, J_q(w_n - x^*)\rangle \\
&\quad + (1 - s_n)\langle(\xi S_n w_n + (1-\xi)Gw_n) - x^*, J_q(w_n - x^*)\rangle \\
&\le s_n\langle x_n - x^*, J_q(w_n - x^*)\rangle \\
&\quad + (1 - s_n)\|w_n - x^*\|^q.
\end{aligned}
$$

Using Lemma 2.2 (b), we get

$$
\|w_n - x^*\|^q \le \langle x_n - x^*, J_q(w_n - x^*)\rangle \le \frac{1}{q}[\|x_n - x^*\|^q + (q-1)\|w_n - x^*\|^q - h_3(\|x_n - w_n\|)].
$$

This together with (3.2) implies that

$$
\|u_n - x^*\|^q \le \|w_n - x^*\|^q \le \|x_n - x^*\|^q - h_3(\|x_n - w_n\|). \tag{3.13}
$$

In a similar way, we have

$$\begin{aligned}\|z_n - x^*\|^q &= \|J^B_{\lambda_n}(u_n - \lambda_n Ay_n + r_n(y_n - u_n)) - J^B_{\lambda_n}(x^* - \lambda_n Ax^*)\|^q \\ &\le \langle (u_n - \lambda_n Ay_n + r_n(y_n - u_n)) - (x^* - \lambda_n Ax^*), J_q(z_n - x^*)\rangle \\ &\le \frac{1}{q}[\|(u_n - \lambda_n Ay_n + r_n(y_n - u_n)) - (x^* - \lambda_n Ax^*)\|^q \\ &\quad + (q-1)\|z_n - x^*\|^q - h_2(\|u_n + r_n(y_n - u_n) - \lambda_n(Ay_n - Ax^*) - z_n\|)],\end{aligned}$$

which together with (3.4), implies that

$$\begin{aligned}\|z_n - x^*\|^q &\le \|(u_n - \lambda_n Ay_n + r_n(y_n - u_n)) - (x^* - \lambda_n Ax^*)\|^q \\ &\quad - h_2(\|u_n + r_n(y_n - u_n) - \lambda_n(Ay_n - Ax^*) - z_n\|) \\ &\le \|u_n - x^*\|^q - h_2(\|u_n + r_n(y_n - u_n) - \lambda_n(Ay_n - Ax^*) - z_n\|).\end{aligned}$$

This together with (3.13), ensures that

$$\begin{aligned}\|x_{n+1} - x^*\|^q &\le \alpha_n\|f(u_n) - x^*\|^q + \beta_n\|u_n - x^*\|^q + \gamma_n\|Gz_n - x^*\|^q \\ &\le \alpha_n\|f(u_n) - x^*\|^q + \beta_n\|u_n - x^*\|^q + \gamma_n[\|u_n - x^*\|^q \\ &\quad - h_2(\|u_n + r_n(y_n - u_n) - \lambda_n(Ay_n - Ax^*) - z_n\|)] \\ &\le \alpha_n\|f(u_n) - x^*\|^q + \beta_n\|x_n - x^*\|^q + \gamma_n[\|x_n - x^*\|^q \\ &\quad - h_3(\|x_n - w_n\|) - h_2(\|u_n + r_n(y_n - u_n) \\ &\quad - \lambda_n(Ay_n - Ax^*) - z_n\|)] \\ &\le \alpha_n\|f(u_n) - x^*\|^q + \|x_n - x^*\|^q - \gamma_n[h_3(\|x_n - w_n\|) \\ &\quad + h_2(\|u_n + r_n(y_n - u_n) - \lambda_n(Ay_n - Ax^*) - z_n\|)],\end{aligned}$$

which immediately leads to

$$\begin{aligned}&\gamma_n[h_3(\|x_n - w_n\|) + h_2(\|u_n + r_n(y_n - u_n) - \lambda_n(Ay_n - Ax^*) - z_n\|)] \\ &\qquad \le \alpha_n\|f(x_n) - x^*\|^q + \Gamma_n - \Gamma_{n+1}.\end{aligned}$$

Since h_2 and h_3 are strictly increasing, continuous and convex functions with $h_2(0) = h_3(0) = 0$, from (3.9) and (3.12) we have

$$\lim_{n\to\infty}\|x_n - w_n\| = \lim_{n\to\infty}\|u_n - z_n\| = 0. \tag{3.14}$$

So, it follows from (3.10), (3.12) and (3.14) that

$$\|x_n - u_n\| \le \|x_n - w_n\| + \|w_n - u_n\| \to 0 \quad (n\to\infty),$$

$$\|x_n - z_n\| \le \|x_n - u_n\| + \|u_n - z_n\| \to 0 \quad (n\to\infty),$$

and hence

$$\begin{aligned}\|x_n - Gx_n\| &\le \|x_n - u_n\| + \|u_n - Gz_n\| + \|Gz_n - Gx_n\| \\ &\le \|x_n - u_n\| + \|u_n - Gz_n\| + \|z_n - x_n\| \to 0 \quad (n\to\infty).\end{aligned} \tag{3.15}$$

Since $w_n = s_n x_n + (1 - s_n)p_n$ and $p_n = \xi S_n w_n + (1-\xi)u_n$, from (3.12) and (3.14) we get

$$\|p_n - w_n\| = \frac{s_n}{1 - s_n}\|x_n - w_n\| \le \frac{d}{1-d}\|x_n - w_n\| \to 0 \quad (n\to\infty),$$

and hence

$$\xi\|S_nw_n-w_n\| = \|p_n-w_n-(1-\xi)(u_n-w_n)\| \leq \|p_n-w_n\|+\|u_n-w_n\| \to 0 \quad (n \to \infty).$$

Since $\{S_n\}_{n=0}^{\infty}$ is ℓ-uniformly Lipschitzian on C, we deduce from (3.14) that

$$\begin{aligned} \|S_nx_n - x_n\| &\leq \|S_nx_n - S_nw_n\| + \|S_nw_n - w_n\| + \|w_n - x_n\| \\ &\leq (\ell+1)\|x_n - w_n\| + \|S_nw_n - w_n\| \to 0 \quad (n \to \infty). \end{aligned} \tag{3.16}$$

Next, we claim that $\|x_n - \widehat{S}x_n\| \to 0$ as $n \to \infty$ where $\widehat{S} := (2I - S)^{-1}$. In fact, it is first clear that $S : C \to C$ is pseudocontractive and ℓ-Lipschitzian where $Sx = \lim_{n\to\infty} S_nx \ \forall x \in C$. We claim that $\lim_{n\to\infty} \|Sx_n - x_n\| = 0$. Using the boundedness of $\{x_n\}$ and setting $D = \overline{\mathrm{conv}}\{x_n : n \geq 0\}$ (the closed convex hull of the set $\{x_n : n \geq 0\}$), by the assumption we have $\sum_{n=1}^{\infty} \sup_{x\in D} \|S_nx - S_{n-1}x\| < \infty$. Hence, by Proposition 2.1 we get $\lim_{n\to\infty} \sup_{x\in D} \|S_nx - Sx\| = 0$, which immediately arrives at

$$\lim_{n\to\infty} \|S_nx_n - Sx_n\| = 0.$$

Thus, from (3.16) we have

$$\|x_n - Sx_n\| \leq \|x_n - S_nx_n\| + \|S_nx_n - Sx_n\| \to 0 \quad (n \to \infty). \tag{3.17}$$

Now, let us show that if we define $\widehat{S} := (2I - S)^{-1}$, then $\widehat{S} : C \to C$ is nonexpansive, $\mathrm{Fix}(\widehat{S}) = \mathrm{Fix}(S) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n)$ and $\lim_{n\to\infty} \|x_n - \widehat{S}x_n\| = 0$. As a matter of fact, put $\widehat{S} := (2I - S)^{-1}$, where I is the identity operator of E. Then it is known that $\widehat{S}$ is nonexpansive and $\mathrm{Fix}(\widehat{S}) = \mathrm{Fix}(S) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n)$ as a consequence of [17, Theorem 6]. From (3.17) it follows that

$$\begin{aligned} &\|x_n - \widehat{S}x_n\| = \|\widehat{S}\widehat{S}^{-1}x_n - \widehat{S}x_n\| \\ &\leq \|\widehat{S}^{-1}x_n - x_n\| = \|(2I - S)x_n - x_n\| = \|x_n - Sx_n\| \to 0 \quad (n \to \infty). \end{aligned} \tag{3.18}$$

For each $n \geq 0$, we put $T_{\lambda_n} := J_{\lambda_n}^B(I - \lambda_nA)$. Then from (3.12) we have

$$\begin{aligned} \|x_n - T_{\lambda_n}x_n\| &\leq \|x_n - u_n\| + \|u_n - T_{\lambda_n}u_n\| + \|T_{\lambda_n}u_n - T_{\lambda_n}x_n\| \\ &\leq 2\|x_n - u_n\| + \|u_n - y_n\| \to 0 \quad (n \to \infty). \end{aligned}$$

Noticing $0 < \lambda \leq \lambda_n$ for all $n \geq 0$ and using Proposition 2.3 (ii), we obtain

$$\|T_\lambda x_n - x_n\| \leq 2\|T_{\lambda_n}x_n - x_n\| \to 0 \quad (n \to \infty). \tag{3.19}$$

We define the mapping $\Psi : C \to C$ by $\Psi x := \theta_1\widehat{S}x + \theta_2Gx + (1-\theta_1-\theta_2)T_\lambda x \ \forall x \in C$ with $\theta_1 + \theta_2 < 1$ for constants $\theta_1, \theta_2 \in (0, 1)$. Then by Lemma 2.8 and Proposition 2.3 (i), we know that Ψ is nonexpansive and

$$\mathrm{Fix}(\Psi) = \mathrm{Fix}(\widehat{S}) \cap \mathrm{Fix}(G) \cap \mathrm{Fix}(T_\lambda) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{Fix}(G) \cap (A + B)^{-1}0 \ (=: \Omega).$$

Taking into account that

$$\|\Psi x_n - x_n\| \leq \theta_1\|\widehat{S}x_n - x_n\| + \theta_2\|Gx_n - x_n\| + (1 - \theta_1 - \theta_2)\|T_\lambda x_n - x_n\|,$$

we deduce from (3.15), (3.18) and (3.19) that

$$\lim_{n\to\infty} \|\Psi x_n - x_n\| = 0. \tag{3.20}$$

Let $z_s = sf(z_s) + (1-s)\Psi z_s\ \forall s \in (0,1)$. Then it follows from Proposition 2.4 that $\{z_s\}$ converges strongly to a point $x^* \in \mathrm{Fix}(\Psi) = \Omega$, which solves the VIP:

$$\langle (I-f)x^*, J(x^*-p)\rangle \le 0 \quad \forall p \in \Omega.$$

Also, from (2.3) we get

$$\begin{aligned}\|z_s - x_n\|^q &= \|s(f(z_s)-x_n) + (1-s)(\Psi z_s - x_n)\|^q \\ &\le (1-s)^q\|\Psi z_s - x_n\|^q + qs\langle f(z_s)-x_n, J_q(z_s-x_n)\rangle \\ &= (1-s)^q\|\Psi z_s - x_n\|^q + qs\langle f(z_s)-z_s, J_q(z_s-x_n)\rangle \\ &\quad + qs\langle z_s - x_n, J_q(z_s-x_n)\rangle \\ &\le (1-s)^q(\|\Psi z_s - \Psi x_n\| + \|\Psi x_n - x_n\|)^q \\ &\quad + qs\langle f(z_s)-z_s, J_q(z_s-x_n)\rangle + qs\|z_s-x_n\|^q \\ &\le (1-s)^q(\|z_s - x_n\| + \|\Psi x_n - x_n\|)^q \\ &\quad + qs\langle f(z_s)-z_s, J_q(z_s-x_n)\rangle + qs\|z_s-x_n\|^q,\end{aligned}$$

which immediately attains

$$\langle f(z_s)-z_s, J_q(x_n-z_s)\rangle \le \frac{(1-s)^q}{qs}(\|z_s-x_n\| + \|\Psi x_n - x_n\|)^q + \frac{qs-1}{qs}\|z_s-x_n\|^q.$$

From (3.20), we have

$$\limsup_{n\to\infty}\langle f(z_s)-z_s, J_q(x_n-z_s)\rangle \le \tfrac{(1-s)^q}{qs}M + \tfrac{qs-1}{qs}M = \left(\tfrac{(1-s)^q+qs-1}{qs}\right)M, \tag{3.21}$$

where M is a constant such that $\|z_s - x_n\|^q \le M$ for all $n \ge 0$ and $s \in (0,1)$. It is easy to see that $((1-s)^q + qs - 1)/qs \to 0$ as $s \to 0$. Since J_q is norm-to-norm uniformly continuous on bounded subsets of E and $z_s \to x^*$, we get

$$\|J_q(x_n-z_s) - J_q(x_n-x^*)\| \to 0 \quad (s \to 0).$$

So we obtain

$$\begin{aligned}&|\langle f(z_s)-z_s, J_q(x_n-z_s)\rangle - \langle f(x^*)-x^*, J_q(x_n-x^*)\rangle| \\ &= |\langle f(z_s)-f(x^*), J_q(x_n-z_s)\rangle + \langle f(x^*)-x^*, J_q(x_n-z_s)\rangle \\ &\quad + \langle x^*-z_s, J_q(x_n-z_s)\rangle - \langle f(x^*)-x^*, J_q(x_n-x^*)\rangle| \\ &\le |\langle f(x^*)-x^*, J_q(x_n-z_s) - J_q(x_n-x^*)\rangle| + |\langle f(z_s)-f(x^*), J_q(x_n-z_s)\rangle| \\ &\quad + |\langle x^*-z_s, J_q(x_n-z_s)\rangle| \\ &\le \|f(x^*)-x^*\|\|J_q(x_n-z_s) - J_q(x_n-x^*)\| + (1+\delta)\|z_s-x^*\|\|x_n-z_s\|^{q-1}.\end{aligned}$$

Thus, for each $n \ge 0$, we have

$$\lim_{s\to 0}\langle f(z_s)-z_s, J_q(x_n-z_s)\rangle = \langle f(x^*)-x^*, J_q(x_n-x^*)\rangle.$$

From (3.21), as $s \to 0$, it follows that

$$\limsup_{n\to\infty}\langle f(x^*)-x^*, J_q(x_n-x^*)\rangle \le 0. \tag{3.22}$$

By (C1) and (3.10), we get

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|\alpha_n f(u_n)+\beta_n u_n+\gamma_n Gz_n - x_n\| \\ &\le \alpha_n\|f(u_n)-x_n\|+\beta_n\|u_n-x_n\| \\ &\quad + \gamma_n(\|Gz_n-u_n\|+\|u_n-x_n\|) \\ &\le \alpha_n\|f(u_n)-x_n\|+\|u_n-x_n\| \\ &\quad + \|Gz_n-u_n\| \to 0 \quad (n\to\infty).\end{aligned} \tag{3.23}$$

Using (3.22) and (3.23), we have

$$\limsup_{n\to\infty}\langle f(x^*)-x^*, J_q(x_{n+1}-x^*)\rangle \le 0. \tag{3.24}$$

Using Lemma 2.9 and (3.24), we can infer that $\Gamma_n \to 0$ as $n\to\infty$. Thus, $x_n \to x^*$ as $n\to\infty$.

Case 2. Suppose that there exists $\{\Gamma_{l_i}\} \subset \{\Gamma_l\}$ s.t. $\Gamma_{l_i} < \Gamma_{l_i+1}$ $\forall i\in\mathbf{N}$, where $\mathbf{N}$ is the set of all positive integers. Define the mapping $\tau:\mathbf{N}\to\mathbf{N}$ by

$$\tau(l) := \max\{i\le l : \Gamma_i < \Gamma_{i+1}\}.$$

Using Lemma 2.7, we get

$$\Gamma_{\tau(l)} \le \Gamma_{\tau(l)+1} \quad \text{and} \quad \Gamma_l \le \Gamma_{\tau(l)+1}.$$

Putting $\Gamma_l = \|x_l - x^*\|^q$ $\forall l\in\mathbf{N}$ and using the same inference as in Case 1, we can obtain

$$\lim_{l\to\infty}\|x_{\tau(l)+1}-x_{\tau(l)}\| = 0 \tag{3.25}$$

and

$$\limsup_{l\to\infty}\langle f(x^*)-x^*, J_q(x_{\tau(l)+1}-x^*)\rangle \le 0. \tag{3.26}$$

Thanks to $\Gamma_{\tau(l)} \le \Gamma_{\tau(l)+1}$ and $\alpha_{\tau(l)}>0$, we conclude from (3.7) that

$$\|x_{\tau(l)}-x^*\|^q \le \frac{q}{1-\delta}\langle f(x^*)-x^*, J_q(x_{\tau(l)+1}-x^*)\rangle$$

and hence

$$\limsup_{l\to\infty}\|x_{\tau(l)}-x^*\|^q \le 0.$$

Thus, we get

$$\lim_{l\to\infty}\|x_{\tau(l)}-x^*\|^q = 0.$$

Using Proposition 2.2 and (3.25), we obtain

$$\begin{aligned}&\|x_{\tau(l)+1}-x^*\|^q - \|x_{\tau(l)}-x^*\|^q \\ &\quad \le q\langle x_{\tau(l)+1}-x_{\tau(l)}, J_q(x_{\tau(l)}-x^*)\rangle + \kappa_q\|x_{\tau(l)+1}-x_{\tau(l)}\|^q \\ &\quad \le q\|x_{\tau(l)+1}-x_{\tau(l)}\|\|x_{\tau(l)}-x^*\|^{q-1} \\ &\qquad + \kappa_q\|x_{\tau(l)+1}-x_{\tau(l)}\|^q \to 0 \quad (l\to\infty).\end{aligned}$$

Noticing $\Gamma_l \le \Gamma_{\tau(l)+1}$, we get

$$\|x_l-x^*\|^q \le \|x_{\tau(l)+1}-x^*\|^q \le \|x_{\tau(l)}-x^*\|^q + q\|x_{\tau(l)+1}-x_{\tau(l)}\|\|x_{\tau(l)}-x^*\|^{q-1}$$

$$+ \kappa_q \|x_{\tau(l)+1} - x_{\tau(l)}\|^q.$$

It is easy to see from (3.25) that $x_l \to x^*$ as $l \to \infty$. This completes the proof. □

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