Applied Analysis and Optimization
Volume 6, Number 2, 2022, 195–197

Yokohama Publishers
ISSN 2189-1664 Online Journal

ADDENDUM TO "A MORE COMPLETE VERSION OF A MINIMAX THEOREM"

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ABSTRACT. It is pointed out that an inequality in the main result of [1] can be improved.

Let X be a topological space and Y a convex set in a real Hausdorff topological vector space. If S is a convex subset of Y, we denote by $\mathcal{A}_S$ the class of all functions $f : X \times Y \to \mathbf{R}$ such that, for each $y \in S$, the function $f(\cdot, y)$ is lower semicontinuous and inf-compact.

Moreover, we denote by $\mathcal{B}$ the class of all functions $f : X \times Y \to \mathbf{R}$ such that either, for each $x \in X$, the function $f(x, \cdot)$ is quasi-concave and continuous, or, for each $x \in X$, the function $f(x, \cdot)$ is concave.

For any $f : X \times Y \to \mathbf{R}$, we set

$$\alpha_f = \sup_Y \inf_X f$$

and

$$\beta_f = \inf_X \sup_Y f \ .$$

Also, we denote by $\mathcal{C}_f$ the family of all sets $S \subseteq Y$ such that

$$\inf_X \sup_S f = \inf_X \sup_Y f \ .$$

In [1], we established the following result (with the usual rules in $\overline{\mathbf{R}}$):

Ttheorem A. *Let $f : X \times Y \to \mathbf{R}$. Assume that there is a function $\psi : Y \to \mathbf{R}$ such that $f + \psi \in \mathcal{B}$ and*

$$\alpha_{f+\psi} < \beta_{f+\psi} \ .$$

Then, for every convex set $S \in \mathcal{C}_{f+\psi}$, for every bounded function $\varphi : X \to \mathbf{R}$ and for every $\lambda > 0$ such that $\lambda f + \varphi \in \mathcal{A}_S$ and

$$\lambda > \frac{2 \sup_X |\varphi|}{\beta_{f+\psi} - \alpha_{f+\psi}} \ , \tag{1}$$

there exists $y^* \in S$ such that the function $\lambda f(\cdot, y^*) + \varphi(\cdot)$ has at least two global minima.

The aim of this addendum is to remark that, in the above statement, the inequality (1) can be improved.

Actually, we have

2020 *Mathematics Subject Classification.* 49J35, 49K35, 90C47, 35J92.

Key words and phrases. Strict minimax inequality, global extremum, multiplicity, farthest point; integral functional, Kirchhhoff-type equation, Neumann problem.

Theorem 1. *Let $f : X \times Y \to \mathbf{R}$. Assume that there is a function $\psi : Y \to \mathbf{R}$ such that $f + \psi \in \mathcal{B}$ and*

$$\alpha_{f+\psi} < \beta_{f+\psi} \ .$$

Then, for every convex set $S \in \mathcal{C}_{f+\psi}$, for every bounded function $\varphi : X \to \mathbf{R}$ and for every $\lambda > 0$ such that $\lambda f + \varphi \in \mathcal{A}_S$ and

$$\lambda > \frac{\sup_X \varphi - \inf_X \varphi}{\beta_{f+\psi} - \alpha_{f+\psi}} \ , \tag{1'}$$

there exists $y^* \in S$ such that the function $\lambda f(\cdot, y^*) + \varphi(\cdot)$ has at least two global minima.

To see this, it is enough to apply Theorem A with $\varphi - a$ instead of φ, where

$$a = \frac{\inf_X \varphi + \sup_X \varphi}{2} \ ,$$

so that

$$\sup_X |\varphi - a| = \frac{\sup_X \varphi - \inf_X \varphi}{2} \ .$$

As a consequence:

- in the statements of Theorem 4 and Corollary 1, the inequality

$$\lambda > \frac{4 \sup_X |\gamma|}{\mathrm{diam}(X)}$$

can be replaced with

$$\lambda > \frac{2(\sup_X \gamma - \inf_X \gamma)}{\mathrm{diam}(X)} \ ;$$

- in the statement of Theorem 5, the inequality

$$\lambda > \frac{2 \sup_U |\omega|}{\int_\Omega h(x, \xi_2, 0)dx - \max\left\{\int_\Omega h(x, \xi_1, 0)dx, \int_\Omega h(x, \xi_3, 0)dx\right\}}$$

can be replaced with

$$\lambda > \frac{\sup_U \omega - \inf_U \omega}{\int_\Omega h(x, \xi_2, 0)dx - \max\left\{\int_\Omega h(x, \xi_1, 0)dx, \int_\Omega h(x, \xi_3, 0)dx\right\}} \ ;$$

- in the statement of Theorem 6, the inequality

$$\lambda > \frac{2 \sup_{[0,+\infty[} |\chi|}{p(f(\xi_2) - \max\{f(\xi_1), f(\xi_3)\}) \int_\Omega \beta(x)dx}$$

can be replaced with

$$\lambda > \frac{\sup_{[0,+\infty[} \chi - \inf_{[0,1]} \chi}{p(f(\xi_2) - \max\{f(\xi_1), f(\xi_3)\}) \int_\Omega \beta(x)dx} \ .$$

References

[1] B. Ricceri, *A more complete version of a minimax theorem*, Appl. Anal. Optim. **5** (2021), 251-261.

Manuscript received January 12 2022
revised February 10 2022

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