*Applied Analysis and Optimization*
Volume 6, Number 2, 2022, 169–194

**Yokohama Publishers**
*ISSN 2189-1664 Online Journal*


# AN INERTIAL APPROXIMATION METHOD FOR GENERALIZED MIXED EQUILIBRIUM AND FIXED POINT PROBLEMS OF BREGMAN TOTAL QUASI - ASYMPTOTICALLY NONEXPANSIVE MULTIVALUED MAPPINGS

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Abstract. In this paper, we introduce a modified inertial iterative algorithm for approximating a common solution of generalized mixed equilibrium problem and fixed points problem for Bregman total quasi - asymptotically nonexpansive multivalued mappings in Banach spaces. Furthermore, we prove strong convergence of the sequence by our algorithm in real Banach space. Our result generalize and improve related results announced by many authors.

## 1. Introduction

Let $E$ be a real Banach space with norm $\| \, . \, \|$, $E^*$ be the dual space of $E$ and let $C$ be a nonempty closed convex subset of $E$. Let $g, b \, : C \times C \longrightarrow \mathbb{R}$ be bifunctions, where $\mathbb{R}$ is the set of real numbers and let $A : C \longrightarrow E^*$ be a nonlinear mapping, The generalized mixed equilibrium problem (see [5, 23]) is to find $x \in C$ such that:

$$g(x, y) + b(x, y) - b(x, x) + \langle Ax, y - x\rangle \geq 0, \forall y \in C. \tag{1.1}$$

The set of solutions of generalized mixed equilibrium problem (1.1) is denoted by

$$GMEP(g, b, A) = \{x \in C : g(x, y) + b(x, y) - b(x, x) + \langle Ax, y - x\rangle \geq 0 \ \forall y \in C\}.$$

If $A = 0$, problem (1.1) reduces to the following mixed equilibrium problem [18] that is to find $x \in C$ such that

$$g(x, y) + b(x, y) - b(x, x) \geq 0, \forall y \in C. \tag{1.2}$$

The set of solutions of mixed equilibrium problem (1.3) is denoted by

$$MEP(g, b) = \{x \in C : g(x, y) + b(x, y) - b(x, x) \geq 0 \ \forall y \in C\}.$$

If $b = 0$, problem (1.1) reduces to the following generalized equilibrium problem [38] that is to find $x \in C$ such that

$$g(x, y) + \langle Ax, y - x\rangle \geq 0, \forall y \in C. \tag{1.3}$$

The set of solutions of generalized equilibrium problem (1.3) is denoted by

$$GEP(g, A) = \{x \in C : g(x, y) + \langle Ax, y - x\rangle \geq 0 \ \forall y \in C\}.$$

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2020 *Mathematics Subject Classification.* 47H09, 47J25.

*Key words and phrases.* Strong convergence, generalized mixed equilibrium problem, Bregman projection, Bregman total quasi - asymptotically nonexpansive multivalued mappings.

If $g(x,y) = 0$ and $b(x,y) = 0$ $\forall x, y \in C$, then (1.1) reduces to the following variational inequality problem [36] that is to find $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \forall y \in C. \tag{1.4}$$

The set of solutions of variational inequality problem (1.4) is denoted by

$$VIP(A) = \{x \in C : \langle Ax, y - x\rangle \geq 0, \forall y \in C\}.$$

If $A = 0$ and $b(x,y) = 0, \forall x, y \in C$, then (1.1) reduces to the equilibrium problem in the sense of Blum and Oettli [11] : Find $x \in C$ such that

$$g(x,y) \geq 0, \forall y \in C. \tag{1.5}$$

The set of solutions of equilibrium problem (1.5) is denoted by

$$EP(g) = \{x \in C : g(x,y) \geq 0 \ \forall y \in C\}.$$

The generalized equilibrium problems are suitable method for investigating various applied problems arising in economics, mathematical physics, engineering and other fields. Moreover equilibrium problems are closely related with other general problems in nonlinear analysis such as fixed point, game theory, variational inequality and optimization problems. It has been shown that variational inequality, complementarity problems, fixed point problems and inclusion problems can be viewed as a special realization of the equilibrium problems; see [7,10,21,22,37,40,42] and the references therein.

In 1967, Bregman [15] introduced an effective technique through Bregman distance function $D_f$ for designing and analyzing feasibility and optimization algorithms. This opened a new area of research in which Bregman's technique is applied in various ways to iterative algorithm for solving not only feasibility and optimization problems, but also algorithms for solving fixed point problems for nonlinear mappings (see, e.g [4,19,27,39] and the references therein).

Let $C$ be a nonempty closed convex subset of a Banach space $E$. Let $\hat{C}B(C)$ be a family of nonempty, closed and bounded subsets of $C$ then the Hausdorff metric on $\hat{C}B(C)$ is defined by

$$H(A,B) = max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\}, \ \forall A, B \in \hat{C}B(C),$$

where $d(x,B) = \inf\{\|x - y\| : y \in B\}$, is the distance from a point $x$ to a $B$. Let $T : C \longrightarrow \hat{C}B(C)$ be a multi valued mapping. A point $\hat{x} \in C$ is called fixed point of $T$ if $\hat{x} \in T\hat{x}$. The set of fixed points of $T$ is denoted by $F(T) = \{\hat{x} \in C : \hat{x} \in T\hat{x}\}$. A multi valued mapping $T : C \longrightarrow \hat{C}B(C)$ is called nonexpansive, if

$$H(Tx,Ty) \leq \| x - y \|, \forall x, y \in C.$$

A multi valued mapping $T : C \longrightarrow \hat{C}B(C)$ is said to be uniformly $L$- Lipschitz continuous, if there exists a constant $L > 0$ such that

$$H(T^n x, T^n y) \leq L \| x - y \|, \ \forall n \geq 1, \ x, y \in C.$$

Several results for fixed point approximations of Bregman quasi nonexpansive mappings and their generalizations are established see for example [3,4,26,35,43].
Various iterative schemes have been used to approximate common elements in the

set of equilibrium problems and fixed point of nonexpansive mapping in various spaces see [1, 5, 6, 40, 42].
In 1964, an inertial algorithms was first proposed by Polyak [31] as an acceleration process in solving a smooth convex minimisation problem. An inertial-types algorithm is a two-step iterative method in which the next iteration is defined by making use of the previous two iterates. Also an inertial-type play a crucial role in speeding up the convergence of the sequence generated by the algorithm. With regards to this importance, a number of researchers have been working on an inertial-type method (see, e.g [13, 14, 20, 24, 28] and the references therein)

In 2009, Takahashi and Zembayashi [37], studied the following algorithm for approximating solutions of an equilibrium problems and fixed points problems of a relatively nonexpansive mappings in Banach spaces :

$$\begin{cases} x_0 = x \in C \\ z_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ u_n \in C \text{ such that } f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jz_n\rangle \geq 0, \forall y \in C \\ H_n = \{z \in C : \psi(z, u_n) \leq \psi(z, x_n)\} \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geq 0\} \\ x_{n+1} = \Pi_{H_n \cap W_n} x, n \geq 0, \end{cases}$$

where $\psi$ is Lyapunov function. They proved that the sequence $\{x_n\}$ converges strong to $\Pi_{F(T)\cap EP(f)}x$.
In 2014, Li and Liu [27] proposed and studied the following hybrid Halpern algorithm for solving a common fixed point for Bregman totally quasi-asymptotically nonexpansive multi-valued mappings in reflexive Banach spaces.

$$\begin{cases} x_1 \in E, C = C_1; \\ z_{n,j} = \nabla f^*(\beta_n \nabla f(x_1) + (1-\beta_n)(\alpha_n \nabla f(x_n) \\ +(1-\alpha_n)\nabla f(u_{n,j})),\ u_{n,j} \in T_j^n x_n; \\ C_{n+1} = \{z \in C_n : \sup\limits_{j\geq 1} D_f(z, z_{n,j}) \leq \beta_n D_f(z, x_1) + (1-\beta_n)(z, x_n) + \xi_n; \\ x_{n+1} = Proj^f_{C_{n+1}} x_0, n \geq 1, \end{cases}$$

where $\xi_n = v_n \sup\limits_{q\in\Omega} \zeta(D_f(q, x_n) + \eta_n, u_{n,j} \in T_j^n x_n, j \geq 1,\ \Omega; = \cap_{j=1}^{\infty} F(T_j) \neq \emptyset$ and $Proj^f_{c_{n+1}}$ is the Bregman projection of $E$ onto $C_{n+1}$. Then, the sequence $\{x_n\}$ converges strongly to $Proj^f_{\Omega}$.

Recently, Kazmi [26], studied the following hybrid algorithm for approximating a common solution for generalized equilibrium problems and the fixed point problems of a Bregman relatively nonexpansive mappings in reflexive Banach spaces

$$\begin{cases} x_0, z_0 \in C; \\ u_n = \nabla f^*(\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(Tx_n)); \\ z_{n+1} = Res^f_{G,\psi}(u_n); \\ C_n = \{z \in C : D_f(z, z_{n+1}) \leq \alpha_n D_f(z, z_n) + (1-\alpha_n) D_f(z, x_n)\}, \\ Q_n = \{z \in C : \langle \nabla f(x_0) - f(x_n), z - x_n\rangle \leq 0\}, \\ x_{n+1} = Proj^f_{C_n \cap Q_n} x_0, n \geq 0, \end{cases}$$

where $\{x_n\}$ is a sequence in $[0,1]$ such that $\lim_{n\to\infty}\alpha_n = 0$. Then, the sequence $\{x_n\}$ converges strongly to $Proj^f_{\Omega}x_1$, where $Proj^f_{\Omega}x_1$ is the projection of $C$ onto $\Omega$ Very recently, Alansari et al [1], established an inertial iterative algorithm for approximating a common solution for a system of variational inequality problem, generalized equilibrium problem and a common fixed point problem of relatively nonexpansive mapping in uniformly smooth and uniformly convex real Banach space. Let the sequences $\{x_n\}$, and $\{z_n\}$ be generated by the algorithm:

$$\begin{cases} x_0 = x_1,\ z_0 \in C,\ C_0 := C, \\ \eta_n = x_n + \Phi_n(x_n - x_{n-1}) \\ y_n = \Pi_C J^{-1}(J\eta_n - \mu_n D\eta_n), \\ u_n = J^{-1}(\alpha_n Jz_n + (1-\alpha_n)JSy_n), \\ z_{n+1} = S_{r_n}u_n, \\ H_n = \{z \in C : \psi(z, z_{n+1}) \le \alpha_n\phi(z, z_n) + (1-\alpha_n)\psi(z, w_n)\}, \\ N_n = \{z \in C : \langle x_n - z, Jx_n - Jx_0\rangle \le 0\}, \\ x_{n+1} = \Pi_{H_n\cap N_n}x_0, n \ge 0, \end{cases}$$

where $\{\alpha_n\} \in [0,1],\ r_n \in [a,\infty)$ for some $a > 0,\ \{\Phi_n\} \in (0,1)$ and $\{\mu_n\} \in (0,\infty)$. Then sequences $\{x_n\}$, converges strongly to a point $\hat{x} \in \Omega$.

Motivated and inspired by the results of Li and Liu [27], Kazmi [26] and Alansari et al [1] mentioned above, we study an inertial algorithm for approximating a common solution of generalized mixed equilibrium problem and fixed points problem for Bregman total quasi - asymptotically nonexpansive multivalued mappings in Banach spaces. Our results extend and improves recent the result of Alansari et al [1] and many results announced by many authors.

## 2. Preliminaries

In this section, we shall consider some basic definitions and lemmas which will be used in the proof of our main results.

Let $f : E \to (-\infty, +\infty]$ is a proper, lower semi-continuous and convex function. We denote by $\text{dom}f := \{x \in E : f(x) < +\infty\}$, the domain of $f$. Let $x \in \text{int}(\text{dom}f)$; the subdifferential of $f$ at $x$ is the convex set defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x\rangle \le f(y),\ \forall y \in E\}.$$

The Fenchel conjugate of $f$ is the function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}.$$

We know that the Young - Fenchel inequality holds:

$$\langle x^*, x\rangle \le f(x) + f^*(x^*), \forall x \in E, x^* \in E^*.$$

A function $f$ on $E$ is coercive [25] if the sublevel set of $f$ is bounded; equivalently,

$$\lim_{\|x\|\to+\infty} f(x) = +\infty.$$

A function $f$ on $E$ is said to be strongly coercive [41] if

$$\lim_{\|x\|\to+\infty} \frac{f(x)}{\|x\|} = +\infty.$$

For any $x \in intdomf$ and $y \in E$, the right-hand derivative of $f$ at $x$ in the direction $y$ is defined by

$$f^0(x,y) := \lim_{t\to 0^+} \frac{f(x+ty)-f(x)}{t}$$

The function $f$ is said to be Gâteaux differentiable at $x$ if $\lim_{t\to 0} \frac{f(x+ty)-f(x)}{t}$ exists for any $y$. In this case, $f^0(x,y)$ coincides with $\nabla f(x)$, the value of the gradient of $f$ at $x$. The function $f$ is said to be Gâteaux differentiable if it is Gâteaux differentiable for any $x \in$ int(dom$f$). The function $f$ is said to be Fréchet differentiable at $x$ if this limit is attained uniformly in $y$ with $\|y\| = 1$. Furthermore $f$ is said to be uniformly Fréchet differentiable on a subset $C$ of $E$ if the limit is attained uniformly for $x \in C$ and $\|y\| = 1$. It is well known that if $f$ is Gâteaux differentiable (resp. Fréchet differentiable) on int(dom$f$), then $f$ is continuous and its Gâteaux derivative $\nabla f$ is norm-to-weak$^*$ continuous (resp. norm-to-norm continuous) on int(dom($f$)) (see [8, 12]).

**Definition 2.1** ( [9]). The function $f$ is said to be:

(1) Essentially smooth, if $\partial f$ is both locally bounded and single-valued on its domain;
(ii) Essentially strictly convex, if $(\partial f)^{-1}$ is locally bounded on its domain and $f$ is strictly convex on every subset of dom$f$;
(iii) Legendre, if it is both essentially smooth and essentially strictly convex.

**Remark 2.2.** If $E$ is a reflexive Banach space, then we have the following results:

(i) $f$ is essentially smooth if and only if $f^*$ is essentially strictly convex (see [9], Theorem 5.4).
(ii) $(\partial f)^{-1} = \partial f^*$ (see [12]).
(iii) $f$ is Legendre if and only if $f^*$ is Legendre (see [9],Corollary 5.5).
(iv) If $f$ is Legendre, then $\nabla f$ is a bijection satisfying $\nabla f = (\nabla f^*)^{-1}$, ran$\nabla f$ = dom$\nabla(f^*)$ = int(dom$f^*$) and ran $\nabla f^*$ = dom$f$ = int(dom$f$) (see [9], Theorem 5.10 ), where ran stands for the range.

The following results gives a relationship between Frechet functions with their derivatives

**Lemma 2.3** ( [32]). *If $f : E \longrightarrow (-\infty, +\infty]$ is uniformly Fréchet differentiable and bounded on bounded subset of $E$, Then, $f$ is uniformly continuous on bounded subsets of $E$ and $\nabla f$ is uniformly continuous on bounded subsets of $E$ from the strong topology of $E$ to the strong topology on $E^*$.*

**Definition 2.4.** Let $f : E \longrightarrow (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function $D_f : dom \times intdomf \longrightarrow [0, +\infty)$ defined by

(2.1) $D_f(y,x) := f(y) - f(x) - \langle \nabla f(x), y-x\rangle, \forall x \in int\ domf, y \in domf,$

is called the Bregman distance with respect to $f$. It should be noted that the Bregman distance satisfies $D_f(x,x) = 0$, but $D_f(x,y) = 0$ may not imply $x = y$. Now if $f$ is Strictly convex Legendre function this indeed holds. In general, $D_f$ is not symmetric and does not satisfy the triangle inequality, hence $D_f$ is not a distance function. Then $D_f$ has the following important properties [34] :

i) the two point identity, for any $x, y \in int\ domf$,

$$D_f(x,y) + D_f(y,x) = \langle \nabla f(x) - \nabla f(y), x - y \rangle;$$

ii) the three point identity, for any $x \in domf$ and $y, z \in int\ domf$,

$$D_f(x,y) + D_f(y,z) - D_f(x,z) = \langle \nabla f(z) - \nabla f(y), x - y \rangle;$$

iii) the four point identity, for any $x \in domf$ and $w, y, z \in int\ domf$,

$$D_f(y,x) + D_f(y,z) - D_f(w,x) + D_f(w,z) = \langle \nabla f(z) - \nabla f(x), y - w \rangle.$$

Let $f : E \longrightarrow \mathbb{R}$ be a convex and Gâteaux differentiable function. Following [2] and [18], the function $V_f : E \times E^* \longrightarrow [0, +\infty)$ asssociated with $f$ is defined by

$$V_f(x, x^*) = f(x) + f^*(x^*) - \langle x, x^* \rangle, \forall x \in E,\ x^* \in E^*.$$

Then the following assertions hold:

(1) $D_f(x, \nabla f^*(x^*)) = V_f(x, x^*)$ for all $x \in E,\ x^* \in E^*$,
(2) $D_f(x,y) = V_f(x, \nabla f(y))$ and $V_f$ is convex in the second variable.

Therefore for $\alpha \in (0,1)$ and $x, y \in E$, we have

$$D_f(z, \nabla f^*(\alpha \nabla f(x) + (1-\alpha)\nabla f(y))) \leq \alpha D_f(z,x) + (1-\alpha) D_f(z,y).$$

Recall that the Bregman projection [15] of $x \in intdomf$ onto a nonempty, closed and convex set $C \subset domf$ is the necessarily unique vector $proj_C^f(x)$ ( for convenience, here we use $p_C^f(x)$ for $proj_C^f(x)$ ) satisfying

$$D_f(proj_C^f(x), x) := \inf\{D_f(y,x) : y \in C\}.$$

The modulus of total convexity of $f$ at $x \in int\ domf$ is the function $v_f(x,t)$ : $[0, +\infty) \longrightarrow [0, +\infty)$ defined by

$$v_f(x,t) := \inf\{Df(y,x) : y \in domf, \|y - x\| = t\}.$$

The function $f$ is called totally convex at $x$, if $v_f(x,t) > 0$ whenever $t > 0$. The function $f$ is called totally convex, if it is totally convex at any point $x \in int\ domf$. Furthermore, $f$ is said to be totally convex on bounded sets, if $v_f(B,t) > 0$, for any nonempty bounded subset $B$ of $E$ and $t > 0$, where the modulus of the total convexity of the function $f$ on the set $B$ is the function $v_f : int\ domf \times [0, +\infty) \longrightarrow [0, +\infty)$ defined by

$$v_f(B,t) := \inf\{v_f(x,t) : x \in B \cap domf\}.$$

It is well known that [17], $f$ is totally convex on bounded sets if and only if $f$ is uniformly convex on bounded.

**Lemma 2.5** ([16]). *If $x \in dom\ f$, then the following statements are equivalent:*

i) *the function $f$ is totally convex at $x$;*
ii) *for any sequence $\{y_n\} \subset dom\ f$,*

$$\lim_{n \to +\infty} D_f(y_n, x) = 0 \Longrightarrow \lim_{n \to +\infty} \| y_n - x \| = 0.$$

Recall that the function $f$ is called sequentially consistent [17] if for any two sequences $\{x_n\}$ and $\{y_n\}$ in $E$ such that the first one is bounded

$$\lim_{t \to +\infty} D_f(y_n, x_n) = 0 \Longrightarrow \lim_{t \to +\infty} \| y_n - x_n \| = 0.$$

**Lemma 2.6** ([30])**.** *Let $E$ be a Banach space and $f : E \longrightarrow \mathbb{R}$ be a Gateaux differentiable) function which is uniformly convex on bounded subsets of $E$. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be bounded sequences in $E$. Then*

$$\lim_{n\to\infty} D_f(x_n, y_n) = 0 \textit{ if and only if } \lim_{n\to\infty} \|x_n - y_n\| = 0.$$

**Definition 2.7.** A mapping $T : C \longrightarrow C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C$. We denote by $F(T)$ the set of fixed points of $T$. A point $p \in C$ is said to be an asymptotic fixed point of $T$ if there exists a sequence $\{x_n\}$ in $C$ which converges weakly to $p$ and $\lim_{n\to\infty} \| x_n - Tx_n \|= 0$. We denote the set of all asymptotic fixed points of $T$ by $\hat{F}(T)$.

**Definition 2.8.** A mapping $T : C \longrightarrow C$ with $F(T) \neq \emptyset$ is called:

(1) quasi-Bregman nonexpansive with respect to $f$ if

$$D_f(p, Tx) \leq D_f(p, x),\ \forall x \in C, p \in F(T).$$

(2) Bregman relatively nonexpansive with respect to $f$ if

$$D_f(p, Tx) \leq D_f(p, x),\ \forall x \in C, p \in F(T) \textit{ and } \hat{F}(T) = F(T).$$

(3) Bregman strongly nonexpansive with respect to $f$ and $\hat{F}(T)$ if

$$D_f(p, Tx) \leq D_f(p, x),\ \forall x \in C, p \in \hat{F}(T)$$

and, if whenever $\{x_n\} \subset C$ is bounded, $p \in \hat{F}(T)$ and

$$\lim_{n\to 0}(D_f(p, x_n) - D_f(p, Tx_n)) = 0,$$

it follows that

$$\lim_{n\to 0} D_f(x_n, Tx_n) = 0.$$

(4) Bregman firmly nonexpansive (BFNE) with respect to $f$ if, for all $x, y \in C$,

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty\rangle \leq \langle \nabla f(x) - \nabla f(y), Tx - Ty\rangle$$

equivalently,

$$\begin{aligned} D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) &\leq D_f(Tx, y) \\ &+ D_f(Ty, x). \end{aligned} \tag{2.2}$$

**Definition 2.9.** Let $C$ be a convex subset of int dom $f$ and let $T$ be a multi-valued mapping of $C$. Apoint $p \in C$ is called an asymptotic fixed point of $T$ if $C$ contains a sequence $\{x_n\}$ which converges weakly to $p$ such that $d(x_n, Tx_n) \longrightarrow 0$ ( as $n \longrightarrow \infty$). We denote by $\hat{F}(T)$ the set of asymptotic fixed points of $T$.

**Definition 2.10.** A multi valued mapping $T : C \longrightarrow \hat{C}B(C)$ with a nonempty fixed point set is said to be:

(i) Bregman strongly nonexpansive with respect to a nonempty $\hat{F}(T)$, if

$$D_f(p, z) \leq D_f(p, x), \forall x \in C, z \in T(x), p \in \hat{F}(T)$$

and if, whenever $\{x_n\} \subset C$ is bounded, $p \in \hat{F}(T)$ and $\lim_{n\to 0}[D_f(p, x_n) - D_f(p, z_n)] = 0$, then $\lim_{n\to 0} D_f(x_n, z_n) = 0$, where $z_n \in Tx_n$.

(ii) Bregman firmly nonexpansive if

$$\langle \nabla f(x^*) - \nabla f(y^*), x^* - y^* \rangle \quad \leq \quad \langle \nabla f(x) - \nabla f(y), x^* - y^* \rangle,$$

for every $x, y \in C, x^* \in Tx, y^* \in Ty$.

(iii) Bregman quasi- asymptotically nonexpansive if there exist a real sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$,(as $n \longrightarrow \infty$), $\hat{F}(T) = F(T) \neq \emptyset$ such that

$$D_f(p, z) \leq k_n D_f(p, x), \forall n \geq 1, x \in C, z \in T^n x, p \in F(T).$$

(iv) Bregman total quasi- asymptotically nonexpansive if there exist nonnegative real sequence $\{\lambda_n\}, \{\mu_n\}$ with $\lambda_n \longrightarrow 0, \mu_n \longrightarrow 0$ ( as $n \longrightarrow \infty$ ) and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with $\zeta(0) = 0$, $\hat{F}(T) = F(T) \neq \emptyset$ such that

$$D_f(p, z) \leq D_f(p, x) + \lambda_n \zeta(D_f(p, x)) + \mu_n,$$
$$\forall n \geq 1, x \in C, z \in T^n x, p \in F(T).$$

(v) Closed, if for any sequence $\{x_n\} \subset C$ with $x_n \longrightarrow x \in \hat{C}B(C)$ and $d(Tx_n, y) \longrightarrow 0$ $(y \in C)$, then $y \in Tx$.

**Remark 2.11** ([19]). (i) The class of Bregman total quasi- asymptotically nonexpansive multivalued mappings contains properly the class of Bregman quasi- asymptotically nonexpansive multivalued mappings.

(ii) The class of Bregman quasi- asymptotically nonexpansive multivalued mappings contains properly the class of Bregman quasi- relatively nonexpansive multivalued mappings.

(iii) The class of Bregman quasi- relatively nonexpansive multivalued mappings contains properly the class of Bregman relatively nonexpansive multivalued mappings. However, converses of these statements are not true.

**Lemma 2.12** ([17]). *Let $C$ be a nonempty, closed and convex subset of a reflexive Banach space $E$. Let $f : E \longrightarrow \mathbb{R}$ be a Gâteaux differentiable and totally convex function and let $x \in E$. Then:*

i) $z = P_C^f(x)$ *if and only if* $\langle \nabla f(x) - \nabla f(z), y - z \rangle \leq 0, \ \forall y \in C$;

ii) $D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \leq D_f(y, x), \ \forall y \in C.$

**Lemma 2.13** ([17]). *Let $f : E \longrightarrow (-\infty, +\infty]$ be a convex function whose domain contains at least two points. Then, $f$ is sequentially consistent if and only if it is totally convex on bounded sets.*

**Lemma 2.14** ( [33]). *Let $f : E \longrightarrow (-\infty, +\infty]$ be a Gateaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

**Lemma 2.15** ( [27, Proposition 3.1]). *Let $f : E \longrightarrow (-\infty, +\infty]$ be a Legendre function and $C$ be a nonempty, closed and convex subset of int dom$f$. Let $T : D \longrightarrow N(D)$ be a Bregman totally quasi-asymptotically nonexpansive multi-valued mapping with respect to $f$. Then $F(T)$ is closed and convex.*

**Lemma 2.16** ([29]). *Let $f : E \to (-\infty, +\infty]$ be a Legendre function such that $\nabla f^*$ is bounded on bounded subsets of $E^*$. Let $x \in E$. if the sequence $\{D_f(x, x_n)\}$ is bounded, then the sequence $\{x_n\}$ is bounded.*

**Lemma 2.17** ([41]). *Let $f : E \to (-\infty, +\infty]$ be a convex function which is bounded on bounded subsets of $E$. Then the following assertions are equivalent:*

(i) *$f$ is strongly coercive and uniformly convex on bounded subsets of $E$;*
(ii) *$f^*$ is Fréchet differentiable and $\nabla f^*$ is uniformly norm-to-norm continuous on bounded subsets of $domf^* = E^*$.*

**Assumption A.** The bifunction $g : C \times C \longrightarrow \mathbb{R}$ satisfies the following assumptions:

$(A_1)$ $g(x,x) = 0, \forall x \in C$;
$(A_2)$ $g$ is monotone, 1.e, $g(x,y) + g(y,x) \leq 0,\ \forall x, y \in C$;
$(A_3)$ For each $x, y, z \in C, \limsup_{t\to 0} g(tz + (1-t)x, y) \leq g(x,y)$;
$(A_4)$ For each $x \in C, y \mapsto g(x,y)$ is convex and lower semicontinuous.

**Assumption B.** Let $b : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying the following assumptions:

$(B_1)$ $b$ is skew-symmetric, i.e., $b(x,x) - b(x,y) - b(y,x) + b(y,y) \geq 0, \forall x, y \in C$;
$(B_2)$ $b$ is convex in the second argument;
$(B_3)$ $b$ is continuous.

**Lemma 2.18** ( [23]). *Let $C$ be a nonempty, closed and convex subset of a real reflexive Banach space $E$. Let $\Theta : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying $(A_1)-(A_4)$, $\psi : C \longrightarrow E^*$ be a continuous monotone mapping and $\varphi : C \longrightarrow \mathbb{R} \cup \{\infty\}$ be proper convex and lower semi-continuous. Let $f : E \longrightarrow (-\infty, +\infty]$ be a coercive Legendre function. For $x \in E$ and define a mapping $Res^f_{\Theta,\psi,\varphi} : E \longrightarrow 2^C$ as follws:*

$$\begin{aligned} Res^f_{\Theta,\psi,\varphi}(x) &= \{z \in C : \Theta(z,y) + \langle \psi z, y - z\rangle + \varphi(y) - \varphi(z) \\ &+ \langle \nabla f(z) - \nabla f(x), y - z\rangle \geq 0,\ \forall y \in C\} \end{aligned}$$

*Then, the following properties holds:*

(a) *$Res^f_{\Theta,\psi,\varphi}$ is single-valued and dom $(Res^f_{\Theta,\psi,\varphi}) = E$;*
(b) *$Res^f_{\Theta,\psi,\varphi}$ is Bregman firmly nonexpansive type mapping;*
(c) *$F(Res^f_{\Theta,\psi,\varphi}) = GMEP(\Theta, \psi, \varphi)$;*
(d) *$GMEP(\Theta, \psi, \varphi)$ is closed and convex;*
(e) *$D_f(p, Res^f_{\Theta,\psi,\varphi}(x)) + D_f(Res^f_{\Theta,\psi,\varphi}(x), x) \leq D_f(p,x),\ \forall p \in F(Res^f_{\Theta,\psi,\varphi}),\ x \in E$.*

**Definition 2.19.** Let $K$ and $C$ be convex sets with $C \subset K$. Then $\text{core}_k C$, the core of $C$ relative to $K$, is defined by

$$a \in core_k C :\Leftrightarrow (a \in C,\ and\ C \cap (a,y) \neq \emptyset\ \forall y \in K \setminus C)$$

where $\text{core}_k K = K$.

In veiw of Definition 2.19, we have the following results;

**Lemma 2.20** ([11, Theorem 1]). *Let the following assumptions (i)-(iv) hold:*

(i) *$X$ is a real topological vector space;*
*$K \subset X$ is a closed, convex, nonvoid set.*

(ii) *$g : K \times K \to \mathbb{R}$ has the following properties:*
*$g(x, x) = 0$ for all $x \in K$;*
*$g(x, y) + g(y, x) \leq 0$ for all $x, y \in K$ ( monotonicity);*
*for all $x, y \in K$ the function $t \in [0, 1] \mapsto g(ty + (1 - t)x, y)$ is upper semi-continuous at $t = 0$ ( hemicontinuity);*
*$g$ is convex and lower semicontinuous in the second argument.*

(iii) *$h : K \times K \to \mathbb{R}$ has the following properties:*
*$h(x, x) = 0$ for all $x \in K$;*
*$h$ is upper semicontinuous in the first argument;*
*$h$ is convex in the second argument;*

(iv) *There exists $C \subset K$ compact, convex, $\neq \emptyset$, such that for every $x \in C \setminus core_k C$ there exists $a \in core_k C$ such that*
$$g(x, a) + h(x, a) \leq 0 \ (coercivity).$$
*Then, there exists $\hat{x} \in C$ such that $0 \leq g(\hat{x}, y) + h(\hat{x}, y)\ \forall y \in K$.*

**Definition 2.21.** Let $C$ be a nonempty, closed and convex subsets of a real reflexive Banach space $E$ and $A : C \longrightarrow E^*$ be a continuous monotone mapping. Let $g : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying $(A_1) - (A_4)$ with respect to bifunction $b : C \times C \longrightarrow \mathbb{R}$ satisfying $(B_1) - (B3)$. The mixed resolvent of $g$ with respect to $b$ is the operator $Res^f_{g,b,A} : E \longrightarrow 2^C$ defined by

$$\begin{aligned} Res^f_{g,b,A}(x) = \{z \in C : g(z, y) + b(z, y) + \langle Az, y - z\rangle \\ +\langle \nabla f(z) - \nabla f(x), y - z\rangle \geq b(z, z), \forall y \in C\}, \forall x \in E. \end{aligned} \tag{2.3}$$

For proving the following lemmas, we use a similar idea of [ [23], Lemma 2.14].

**Lemma 2.22.** *Let $f : E \longrightarrow (-\infty, +\infty]$ be a coercive and Gateaux differentiable function. Let $C$ be a nonempty, closed and convex subset of $E$ and $A : C \longrightarrow E^*$ be a continuous monotone mapping. Assume that $g : C \times C \longrightarrow \mathbb{R}$ is a bifunction satisfying $(A_1) - (A_4)$ with respect to bifunction $b : C \times C \longrightarrow \mathbb{R}$ satisfying $(B_1) - (B3)$, then $dom(Res^f_{g,b,A}) = E$.*

*Proof.* Since $f$ is a coercive function, the function $h : E \times E \longrightarrow (-\infty, +\infty]$ defined by
$$h(\overline{x}, y) := f(y) - f(\overline{x}) - \langle x^*, y - \overline{x}\rangle,$$
satisfies the following for all $x^* \in E^*$ and $y \in C$
$$\lim_{\|\overline{x}-y\|\to+\infty} \frac{h(\overline{x}, y)}{\|\overline{x} - y\|} = +\infty$$
Now, it follows from Lemma 2.20(iv), that there exists $x \in C$ such that

$$\begin{aligned} g(x, y) + \langle Ax, y - x\rangle \quad + \quad & b(x, y) - b(x, x) + f(y) - f(\hat{x}) \\ - \quad & \langle x^*, y - x\rangle \geq 0, \forall y \in C. \end{aligned} \tag{2.4}$$

Let $y = tx + (1 - t)\hat{y}$, for $\hat{y} \in C$. Therefore from (2.4), we have

$$g(x, tx + (1 - t)\hat{y}) \quad + \quad \langle Ax, tx + (1 - t)\hat{y} - x\rangle + b(x, tx + (1 - t)\hat{y})$$

$$\begin{aligned} &+ \quad f(tx+(1-t)\hat{y}) - f(x) - \langle x^*, tx+(1-t)\hat{y}-x\rangle \\ &\geq \quad b(x,x). \end{aligned}$$

By convexity of $b$, we obtain

$$\begin{aligned} g(x, tx+(1-t)\hat{y}) \quad &+ \quad (1-t)\langle Ax, \hat{y}-x\rangle + (1-t)b(x,\hat{y}) \\ &+ \quad f(tx+(1-t)\hat{y}) - f(x) - \langle x^*,\ tx+(1-t)\hat{y}-x\rangle \\ &\geq \quad (1-t)b(x,x). \end{aligned} \tag{2.5}$$

Now, since

$$f(tx+(1-t)\hat{y}) - f(x) \geq \langle \nabla f(tx+(1-t)\hat{y}), tx+(1-t)\hat{y}-x\rangle,$$

then using (2.4) and $(A_4)$, we obtain

$$\begin{aligned} tg(x,x)+(1-t)g(x,\hat{y}) \quad &+ \quad (1-t)\langle Ax, \hat{y}-x\rangle + (1-t)b(x,\hat{y}) + \langle \nabla f(tx \\ &+ \quad (1-t)\hat{y}), tx+(1-t)\hat{y}-x\rangle - \langle x^*, tx+(1-t)\hat{y}-x\rangle \\ &\geq \quad (1-t)b(x,x). \end{aligned}$$

By $(A_1)$ and for all $\hat{y} \in C$, we have

$$\begin{gathered} (1-t)g(x,\hat{y}) + (1-t)\langle Ax, \hat{y}-x\rangle + (1-t)b(x,\hat{y}) \\ +\langle \nabla f(tx+(1-t)\hat{y}), (1-t)(\hat{y}-x)\rangle - \langle x^*, (1-t)(\hat{y}-x)\rangle \\ \geq (1-t)b(x,x), \end{gathered}$$

which implies that

$$\begin{aligned} (1-t)[g(x,\hat{y}) + \langle Ax, \hat{y}-x\rangle \quad &+ \quad b(x,\hat{y}) + \langle \nabla f(tx+(1-t)\hat{y}), \hat{y}-x\rangle - \langle x^*, \hat{y}-x\rangle] \\ &\geq \quad (1-t)b(x,x). \end{aligned}$$

Hence,

$$\begin{aligned} g(x,\hat{y}) + \langle Ax, \hat{y}-x\rangle \quad &+ \quad b(x,\hat{y}) + \langle \nabla f(tx+(1-t)\hat{y}), \hat{y}-x\rangle - \langle x^*, \hat{y}-x\rangle \\ &\geq \quad b(x,x), \end{aligned}$$

for all $\hat{y} \in C$. Since $f$ is Gateaux differentiable function, then $\nabla f$ is norm-to-weak$^*$ continuous. Therefore, allowing $t \to 1^{-1}$, we obtain

$$g(x,\hat{y}) + \langle Ax, \hat{y}-x\rangle + b(x,\hat{y}) + \langle \nabla f(x), \hat{y}-x\rangle - \langle x^*, \hat{y}-x\rangle \geq b(x,x), \forall x^* \in E^*.$$

Putting $x^* = \nabla f(\overline{x})$ such that for $x \in C$, we have

$$g(x,\hat{y}) + \langle Ax, \hat{y}-x\rangle + b(x,\hat{y}) + \langle \nabla f(x) - \nabla f(\overline{x}), \hat{y}-x\rangle \geq b(x,x),$$

for all $\hat{y} \in C$, i.e, $x \in Res^f_{g,b,A}(\overline{x})$. Hence $dom(Res^f_{g,b,A}) = E$. □

**Lemma 2.23.** *Let $C$ be a nonempty, closed and convex subset of a real reflexive Banach space $E$. Let $g : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying $(A_1) - (A_4)$ and bifunction $b : C \times C \longrightarrow \mathbb{R}$ satisfying $(B_1) - (B3)$. Let $A : C \longrightarrow E^*$ be a continuous monotone mapping, $f : E \longrightarrow (-\infty, +\infty]$ be a coercive Legendre function. Let $Res^f_{g,b,A} : E \longrightarrow 2^C$ be a mixed resolvent operator defined by (2.3). Then, the following properties holds:*

(a) *$Res^f_{g,b,A}$ is single-valued;*

(b) *$Res^f_{g,b,A}$ is Bregman firmly nonexpansive type mapping (BFNE);*

(c) $F(Res^f_{g,b,A}) = GMEP(g,b,A)$;
(d) $GMEP(g,b,A)$ *is closed and convex;*
(e) $D_f(p, Res^f_{g,b,A}(x)) + D_f(Res^f_{g,b,A}(x), x) \leq D_f(p,x),\ \forall p \in F(Res^f_{g,b,A}),\ x \in E$.

*Proof.* (a) Let $z_1, z_2 \in Res^f_{g,b,A}(x)$ then by definition of the mixed resolvent, we have

$$g(z_1,z_2) + \langle Az_1, z_2 - z_1\rangle + b(z_1,z_2) + \langle \nabla f(z_1) - \nabla f(x), z_2 - z_1\rangle \geq b(z_1,z_1)$$

and

$$g(z_2,z_1) + \langle Az_2, z_1 - z_2\rangle + b(z_2,z_1) + \langle \nabla f(z_2) - \nabla f(x), z_1 - z_2\rangle \geq b(z_2,z_2).$$

Adding the above two inequalities, we get

$$\begin{aligned} g(z_1,z_2) + g(z_2,z_1) \quad &+ \quad \langle Az_1, z_2 - z_1\rangle + \langle Az_2, z_1 - z_2 + b(z_1,z_2) + b(z_2,z_1) \\ &+ \quad \langle \nabla f(z_2) - \nabla f(z_1), z_1 - z_2\rangle \\ &\quad \geq b(z_1,z_1) + b(z_2,z_2), \end{aligned}$$

which implies

$$\begin{aligned} g(z_1,z_2) + g(z_2,z_1) + b(z_1,z_2) \quad &+ \quad b(z_2,z_1) - b(z_1,z_1) - b(z_2,z_2) \\ &+ \quad \langle \nabla f(z_2) - \nabla f(z_1), z_1 - z_2\rangle \geq 0 \end{aligned}$$

By $(A_2)$ and $(B_1)$, we obtain $\langle \nabla f(z_2) - \nabla f(z_1), z_2 - z_1\rangle \leq 0$. And As $f$ is convex and Gâteaux differentiabe we also have $\langle \nabla f(z_2) - \nabla f(z_1), z_2 - z_1\rangle \geq 0$. Hence $\langle \nabla f(z_2) - \nabla f(z_1), z_2 - z_1\rangle = 0$. Since $f$ is Legendre then it is strictly convex. So, $\nabla f$ is strictly monotone and therefore $z_1 = z_2$. Thus, $Res^f_{g,b,A}$ is single-valued.

(b) Let $x, y \in E$, then

$$\begin{aligned} g(Res^f_{g,b,A}(x), Res^f_{g,b,A}(y)) &+ \langle A(Res^f_{g,b,A})(x), Res^f_{g,b,A}(y) - Res^f_{g,b,A}(x)\rangle \\ &+ b(Res^f_{g,b,A}(x), Res^f_{g,b,A}(y)) + \langle \nabla f(Res^f_{g,b,A}(x)) \\ &- \nabla f(x), Res^f_{g,b,A}(y) - Res^f_{g,b,A}(x)\rangle \\ &\geq b(Res^f_{g,b,A}(x), Res^f_{g,b,A}(x)) \end{aligned}$$

and

$$\begin{aligned} g(Res^f_{g,b,A}(y), Res^f_{g,b,A}(x)) &+ \langle A(Res^f_{g,b,A})(y), Res^f_{g,b,A}(x) - Res^f_{g,b,A}(y)\rangle \\ &+ b(Res^f_{g,b,A}(y), Res^f_{g,b,A}(x)) + \langle \nabla f(Res^f_{g,b,A}(y)) \\ &- \nabla f(y), Res^f_{g,b,A}(x) - Res^f_{g,b,A}(y)\rangle \\ &\geq b(Res^f_{g,b,A}(y). Res^f_{g,b,A}(y)). \end{aligned}$$

By adding the above two inequalities , we get

$$\begin{aligned} g(Res^f_{g,b,A}(x), Res^f_{g,b,A}(y)) + g(Res^f_{g,b,A}(y), Res^f_{g,b,A}(x)) + b(Res^f_{g,b,A}(x), Res^f_{g,b,A}(y)) \\ + b(Res^f_{g,b,A}(y), Res^f_{g,b,A}(x)) - b(Res^f_{g,b,A}(x), Res^f_{g,b,A}(x)) \end{aligned}$$

$$+ b(Res^f_{g,b,A}(y), Res^f_{g,b,A}(y)) + \langle \nabla f(Res^f_{g,b,A}(x)) - \nabla f(x) + \nabla f(y) - \nabla f(Res^f_{g,b,A}(y)), Res^f_{g,b,A}(y) - Res^f_{g,b,A}(x)\rangle \geq 0.$$

Also, by $(A_2)$ and $(B_1)$, we obtain

$$\begin{aligned}&\langle \nabla f(Res^f_{g,b,A}(x)) - \nabla f(Res^f_{g,b,A}(y)), Res^f_{g,b,A}(x) - Res^f_{g,b,A}(y)\rangle \\ &\qquad \leq \langle \nabla f(x) - \nabla f(y), Res^f_{g,b,A}(x) - Res^f_{g,b,A}(y)\rangle\end{aligned}$$

Showing that, $Res^f_{g,b,A}$ is Bregman firmly nonexpansive mapping.

(c)

$$\begin{aligned} x \in F(Res^f_{g,b,A}) &\iff x = Res^f_{g,b,A}(x) \\ &\iff g(x,y) + \langle Ax, y-x\rangle + b(x,y) \\ &+ \quad \langle \nabla f(x) - \nabla f(x), y - x\rangle \geq b(x,x), \forall y \in C \\ &\iff g(x,y) + \langle Ax, y-x\rangle + b(x,y) - b(x,x) \geq 0, \forall y \in C \\ &\iff x \in GMEP(g,b,A).\end{aligned}$$

(d) Since $Res^f_{g,b,A}$ is a $BFNE$ mapping, it follows from Lemma 2.18 that $F(Res^f_{g,b,A})$ is a closed and convex subset of $C$, then using (c), we have $GMEP(g,b,A) = F(Res^f_{g,b,A})$ is a closed and convex subset of $C$.

(e) Since $Res^f_{g,b,A}$ is a $BFNE$ operator, using (2.2) and for all $x, y \in E$, we obtain

$$\begin{aligned} D_f(Res^f_{g,b,A}(x), Res^f_{g,b,A}(y)) &+ D_f(Res^f_{g,b,A}(y), Res^f_{g,b,A}(x)) \\ &\leq D_f(Res^f_{g,b,A}(x), y) - D_f(Res^f_{g,b,A}(x), x) \\ &+ D_f(Res^f_{g,b,A}(y), x) - D_f(Res^f_{g,b,A}(y), y).\end{aligned}$$

Let $y = p \in F(Res^f_{g,b,A})$, then we obtain

$$\begin{aligned} D_f(Res^f_{g,b,A}(x), p) &+ D_f(p, Res^f_{g,b,A}(x)) \\ &\leq D_f(Res^f_{g,b,A}(x), p) - D_f(Res^f_{g,b,A}(x), x) \\ &+ D_f(p, x) - D_f(p,p).\end{aligned}$$

Hence,

$$D_f(p, Res^f_{g,b,A}(x)) + D_f(Res^f_{g,b,A}(x), x) \leq D_f(p, x).$$

This completes the proof. □

## 3. STRONG CONVERGENCE THEOREM

In this section, we prove a strong convergence theorem for the inertial iterative algorithm to approximate a common solution of generalized mixed equilibrium problems and fixed points problem for Bregman total quasi - asymptotically nonexpansive multivalued mappings in Banach spaces

**Theorem 3.1.** *Let $E$ be a real uniformly smooth, uniformly convex and reflexive Banach space with dual $E^*$. Let $C$ be a nonempty closed and convex subset of $E$. Let $f : E \longrightarrow \mathbb{R}$ be a coercive legendre function which is bounded, uniformly Frechet differentiable and totally convex on bounded subset of $E$. Let $g : C \times C \longrightarrow \mathbb{R}$ be a bifunction which satisfying assumptions $(A_1) - (A_4)$, $b : C \times C \longrightarrow \mathbb{R}$ be a bi function which satisfying assumptions $(B_1) - (B_3)$ and $A : C \longrightarrow E^*$ be continuous monotone mapping. Let $T : E \longrightarrow \hat{C}B(E)$ be a closed Bregman total quasi-asymptotically nonexpansive multivalued mapping with nonnegative real sequences $\{\lambda_n\}, \{\mu_n\}$ satisfying $\lambda_n \longrightarrow 0$ and $\mu_n \longrightarrow 0$ ( as $n \longrightarrow \infty$) and strictly increasing and continuous function $\zeta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with $\zeta(0) = 0$. Assume that $T$ is uniformly $L$- Lipschitz continuous such that $\Omega := GMEP(g, b, A) \cap \cap F(T) \neq \emptyset$. Let $\{x_n\}$ and $\{z_n\}$ be sequences generated by the iterative schemes:*

$$
\begin{cases}
x_0, z_0 \in C = E \\
w_n = \nabla f^*(\nabla f(x_n) + \theta_n(\nabla f(x_n) - \nabla f(x_{n-1})), \\
\omega_n = \nabla f^*(\beta_n \nabla f(w_n) + (1-\beta_n)\nabla f(v_n)), \\
u_n = \nabla f^*(\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla h_n), \\
z_{n+1} = Res^f_{g,b,A} u_n, \\
C_n = \{z \in C : D_f(z, z_{n+1}) \leq \alpha_n D_f(z, z_n) \\
+(1-\alpha_n) D_f(z, w_n) + \xi_n\}, \\
Q_n = \{z \in C : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n\rangle \leq 0\}, \\
x_{n+1} = P^f_{C_n \cap Q_n} x_0, \ \forall n \geq 0,
\end{cases} \tag{3.1}
$$

*where $v_n \in T^n w_n$, $h_n \in T^n \omega_n$ $\{\alpha_n\}$, $\theta_n$ and $\{\beta_n\}$ are sequences in $[0, 1]$ satisfying the following conditions;*

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$;
(ii) $0 < \liminf\limits_{n\to\infty} \beta_n \leq \limsup\limits_{n\to\infty} \beta_n < 1$;

*$\xi_n = \lambda_n \zeta(D_f(p, w_n) + D_f(p, \omega_n)) + 2\mu_n$. Then, $\{x_n\}$ converges strongly to $P^f_\Omega x_0$, where $P^f_\Omega x_0$ is the Bregman projection of $C$ onto $\Omega$.*

*Proof.* Let two functions: $\tau : C \times C \longrightarrow \mathbb{R}$ and $Res^f_{g,b,A} : E \longrightarrow 2^C$ be defined by

$$\tau(z, y) = g(z, y) + \langle Az, y - z\rangle, \forall z, y \in C,$$

and

$$Res^f_{g,b,A}(x) = \{z \in C : \tau(z, y) + b(z, y) - b(z, z) + \langle \nabla f(z) - \nabla f(x), y - z\rangle \geq 0, \forall y \in C\}, \forall x \in E.$$

respectively. Then, the function $\tau$ satisfies assumptions $(A_1) - (A_4)$ and $Res^f_{g,b,A}$ has the properties $(a) - (e)$ of Lemma 2.23.

We divide the proof into the following:

*Step* 1 : We show that $\Omega$ is closed and convex. It follows from Lemma 2.15 and

Lemma 2.23 that $\Omega$ is a closed and convex set so that $p^f_{\Omega}x_0$ is well defined.

*Step* 2 : We show that $C_n \cap Q_n$ is closed and convex for all $n \geq 0$. In view of the definition of $Q_n$, it follows that $Q_n$ is closed and convex. Next, we show that $C_n$ is closed and convex for all $n \geq 0$. Let $a_1, a_2 \in C_n$, then $a_1, a_2 \in C$ and

$$D_f(a_1, z_{n+1}) \leq \alpha_n D_f(a_1, z_n) + (1-\alpha_n)D_f(a_1, w_n) + \xi_n$$

and

$$D_f(a_2, z_{n+1}) \leq \alpha_n D_f(a_2, z_n) + (1-\alpha_n)D_f(a_2, w_n) + \xi_n.$$

Recall that $D_f(y,x) = f(y) - f(x) - \langle \nabla f(x), y - x\rangle$. Now, using this definition the above two inequalities are equivalent to

$$\begin{aligned} &\alpha_n\langle \nabla f(z_n), a_1 - z_n\rangle + (1-\alpha_n)\langle \nabla f(w_n), a_1 - w_n\rangle - \langle \nabla f(z_{n+1}), a_1 - z_{n+1}\rangle \\ &\qquad \leq f(z_{n+1}) - f(z_n) - (1-\alpha_n)f(w_n) + \xi_n \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} &\alpha_n\langle \nabla f(z_n), a_2 - z_n\rangle + (1-\alpha_n)\langle \nabla f(w_n), a_2 - w_n\rangle - \langle \nabla f(z_{n+1}), a_2 - z_{n+1}\rangle \\ &\qquad \leq f(z_{n+1}) - f(z_n) - (1-\alpha_n)f(w_n) + \xi_n. \end{aligned} \tag{3.3}$$

Multiplying $t$ and $(1-t)$ on both sides of (3.2) and (3.3) respectively, we get

$$\begin{aligned} \alpha_n\langle \nabla f(z_n), ta_1 + (1-t)a_2 - z_n\rangle &+ (1-\alpha_n)\langle \nabla f(w_n), ta_1 + (1-t)a_2 - w_n\rangle \\ &- \langle \nabla f(z_{n+1}), ta_1 + (1-t)a_2 - z_{n+1}\rangle \\ &\leq f(z_{n+1}) - f(z_n) - (1-\alpha_n)f(w_n) + \xi_n. \end{aligned} \tag{3.4}$$

Therefore (3.4) yields

$$\begin{aligned} D_f(ta_1 + (1-t)a_2, z_{n+1}) &\leq \alpha_n D_f(ta_1 + (1-t)a_2, z_n) + (1-\alpha_n)D_f(ta_1 \\ &\quad +(1-t)a_2, w_n) + \xi_n. \end{aligned}$$

This implies $ta_1 + (1-t)a_2 \in C_n$ for $t \in [0,1]$ and hence $C_n$ is closed and convex for all $n \geq 0$ and thus consequently $C_n \cap Q_n$ is closed and convex for all $n \geq 0$. This shows that the iterative scheme (3.1) is well defined.

*Step* 3 : We show that $\Omega \subset C_n \cap Q_n, \forall\ n \geq 0$. For any $p \in \Omega$ and from (3.1),and definition 2.8(4), we have

$$\begin{aligned} D_f(p, z_{n+1}) &= D_f(p, Res^f_{g,b,A}u_n) \\ &\leq D_f(p, u_n) \\ &= D_f(p, \nabla f^*[\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(h_n)]) \\ &\leq \alpha_n D_f(p, z_n) + (1-\alpha_n)D_f(p, h_n). \end{aligned} \tag{3.5}$$

Since $h_n \in T^n(\omega_n)$ and $T$ is Bregman total quasi asymptotically nonexpansive multivalued mappings, we have

$$\begin{aligned} D_f(p, z_{n+1}) &\leq \alpha_n D_f(p, z_n) + (1-\alpha_n)(D_f(p,\omega_n) + \lambda_n\zeta(D_f(p,\omega_n)) + \mu_n) \\ &= \alpha_n D_f(p, z_n) + (1-\alpha_n)D_f(p,\omega_n) + (1-\alpha_n)\lambda_n\zeta(D_f(p,\omega_n)) \\ &\quad + (1-\alpha_n)\mu_n. \end{aligned}$$

But

$$\begin{aligned} D_f(p,\omega_n) &= D_f(p,\nabla f^*(\beta_n\nabla f(w_n)+(1-\beta_n)\nabla f(v_n))) \\ &\leq \beta_n D_f(p,w_n)+(1-\beta_n)D_f(p,v_n). \end{aligned}$$

Again $T$ is Bregman total quasi asymptotically nonexpansive mapping and $v_n \in T^n w_n$, we have

$$\begin{aligned} D_f(p,\omega_n) &\leq \beta_n D_f(p,w_n)+(1-\beta_n)[D_f(p,w_n)+\lambda_n\zeta(D_f(p,w_n))+\mu_n] \\ &= D_f(p,w_n)+(1-\beta_n)\lambda_n\zeta(D_f(p,w_n))+(1-\beta_n)\mu_n. \end{aligned} \tag{3.6}$$

Therefore

$$\begin{aligned} & D_f(p,z_{n+1}) \leq \alpha_n D_f(p,z_n) \\ &+ (1-\alpha_n)[D_f(p,w_n)+(1-\beta_n)\lambda_n\zeta(D_f(p,w_n))+(1-\beta_n)\mu_n] \\ &+ (1-\alpha_n)\lambda_n\zeta(D_f(p,\omega_n))+(1-\alpha_n)\mu_n \\ &\leq \alpha_n D_f(p,z_n)+(1-\alpha_n)D_f(p,w_n)+\lambda_n\zeta(D_f(p,w_n)+D_f(p,\omega_n))+2\mu_n \\ &= \alpha_n D_f(p,z_n)+(1-\alpha_n)D_f(p,w_n)+\xi_n. \end{aligned}$$

This implies that $p \in C_n$. Hence, $\Omega \subset C_n, \forall n \geq 0$. Next, we show by induction that $\Omega \subset C_n \cap Q_n, \forall n \geq 0$. Since $Q_0 = C$, we get $\Omega \subset C_0 \cap Q_0$. Suppose that $\Omega \subset C_k \cap Q_k$, for some $k > 0$. Then, we have $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = p^f_{C_k\cap Q_k} x_0$, for some $k \in \mathbb{N}$. Using the definition $x_{k+1}$, we obtain, for all $z \in C_k \cap Q_k$, that $\langle \nabla f(x_0) - \nabla f(x_{k+1}), x_{k+1} - z\rangle \geq 0$. Since $\Omega \subset C_k \cap Q_k$, we get

$$\langle \nabla f(x_0) - \nabla f(x_{k+1}), p - x_{k+1}\rangle \leq 0,\ \forall p \in \Omega$$

and then, $p \in Q_{k+1}$. Hence, we obtain $\Omega \subset C_{k+1} \cap Q_{k+1}$, since $\Omega \subset C_n, \forall\ n \geq 0$. Thus, we get $\Omega \subset C_n \cap Q_n, \forall\ n \geq 0$.

*Step* 4 : We show that $\{x_n\}$ converges strongly to some point $\hat{x}$. Since $x_n = p^f_{Q_n} x_0$, using Lemma 2.12(i), we have

$$\langle \nabla f(x_0) - \nabla f(x_n), z - x_n\rangle \leq 0, \forall z \in C.$$

Since $\Omega \subset Q_n$, we obtain

$$\langle \nabla f(x_0) - \nabla f(x_n), u - x_n\rangle \leq 0, \forall u \in \Omega.$$

It follows from Lemma 2.12(i) that, for each $u \in \Omega$, we get

$$\begin{aligned} D_f(x_n,x_0) &= D_f(p^f_{Q_n}x_0,x_0) \\ &\leq D_f(u,x_0) - D_f(u,p^f_{Q_n}x_0) \\ &\leq D_f(u,x_0), \forall\ u \in \Omega \subset Q_n. \end{aligned}$$

This implies that $\{D_f(x_n,x_0)\}$ is bounded, and by Lemma 2.14, the sequence $\{x_n\}$ is bounded. Now, since $x_{n+1} = p^f_{C_n\cap Q_n}x_0 \in Q_n$ and $x_n = p^f_{Q_n}x_0$, we obtain

$$D_f(x_n,x_0) \leq D_f(x_{n+1},x_0), \forall n \geq 0.$$

This implies that $\{D_f(x_n,x_0)\}$ is nondecreasing. Hence, $\lim_{n\to\infty} D_f(x_n,x_0)$ exists. Furthermore

$$D_f(x_{n+1},x_n) = D_f(x_{n+1},p^f_{Q_n}x_0)$$

$$\begin{aligned} &\le D_f(x_{n+1}, x_0) - D_f(p^f_{Q_n} x_0, x_0) \\ &= D_f(x_{n+1}, x_0) - D_f(x_n, x_0) \end{aligned}$$

which implies that

$$\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0 \tag{3.7}$$

Since $f$ is totally convex on bounded sets, it follows from Lemma 2.5 that $f$ is sequentially consistent and so, we have

$$\lim_{n\to\infty} \| x_{n+1} - x_n \| = 0 \tag{3.8}$$

Also, since $f$ is uniformly Frenchet differentiable, it follows from Lemma 2.3 that $\nabla f$ is uniformly norm-to- norm continuous on bounded subset of $E$. Therefore, we have

$$\lim_{n\to\infty} \| \nabla f(x_{n+1}) - \nabla f(x_n) \| = 0 \tag{3.9}$$

By the construction of $\{Q_n\}$, for any $m \ge n$, we have $Q_m \subset Q_n$ and $x_m = p^f_{Q_m} x_0 \in Q_n$. This shows that

$$D_f(x_m, x_n) = D_f(x_m, p^f_{Q_n} x_0) \le D_f(x_m, x_0) - D_f(x_n, x_0) \longrightarrow 0, \ (as \ n \to \infty).$$

It follows from Lemma 2.5 that, $\lim_{n\to\infty} \| x_m - x_n \| = 0$. Hence $\{x_n\}$ is a Cauchy sequence in $C$. Since $C$ is complete, without loss of generality we can assume that

$$\lim_{n\to\infty} x_n = \hat{x} \ (\textit{some point in } C). \tag{3.10}$$

*Step* 5 : We show that $\hat{x} \in \Omega$, First, we show that $\hat{x} \in F(T)$. Since $\{x_n\}$ is bounded and $f$ is uniformly Frechet differentiable and bounded on bounded subset of $E$, then by Lemma 2.3 $\nabla f$ is uniformly continuous and therefore bounded. Moreover, the function $f$ is strongly coercive and totally convex, then by Lemma 2.17 $\nabla f^*$ is uniformly continuous and consequently bounded. Therefore in view of this and definition of $w_n$, we have $\{w_n\}$ is bounded. Also since $v_n \in T^n w_n$ and $T$ is Bregman total quasi asymptotically nonexpansive, it follows from Lemma 2.16 that $\{v_n\}$ is bounded. Therefore from the definition of $\xi_n$, we have

$$\lim_{n\to\infty} \xi_n = \lim_{n\to\infty} (\lambda_n \zeta(D_f(p, w_n) + D_f(p, \omega_n)) + 2\mu_n) = 0. \tag{3.11}$$

Also, from the definition of $w_n$, we obtain

$$\nabla f(w_n) - \nabla f(x_n) = \theta_n(\nabla f(x_n) - \nabla f(x_{n-1})).$$

This implies that

$$\begin{aligned} \lim_{n\to\infty} \| \nabla f(w_n) - \nabla f(x_n) \| &= \lim_{n\to\infty} \| \theta_n(\nabla f(x_n) - \nabla f(x_{n-1})) \| \\ &\le \lim_{n\to\infty} \| \nabla f(x_n) - \nabla f(x_{n-1}) \| . \end{aligned}$$

Now, by using (3.9) we get

$$\lim_{n\to\infty} \| \nabla f(w_n) - \nabla f(x_n) \| = 0.$$

Since $\nabla f$ is norm uniformly continuous on bounded subset of $E^*$, we have

$$\lim_{n\to\infty} \| w_n - x_n \| = 0. \tag{3.12}$$

Since $\{w_n\}$ is bounded, using Lemma 2.6, we get

$$\lim_{n\to\infty} D_f(w_n, x_n) = 0. \tag{3.13}$$

Using (3.10) and (3.12), we obtain

$$\lim_{n\to\infty} w_n = \hat{x}. \tag{3.14}$$

Using (3.8) and (3.12), we have

$$\lim_{n\to\infty} \| x_{n+1} - w_n \| = 0. \tag{3.15}$$

It follows from Lemma 2.6 that

$$\lim_{n\to\infty} D_f(x_{n+1}, w_n) = 0. \tag{3.16}$$

Setting $M = max\{D_f(p, z_0), sup_n D_f(p, w_n)\}$. Then obviously $D_f(p, z_0) \leq M$. Let $D_f(p, z_n) \leq M$ for some $n$, then it follows from (3.7) that

$$\begin{aligned} D_f(p, z_{n+1}) &\leq \alpha_n M + (1-\alpha_n)M + \xi_n \\ &\leq M + \xi_n \end{aligned}$$

Thus, $\{D_f(p, z_{n+1})\}$ is bounded which implies that $\{z_n\}$ is bounded.
From the three point identity of the Bregman distance, we have

$$D_f(x_{n+1}, z_n) = \langle \nabla f(z_n) - \nabla f(x_{n+1}), p - x_{n+1} \rangle + D_f(p, z_n) - D_f(p, x_{n+1}).$$

Since $f$ is bounded on bounded subsets of $E^*$, then $\nabla f$ is bounded on bounded subsets of $E^*$ and hence it follows from boundedness of $\{x_n\}$ and $\{z_n\}$ that the sequences $\{\nabla f(x_n)\}$ and $\{\nabla f(z_n)\}$ are bounded in $E^*$, which implies that $\{D_f(x_{n+1}, z_n)\}$ is bounded. Now, since $x_{n+1} = p^f_{C_n \cap Q_n} x_0 \in C_n$, we get

$$\begin{aligned} D_f(x_{n+1}, z_{n+1}) &\leq \alpha_n D_f(x_{n+1}.z_n) \\ &\quad +(1-\alpha_n) D_f(x_{n+1}, w_n) + \xi_n \end{aligned} \tag{3.17}$$

From (3.16), $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \xi_n = 0$ in (3.17), we obtain

$$\lim_{n\to\infty} D_f(x_{n+1}, z_{n+1}) = 0.$$

Since $f$ is totally convex on bounded sets, it follows from Lemma 2.6 that $f$ is sequentially consistent and so, we have

$$\lim_{n\to\infty} \| x_{n+1} - z_{n+1} \| = 0. \tag{3.18}$$

Now,

$$\| x_n - z_{n+1} \| \leq \| x_n - x_{n+1} \| + \| x_{n+1} - z_{n+1} \| .$$

Using (3.8) and (3.18), we get

$$\lim_{n\to\infty} \| x_n - z_{n+1} \| = 0, \tag{3.19}$$

which implies that $z_{n+1} \longrightarrow \hat{x} \in C$, since $x_n \longrightarrow \hat{x} \in C$.
On the other hand, we obtain

$$\begin{aligned} D_f(x_{n+1}, u_n) &\leq \alpha_n D_f(x_{n+1}.z_n) \\ &\quad +(1-\alpha_n) D_f(x_{n+1}, w_n) + \xi_n. \end{aligned} \tag{3.20}$$

Putting $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \xi_n = 0$ in (3.20), we obtain

$$D_f(x_{n+1}, u_n) = 0.$$

Since $f$ is totally convex on bounded sets, it follows from Lemma 2.6 that $f$ is sequentially consistent and so, we have

$$\lim_{n\to\infty} \| x_{n+1} - u_n \| = 0. \tag{3.21}$$

Then,

$$\| x_n - u_n \| \leq \| x_n - x_{n+1} \| + \| x_{n+1} - u_n \| .$$

It follows from (3.8) and (3.21), that

$$\lim_{n\to\infty} \| x_n - u_n \| = 0, \tag{3.22}$$

which implies that $u_n \longrightarrow \hat{x} \in C$, since $x_n. \longrightarrow \hat{x} \in C$.
Using (3.18) and (3.21), we obtain

$$\lim_{n\to\infty} \| z_{n+1} - u_n \| = 0. \tag{3.23}$$

Since $f$ is uniformly Frechet differentiable, it follows from Lemma 2.3 that

$$\lim_{n\to\infty} \| \nabla f(z_{n+1}) - \nabla f(u_n) \| = 0. \tag{3.24}$$

Since $\omega_n = \nabla f^*(\beta_n \nabla f(w_n) + (1-\beta_n)\nabla f(v_n)$, $\{w_n\}$ and $\{v_n\}$ are bounded, it follows from Lemma 2.3 and Lemma 2.17 that $\{\omega_n\}$ is bounded. Furthermore, since $h_n \in T^n \omega_n$ We have $h_n$ is bounded. Also from the scheme (3.1), we have

$$\nabla f(u_n) = \alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(h_n).$$

Therefore

$$\nabla f(u_n) - \nabla f(h_n) = \alpha_n(\nabla f(z_n) - \nabla f(h_n)). \tag{3.25}$$

In view of the assumption that $\alpha_n \longrightarrow 0$, ( $as\ n \to \infty$), we have

$$\lim_{n\to\infty} \| \nabla f(u_n) - \nabla f(h_n) \| = \lim_{n\to\infty} \alpha_n \| \nabla f(z_n) - \nabla f(h_n) \| = 0 \tag{3.26}$$

and consequently

$$\lim_{n\to\infty} \| u_n - h_n \| = 0. \tag{3.27}$$

Furthermore, using (3.24) and(3.26), we get

$$\lim_{n\to\infty} \| \nabla f(z_{n+1}) - \nabla f(h_n) \| = 0. \tag{3.28}$$

Since $\nabla f^*$ is uniformly continuous on each bounded subset of $E^*$, we obtain

$$\lim_{n\to\infty} \| z_{n+1} - h_n \| = 0. \tag{3.29}$$

Now, using the four point identity, we have

$$D_f(w_n, u_n) + D_f(w_n, v_n) - D_f(x_n, u_n) + D_f(x_n, v_n) = \langle \nabla f(v_n) - \nabla f(u_n), w_n - x_n \rangle.$$

Therefore,

$$
\begin{aligned}
D_f(w_n, v_n) &= \langle \nabla f(v_n) - \nabla f(u_n), w_n - x_n \rangle + D_f(x_n, u_n) - D_f(w_n, u_n) \\
&\quad - D_f(x_n, v_n) \\
&\leq \langle \nabla f(v_n) - \nabla f(u_n), w_n - x_n \rangle + D_f(x_n, u_n).
\end{aligned} \tag{3.30}
$$

From (3.12), (3.22) and Lemma 2.6, we obtain

$$\lim_{n\to\infty} D_f(w_n, v_n) = 0. \tag{3.31}$$

Again Lemma 2.6, we obtain

$$\lim_{n\to\infty} \|w_n - v_n\| = 0. \tag{3.32}$$

We also have from the scheme (3.1)

$$
\begin{aligned}
D_f(w_n, \omega_n) &= D_f(w_n, \nabla f^*(\beta_n \nabla f(w_n) + (1-\beta_n)\nabla f(v_n))) \\
&\leq \beta_n D_f(w_n, w_n) + (1-\beta_n) D_f(w_n, v_n).
\end{aligned}
$$

It follows from (3.31) that

$$\lim_{n\to\infty} D_f(w_n, \omega_n) = 0$$

and consequently

$$\lim_{n\to\infty} \|w_n - \omega_n\| = 0. \tag{3.33}$$

Observe that

$$\|w_n - h_n\| \leq \|w_n - x_n\| + \|x_n - u_n\| + \|u_n - h_n\|.$$

It follows from (3.12), (3.22) and (3.27) that

$$\lim_{n\to\infty} \|w_n - h_n\| = 0. \tag{3.34}$$

Using (3.33) and (3.34), we get

$$\lim_{n\to\infty} \|\omega_n - h_n\| = 0. \tag{3.35}$$

Hence from (3.29) and (3.35), we obtain

$$\lim_{n\to\infty} \|z_{n+1} - \omega_n\| = 0. \tag{3.36}$$

Since $z_{n+1} \longrightarrow \hat{x}$,then from (3.36), we obtain

$$\lim_{n\to\infty} \omega_n = \hat{x}. \tag{3.37}$$

From (3.37) and norm-to-norm uniform continuity of $\nabla f$ on bounded subset of $E$ we have

$$
\begin{aligned}
0 &= \lim_{n\to\infty} \| \nabla f(\omega_n) - \nabla f(\hat{x}) \| \\
&= \lim_{n\to\infty} \| \beta_n \nabla f(w_n) + (1-\beta_n)\nabla f(v_n) - \nabla f(\hat{x}) \| \\
&= \lim_{n\to\infty} \| \beta_n(\nabla f(w_n) - \nabla f(\hat{x})) + (1-\beta_n)(\nabla f(v_n) - \nabla f(\hat{x})) \| .
\end{aligned}
$$

Using (3.14), we have

$$0 = \lim_{n\to\infty} (1-\beta_n) \| \nabla f(v_n) - \nabla f(\hat{x}) \| .$$

By condition (ii), we obtain

$$\lim_{n\to\infty} \| \nabla f(v_n) - \nabla f(\hat{x}) \| = 0.$$

Also, since $\nabla f^*$ is uniformly continuous on each bounded subset of $E^*$, we get

$$\lim_{n\to\infty} v_n = \hat{x}. \tag{3.38}$$

Now, by the assumption that $T : C \longrightarrow \hat{C}B(C)$ is uniformly $L-$ Lipschitz continuous, we have the following estimate;

$$\begin{aligned} H(T^{n+1}w_n, T^n w_n) &\leq H(T^{n+1}w_n, T^{n+1}w_{n+1}) + d(T^{n+1}w_{n+1}, w_{n+1}) \\ &+ d(w_{n+1}, w_n) + d(w_n, T^n w_n) \\ &\leq (L+1)d(w_{n+1}, w_n) + d(T^{n+1}w_{n+1}, w_{n+1}) + d(w_n, T^n w_n). \end{aligned}$$

We obtain $\lim_{n\to\infty} H(T^{n+1}w_n, T^n w_n) = 0$. Since $\lim_{n\to\infty} v_n = \hat{x}$ and $\lim_{n\to\infty} w_n = \hat{x}$, we get

$$\lim_{n\to\infty} d(TT^n w_n, \hat{x}) = 0.$$

In view of the closedness of $T$, it yield $d(T\hat{x}, \hat{x}) = 0$. Since $\hat{x} \in C,\ \hat{x} \in T\hat{x}$, hence

$$\hat{x} \in F(T).$$

Next, we show that $\hat{x} \in GMEP(g, b, A)$. Since $z_{n+1} = Res^f_{g,b,A} u_n$, we have

$$\tau(z_{n_{k+1}}, y) + \langle \nabla f(z_{n_{k+1}}) - \nabla f(u_{n_k}), y - z_{n_{k+1}} \rangle + b(y, z_{n_{k+1}}) - b(z_{n_{k+1}}, z_{n_{k+1}}) \geq 0,$$

for all $y \in C$. Using Assumption 2, we obtain

$$\begin{aligned} \langle \nabla f(z_{n_{k+1}}) - \nabla f(u_{n_k}), y - z_{n_{k+1}} \rangle &\geq \tau(y, z_{n_{k+1}}) - b(y, z_{n_{k+1}}) \\ &+ b(z_{n_{k+1}}, z_{n_{k+1}}) \geq 0, \forall y \in C. \end{aligned} \tag{3.39}$$

Now, since $\tau$ is lower semicontinuous in the second argument, $b$ is continuous and using (3.24) and taking $k \to \infty$ in (3.39), we get

$$0 \geq \tau(y, \hat{x}) - b(y, \hat{x}) + b(\hat{x}, \hat{x}).$$

Setting $y_t := ty + (1-t)\hat{x}, \forall t \in (0, 1]$ and $y \in C$. Then, we obtain $y_t \in C$ and hence

$$\tau(y_t, \hat{x}) - b(y_t, \hat{x}) + b(\hat{x}, \hat{x}) \leq 0.$$

Therefore,

$$\begin{aligned} 0 &= \tau(y_t, y_t) \\ &\leq t\tau(y_t, y) + (1-t)\tau(y_t, \hat{x}) \\ &\leq t\tau(y_t, y) + (1-t)[b(y_t, \hat{x}) - b(\hat{x}, \hat{x})] \\ &\leq t\tau(y_t, y) + (1-t)[b(y, \hat{x}) - b(\hat{x}, \hat{x})]. \end{aligned}$$

Letting $t > 0$, we have from $(A_3)$ that

$$\tau(\hat{x}, y) + b(y, \hat{x}) - b(\hat{x}, \hat{x}) \geq 0, \forall y \in C.$$

Hence,

$$\hat{x} \in GMEP(g, b, A).\ Thus,\ \hat{x} \in \Omega.$$

*Step* 6 : We show that $\hat{x} = Proj^f_\Omega x_0$. Let $\hat{u} = Proj^f_\Omega x_0$, since $\hat{x} \in \Omega$ then

$$D_f(\hat{u}, x_0) \leq D_f(\hat{x}, x_0) \tag{3.40}$$

Since $x_n = Proj^f_{Q_n} x_0$, then

$$D_f(x_n, x_0) \leq D_f(\hat{u}, x_0).$$

Since $x_n \longrightarrow \hat{x}$, we obtain

$$D_f(\hat{x}, x_0) \leq D_f(\hat{u}, x_0). \tag{3.41}$$

By (3.40) and (3.41) we get

$$D_f(\hat{x}, x_0) = D_f(\hat{u}, x_0).$$

Hence $\hat{x} = \hat{u} = Proj^f_\Omega x_0$.
This completes the proof. □

If $A = 0$, then the generalized mixed equilibrium probem reduces to the equilibrium problem. By remark 2.11(iii), Theorem 3.1 reduces to the following Corollary

**Corollary 3.2.** *Let $E$ be a real uniformly smooth, uniformly convex and reflexive Banach space with dual $E^*$. Let $C$ be a nonempty closed and convex subset of $E$. Let $f : E \longrightarrow \mathbb{R}$ be a coercive legendre function which is bounded, uniformly Frechet differentiable and totally convex on bounded subset of $E$. Let $g : C \times C \longrightarrow \mathbb{R}$ be a bifunction which satisfying assumptions $(A_1)-(A_4)$, $b : C\times C \longrightarrow \mathbb{R}$ be a bi function which satisfying assumptions $(B_1) - (B_3)$. Let $T : E \longrightarrow \hat{C}B(E)$ be Bregman relatively nonexpansive multivalued mapping such that $\Omega := GEP(g, b)\cap\cap F(T) \neq \emptyset$. Let $\{x_n\}$ and $\{z_n\}$ be sequences generated by the iterative schemes:*

$$\begin{cases} x_0, z_0 \in C = E \\ w_n = \nabla f^*(\nabla f(x_n) + \theta_n(\nabla f(x_n) - \nabla f(x_{n-1})), \\ \omega_n = \nabla f^*(\beta_n \nabla f(w_n) + (1-\beta_n)\nabla f(v_n)), \\ u_n = \nabla f^*(\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla h_n), \\ z_{n+1} = Res^f_{g,b} u_n, \\ C_n = \{z \in C : D_f(z, z_{n+1}) \leq \alpha_n D_f(z, z_n) \\ +(1-\alpha_n) D_f(z, w_n)\}, \\ Q_n = \{z \in C : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n\rangle \leq 0\}, \\ x_{n+1} = P^f_{C_n \cap Q_n} x_0, \ \forall n \geq 0, \end{cases} \tag{3.42}$$

*where $v_n \in Tw_n$, $h_n \in T\omega_n$ $\{\alpha_n\}$, $\theta_n$ and $\{\beta_n\}$ are sequences in $[0, 1]$ satisfying the following conditions;*

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

*Then, $\{x_n\}$ converges strongly to $P^f_\Omega x_0$, where $P^f_\Omega x_0$ is the Bregman projection of $C$ onto $\Omega$.*

Observe from (2.1) if $E$ is smooth and strictly convex space and $f : E \longrightarrow (-\infty, +\infty]$ defined by $f(x) = \|x\|^2$, $x \in E$, then it is easy to see that $\nabla f(x) = 2J(x)$ and $D_f(x, y) = \phi(x, y)$, $\forall x, y \in E$, where $\phi$ is the Lyapunov functional introduced in [2]. Hence the Bregman projection reduces to generalized projection $\Pi$ defined by $\phi(\Pi_C(x), x) = \inf\{\phi(y, x) : y \in C\}$. In view of this, Theorem 3.1 reduces to the following corollary;

**Corollary 3.3.** *Let $E$ be a real uniformly smooth, uniformly convex and reflexive Banach space with dual $E^*$. Let $C$ be a nonempty closed and convex subset of $E$. Let $g : C \times C \longrightarrow \mathbb{R}$ be a bi function which satisfying assumptions $(A_1) - (A_4)$, $b : C \times C \longrightarrow \mathbb{R}$ be a bi function which satisfying assumptions $(B_1) - (B_3)$ and $A : C \longrightarrow E^*$ be continuous monotone mapping. Let $T : E \longrightarrow \hat{C}B(E)$ be a total quasi- $\phi$- asymptotically nonexpansive multivalued mapping with nonnegative real sequences $\{\lambda_n\}, \{\mu_n\}$ satisfying $\lambda_n \longrightarrow 0$ and $\mu_n \longrightarrow 0$ ( as $n \longrightarrow \infty$) and strictly increasing and continuous function $\zeta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with $\zeta(0) = 0$. Assume that $T$ is uniformly $L$- Lipschitz continuous such that $\Omega := GMEP(g, b, A) \cap \cap F(T) \neq \emptyset$. Let $\{x_n\}$ and $\{z_n\}$ be a sequences generated by the iterative schemes:*

$$\begin{cases} x_0, z_0 \in C = E \\ w_n = J^{-1}(J(x_n) + \theta_n(J(x_n) - J(x_{n-1})), \\ \omega_n = J^{-1}(\beta_n J(w_n) + (1-\beta_n)J(v_n)), \\ u_n = J^{-1}(\alpha_n J(z_n) + (1-\alpha_n J(h_n), \\ z_{n+1} = Res_{g,b,A} u_n, \\ C_n = \{z \in C : \phi(z, z_{n+1}) \leq \alpha_n \phi(z, z_n) \\ +(1-\alpha_n)\phi(z, w_n) + \xi_n\}, \\ Q_n = \{z \in C : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n \rangle \leq 0\}, \\ x_{n+1} = P^f_{C_n \cap Q_n} x_0, \ \forall n \geq 0, \end{cases} \tag{3.43}$$

*where $J$ is the normalised duality mapping on $E$, $\{\alpha_n\}$ is a sequence in $[0,1]$ and $\{\beta_n\}$ is a sequence in $(0,1)$ satisfying the following conditions;*

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

*$\xi_n = \lambda_n \zeta((\phi(p, w_n) + \phi(p, \omega_n)) + 2\mu_n$, $\theta_n \in (0,1)$ and $v_n \in T^n w_n$, $h_n \in T^n \omega_n$. Then, $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$, where $\Pi_\Omega x_0$ is the generalized projection of $C$ onto $\Omega$.*

**Remark 3.4.** Theorem 3.1 improves and extends some recent results in the following sense:

(1) Improves the results of Li and Liu [27], since our results involved fixed point and generalized mixed equilibrium problems as against only fixed point problem studied in [27]. Furthermore, our scheme incoorperates inertial term that speed up the convergence rate of iterative sequence ;

(2) Extends the work of Kazmi, Ali and Yousuf [26] from Bregman relatively nonexpansive single valued mapping to Bregman total quasi - asymptotically nonexpansive;

(3) Extends the structure of the Banach space in Alansari et al. [1] from duality mapping to more general case: That is a Legendre, strongly coercive, uniformly Fréchet differentiable and totally convex function.

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*Manuscript received June 30 2022*
*revised June 30 2022*

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