Applied Analysis and Optimization
Volume 6, Number 1, 2022, 95–107

Yokohama Publishers
ISSN 2189-1664 Online Journal

ON A NONLINEAR CONSTRAINED PROBLEM OF A NONLINEAR FUNCTIONAL INTEGRAL EQUATION

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ABSTRACT. In this article, we study a nonlinear constrained problem of a nonlinear functional integral equation constrained with a nonlinear positive constraint of functional equation with parameters in the class of Lebegue integrable functions $L_1[0, T]$.

1. Introduction

It is well-known that a lot of problems investigated in engineering, mechanics, mathematical physics, vehicular traffic theory, queuing theory and also several real world problems can be described with help of various functional integral equations. The theory of functional integral equations is highly developed and constitutes a significant and important branch of nonlinear analysis. It is also known control theory in control systems engineering deals with the control of continuously operating dynamical systems in engineered processes and machines. There have been published, up to now, numerous research papers; see [6–10, 12, 14–16].

In this paper, we are concerned with the nonlinear positive constrained problem of the nonlinear functional integral equation

$$x(t) = f_1(t, \int_0^t f_2(t, s, x(s), u(s))ds), \quad t \in [0, T], \tag{1.1}$$

under the positive constraint u satisfies the (constraint) functional equation with parameter λ

$$u(t) = g(t, \lambda u(t)), \quad t \in [0, T]. \tag{1.2}$$

The existence of at least and exactly one solution $x \in L_1[0, T]$ under certain conditions will be proved. The continuous dependence of the solution $x \in L_1[0, T]$ on the set of solutions u of the (constraint) functional equation (1.2) and on the parameter λ and the functional g will be studied.

2. Existence of solution

Consider the constrained problem of the nonlinear functional integral equation (1.1) and the nonlinear positive constraint (1.2) with the following assumptions:

(1) $f_1 : [0, T] \times \mathbb{R} \to \mathbb{R}$ satisfies Caratheodory condition i.e., measurable in $t \in [0, T]$ for all $x \in R$ and continuous in $x \in R$ for all $t \in [0, T]$. There

2020 *Mathematics Subject Classification.* 34H05, 49J15, 34A12, 34k20, 34k25.

Key words and phrases. Nonlinear constraint; nonlinear functional integral equations, solutions in $L_1[0, T]$, continuous dependence.

exist an integrable function $a_1(t) \in L^1[0,T]$ and a positive constant $b_1 > 0$, such that

$$|f_1(t,x)| \leq |a_1(t)| + b_1|x|.$$

(2) $f_2 : [0,T] \times [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies Carathéodory condition i.e., measurable in $t, s \in [0,T]$ for all $x \in R$ and continuous in $x \in R$ for all $t, s \in [0,T]$. There exist an integrable function $k(t,s) \in L^1[0,T]$ and a positive constant $b_2 > 0$, such that

$$|f_2(t,s,x,u)| \leq |k(t,s)| + b_2(|x| + |u|).$$

(3)

$$\int_0^T \int_0^t k(t,s)dsdt \leq M.$$

(4) $g : [0,T] \times \mathbb{R} \to \mathbb{R}_+$ satisfies Carathéodory condition. There exist a measurable function $a_2(t) \in L^1[0,T]$ and a positive constant $b_3 > 0$, such that

$$g(t,u) \leq |a_2(t)| + b_3(|u|).$$

(5) $b_1 b_2 T < 1$, $b_3|\lambda| < 1$.

Definition 2.1. By a solution of the functional integral equation (1.1) we mean a function $x \in L_1[0,T]$ that satisfies (1.1).

Theorem 2.2. *Let the assumptions* 1–5 *be satisfied, then the functional integral equation* (1.1)*, has at least one solution $x \in L_1[0,T]$ depending on the existence of at least one solution $u \in L_1[0,T]$ of the functional equation* (1.2).

Proof. Let the operators A_1, A_2 associated with the functional equation (1.2) and the functional integral equation (1.1) respectively by

$$\begin{aligned} A_1u(t) &= g(t, \lambda u(t)), \\ A_2x(t) &= f_1(t, \int_0^t f_2(t,s,x(s),u(s))ds). \end{aligned}$$

Let $Q_r = \{u \in L_1[0,T] : ||u||_{L_1} \leq r\}$, where $r = \frac{\|a_2\|_{L_1}}{1-b_3|\lambda|}$.

Then we have, for $u \in Q_r$

$$\begin{aligned} |A_1u(t)| &\leq |g(t, \lambda u(t))| \\ &\leq a_2(t) + b_3|\lambda u(t)| \end{aligned}$$

Integrating the above inequality from 0 to T and making the change of variable we have

$$\begin{aligned} \int_0^T |A_1u(t)|dt &\leq \int_0^T (a_2(t) + b_3|\lambda u(t)|)dt \\ &\leq \int_0^T |a_2(t)|dt + b_3|\lambda| \int_0^T |u(t)|dt. \end{aligned}$$

Hence

$$\begin{aligned} \|A_1u\|_{L_1} &\leq \|a_2\|_{L_1} + b_3|\lambda|\|u\|_{L_1} \\ &\leq \|a_2\|_{L_1} + b_3|\lambda|r = r. \end{aligned}$$

Therefor $A_1 : Q_r \to Q_r$ and the class of functions $\{A_1 u\}$ is uniformly bounded in Q_r.

Let Ω be bounded subset of Q_r, then $A_1(\Omega)$ is also bounded on Q_r.

Let $u \in \Omega$, then

$$\begin{aligned}\|(A_1u)_h - A_1u\|_{L_1} &= \int_0^T |(A_1u)_h(t) - (A_1u)(t)|dt \\ &= \int_0^T |\frac{1}{h}\int_t^{t+h} |(A_1u)(s)ds - (A_1u)(t)|dt \\ &\leq \int_0^T \frac{1}{h}\int_t^{t+h} |g(s,\lambda u(s)) - g(t,\lambda u(t))|dsdt\end{aligned}$$

since $f \in L_1[0,T]$, It follows that

$$\frac{1}{h}\int_t^{t+h} |g(s,\lambda u(s)) - g(t,\lambda u(t))|dsdt \to 0 \quad \text{as} \quad h \to 0,$$

From Egorov's theorem [13]

$$\begin{aligned}&\exists \quad \delta > 0, \quad E_\delta \subset [0,1], \quad \mu(E) < \frac{\delta}{4r} \quad s.t \\ &A_1(u)_h - A_1(u) \to 0 \quad \text{uniformly} \quad on \quad I - E_\delta.\end{aligned}$$

$$\begin{aligned}\|A_1(u)_h - A_1u\|_{L_1} &= \int_0^1 |(A_1(u))_h - A_1(u)|dt \\ &= \int_{I-E_\delta} |(A_1(u))_h - A_1(u)|dt + \int_{E_\delta} |(A_1(u))_h - A_1(u)|dt, \qquad (2.1)\end{aligned}$$

using assumptions 1 and 2 we obtain

$$\begin{aligned}\frac{1}{h}\int_t^{t+h}\Big|g(s,\lambda u(s)) - g(t,\lambda u(t))|dsdt &\leq 2\frac{1}{h}\int_t^{t+h}\Big[m_3(s) + \lambda b_3|u(s)]dt \\ &\leq 2(\|m_3\| + \lambda r)\frac{1}{h}\int_t^{t+h} ds \leq 2r. \qquad (2.2)\end{aligned}$$

from (2.1) and (2.2), we have

$$\begin{aligned}\|(A_1u)_h - A_1u\|_{L_1} &\leq \frac{\epsilon}{2\mu(I-E_\delta)}\int_{I-E_\delta} dt + 2r\int_{E_\delta} dt \\ &= \frac{\epsilon}{2\mu(I-E_\delta)}\mu(I-E_\delta) + 2r\mu(E_\delta) \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,\end{aligned}$$

then $(A_1u)_h \to (A_1u)$ uniformly. Hence, by Arzela Theorem [13], $A_1(\Omega)$ is relatively compact. Hence A_1 is compact operator.

Let $\{u_n\} \subset Q_r$ and $u_n \to u$, then from (assumption 4) the continuity of the function g we obtain

$$\lim_{n\to\infty} A_1u_n \quad = \quad \lim_{n\to\infty} g(t,\lambda u_n(t),\mu)$$

$$= \quad g(t, \lambda \lim_{n\to\infty} u_n(t), \mu) = A_1 u.$$

Then $u_n \to u \Rightarrow A_1 u_n \to A_1 u$ as $n \to \infty$. This mean that the operator A_1 is continuous operator.

Then by Schauder fixed point Theorem [11] there exist at least one solution $u \in L_1[0,T]$ of the (constraint) functional equations (1.2).

Now, for the existence of solutions of the functional integral equation (1.1) we have the following.

Let $Q_{r_1} = \{x \in L_1[0,T] : ||x||_{L_1} \leq r_1\}$, where $r_1 = \frac{\|a_1\|_{L_1} + b_1 M + b_1 b_2 T r_1}{1 - b_1 b_2 T}$.

Then we have, for $x \in Q_{r_1}$

$$\begin{aligned} |A_2 x(t)| &\leq |f_1(t, \int_0^t f_2(t, s, x(s), u(s)) ds)| \\ &\leq a_1(t) + b_1 \int_0^t |f_2(t, s, x(s), u(s))| ds \\ &\leq a_1(t) + b_1 \int_0^t (k(t,s) + b_2 |x(s)| + b_2 |u(s)|) ds \end{aligned}$$

Integrating the above inequality from 0 to T and making the change of variable we have

$$\begin{aligned} \int_0^T |A_2 x(t)| dt &\leq \int_0^T (a_1(t) + b_1 \int_0^t (k(t,s) + b_2 |x(s)| + b_2 |u(s)|) ds) dt \\ &\leq \int_0^T |a_1(t)| dt + b_1 \int_0^T \int_0^t k(t,s) ds dt \\ &\quad + b_1 b_2 \int_0^T \int_0^t (|x(s)| + |u(s)|) ds dt. \end{aligned}$$

Hence

$$\begin{aligned} \|A_2 x\|_{L_1} &\leq \|a_1\|_{L_1} + b_1 M + b_1 b_2 T (\|x\|_{L_1} + \|u\|_{L_1}) \\ &\leq \|a_1\|_{L_1} + b_1 M + b_1 b_2 T (r_1 + r) = r_1, \end{aligned}$$

Therefor $A_2 : Q_{r_1} \to Q_{r_1}$ and the class of functions $\{A_2 x\}$ is uniformly bounded in Q_{r_1}.

Let Ω_1 be bounded subset of Q_{r_1}, then $A_2(\Omega_1)$ is also bounded on Q_{r_1}.

Let $x \in \Omega_1$, then

$$\begin{aligned} \|(A_2 x)_h - A_2 x\|_{L_1} &= \int_0^T |(A_2 x)_h(t) - (A_2 x)(t)| dt \\ &= \int_0^T |\frac{1}{h} \int_t^{t+h} |(A_2 x)(s) ds - (A_2 x)(t)| dt \\ &\leq \int_0^T \frac{1}{h} \int_t^{t+h} |f_1(s, \int_0^s f_2(s, \theta, x(\theta), u(\theta)) d\theta) \\ &\quad - f_1(t, \int_0^t f_2(t, s, x(s), u(s)) ds)| ds dt \end{aligned}$$

since $f_1, f_2 \in L_1[0,T]$, It follows that

$$\begin{aligned} \frac{1}{h}\int_t^{t+h} & |f_1(s, \int_0^s f_2(s,\theta,x(\theta),u(\theta))d\theta) \\ & - f_1(t, \int_0^t f_2(t,s,x(s),u(s))ds)|ds \to 0 \quad \text{as} \quad h \to 0, \end{aligned}$$

then $(A_2x)_h \to (A_2x)$ uniformly. Hence, by Arzela Theorem [13], $A_2(\Omega_1)$ is relatively compact. Hence A_2 is compact operator.

Let $\{x_n\} \subset Q_{r_1}$ and $x_n \to x$

$$\lim_{n\to\infty} A_2x_n \quad = \quad \lim_{n\to\infty} f_1(t, \int_0^t f_2(t,s,x_n(s),u(s))ds) = A_2x.$$

Then $x_n \to x \Rightarrow A_2x_n \to A_2x$ as $n \to \infty$. This mean that the operator A_2 is continuous operator.

Then by Schauder fixed point Theorem [11] there exist at least one solution $x \in L_1[0,T]$ of the functional integral equation (1.1). □

3. Measure of non compactness

The usefulness of the measure of noncompactness was pointed out by [2]. For papers studied such kind of equations (see [1,3,5], and references therein).

Consider Problem (1.1) under the constrain (1.2) with the following assumptions:

- **$1'$:** $f_1 : I = [0,T] \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfies Carathéodory condition, that is, f_1 is measurable with respect to t for all $x \in \mathbb{R}^+$ and continuous in $x \in \mathbb{R}^+$ for almost all $t \in [0,T]$.

  $$f_1(t,x) \leq |m_1(t)| + k_1|x|.$$

  where $m_1 \in L_1[0,T]$ and k_1 is a positive constant. Moreover, f_1 is nondecreasing with respect to all variables.
- **$2'$:** $f_2 : I \times I \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfies Carathéodory condition, that is, f_2 is measurable with respect to t, s for all $(x,y) \in \mathbb{R}^+ \times \mathbb{R}^+$ and continuous in $x, y \in \mathbb{R}^+$ for almost all $t \in [0,T]$.

  $$f_2(t,s,x,y) \leq |m_2(t,s)| + k_2(|x| + |y|).$$

  where $m_2 \in L_1[0,T]$ and k_2 is a positive constant. Moreover, f_2 is nondecreasing with respect to all variables.
- **$3'$:** $g : I \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfies Carathéodory condition, there exist an integrable function $b_3 > 0$, such that

  $$|g(t,x)| \leq |m_3(t)| + b(|x|),$$

  where $m_3(t) \in L^1[0,T]$ and b is a positive constant. Moreover g is nondecreasing with respect to all variable.
- **$4'$:**

Theorem 3.1. *Let the assumptions $1'$–$2'$ be satisfied, if $b_3\lambda < 1$, then problem (1.1)–(1.2) has at least one positive monotonic nondecreasing solution $x \in L_1[0,T]$.*

Proof. Let the operator F_1 defined by the formula

$$(F_1u)(t) = g(t, \lambda u(t)), \qquad t \in [0, T].$$

$B_{r_1} = \{u \in L_1[0,1] : ||u||_{L_1} \leq r_1\}$. Let $u \in L_1[0, T]$, then we have

$$|(F_1u)(t)| \leq |m_3(t)| + b_3\lambda|u|.$$

This implies that

$$\begin{aligned} \|F_1u\| &= \int_0^T |(F_1u)(t)|dt \\ &\leq \int_0^1 [|m_3(t)| + b_3\lambda|u|]dt \\ &\leq \|m_3\| + b_3\lambda\|u\| \\ &\leq \|m_3\| + b_3\lambda r_1 = r_1. \end{aligned}$$

Hence $F_1u \in L_1[0, 1]$, moreover the operator F_1 maps B_{r_1} into itself, where

$$r_1 = \frac{\|m_3\|}{1 - b_3\lambda}.$$

Now, let $Q_{r_1} \subset B_{r_1}$ containing of all functions positive and nondecreasing on $[0, 1]$. Clear that Q_{r_1} is nonempty, closed, bounded and convex. This mean that Q_{r_1} is a bounded subset of L_1 consisting of all functions positive and nondecreasing on $[0, 1]$. Then by [4] Q_{r_1} is compact in measure. Now, we show that F_1 transform the a positive nondecreasing function into functions of the same type. If $x \in Q_{r_1}$, then $u(t)$ is positive nondecreasing function on $[0, T]$ and $g(s, \lambda u(s))ds,\ t \in [0, T]$ is positive and nondecreasing function on $[0, T]$ [4]. Thus the operator $F_1 : Q_{r_1} \to Q_{r_1}$.

Now, we show that F_1 is continuous operator on Q_{r_1}. Let $u_n \in Q_{r_1}$ such that $u_n \to u$. Then from our assumptions, we get

$$\begin{aligned} \lim_{n\to\infty} F_1u_n &= \lim_{n\to\infty} (g(t, \lambda u_n(t))) \\ &= g(t, \lambda \lim_{n\to\infty} u_n(s))ds = F_1u. \end{aligned}$$

Thus F_1 is continuous on Q_{r_1}.

Finally, we show that F_1 is contraction with respect to the measure of non compactness χ.

Let U be a nonempty subset of Q_{r_1}. Fix $\epsilon > 0$ and take a measurable subset $D \subset I$ such that meas $D \leq \epsilon$. Then for any $u \in U$, we get

$$\begin{aligned} \|F_1u\|_D &\leq \int_D |g(t, \lambda u(t))|dt \\ &\leq \|m_3\|_D + b|\lambda|\|u\|_D + . \end{aligned}$$

But

$$\lim_{\epsilon\to 0}\{\sup[\int_D |m_3(t)|dt : \ D \subset [0, T],\ meas.\ D < \epsilon]\} = 0.$$

Thus, we obtain

$$\beta(F_1u)(t) \leq b\beta\lambda(u(t)),$$

and

$$\beta(F_1U) \leq b\beta\lambda(U),$$

since Q_{r_1} is compact in measure, thus

$$\chi(F_1U) \leq b\lambda\chi(U),$$

since $b < 1$, it follows that F_1 is contraction. Now by Darbo fixed point theorem, then exist at least one fixed point in Q_{r_1}. Consequently, there exist at least one solution $u \in [0,T]$ of problem (1.2) and solution is positive and nondecreasing on $[0,T]$.

Now, let the operator F_2 defined by the formula

$$(F_2x)(t) = f_1(t, \int_0^t f_2(t,s,x(s),u(s))ds), \qquad t \in [0,T].$$

$B_{r_2} = \{x \in L_1[0,1] : ||x||_{L_1} \leq r_2\}$. Let $x \in L_1$, then we have

$$\begin{aligned} |(F_2x)(t)| &\leq |f_1(t, \int_0^t f_2(t,s,x(s),u(s))ds)| \\ &\leq |m_1(t)| + k_1 \int_0^t |f_2(t,s,x(s),u(s))|ds \\ &\leq |m_1(t)| + k_1 \int_0^t [|m_2(t,s)| + k_2(|x(s)| + |u(s)|)]ds. \end{aligned}$$

This implies that

$$\begin{aligned} \|(F_2x)\| &\leq \int_0^T |m_1(t)|dt + k_1 \int_0^T \int_0^t [|m_2(t,s)| + k_2(|x(s)| + |u(s)|)]dsdt \\ &\leq \|m_1\| + k_1M + k_1k_2T\|x\| + k_1k_2T\|u\|. \end{aligned}$$

Hence $F_2x \in L_1$, moreover the operator F_2 maps B_{r_2} into itself, where

$$r_2 = \frac{\|m_1\| + k_1M + k_1k_2Tr_1}{1 - k_1k_2T}.$$

Now, let $Q_{r_2} \subset B_{r_2}$ containing of all functions positive and nondecreasing on $[0,1]$, thus Q_{r_2} is compact in measure.

Similarly, the operator $F_2 : Q_{r_2} \to Q_{r_2}$ and it is continuous operator on Q_{r_2}.

Finally, we show that F_2 is contraction with respect to the measure of non compactness χ.

Let X be a nonempty subset of Q_{r_2}. Fix $\epsilon > 0$ and take a measurable subset $D \subset I$ such that meas. $D \leq \epsilon$. Then for any $x \in X$, we get

$$\|F_2x\|_D \leq \int_D |m_1(t)|dt + k_1 \int_D \int_0^t [|m_2(t,s)| + k_2(|x(s)| + |u(s)|)]dsdt.$$

But

$$\lim_{\epsilon\to 0}\{\sup[\int_D |m_1(t)|dt : \ D \subset [0,T], meas. \ D < \epsilon]\} = 0,$$

$$\lim_{\epsilon\to 0}\{\sup[\int_D |m_2(t,s)|dt : \ D \subset [0,T], meas. \ D < \epsilon]\} = 0,$$

and

$$\lim_{\epsilon\to 0}\{\sup[\int_D |u(t)|dt : \ D \subset [0,T], meas. \ D < \epsilon]\} = 0.$$

Thus, we obtain

$$\beta(F_2x)(t) \leq k_1k_2\beta(x(t)),$$

and

$$\beta(F_2X) \leq k_1k_2\beta(X),$$

since Q_{r_2} is compact in measure, thus

$$\chi(F_2X) \leq k_1k_2\chi(X),$$

since $(\frac{k_1}{B} + k_2) < 1$, it follows that F_2 is contraction. Now by Darbo fixed point theorem, there exist at least one solution $x \in [0,T]$ of Problem (1.1) under the constrain (1.2) and the solution is positive and nondecreasing on $[0,T]$. This completes the proof. □

4. Continuous dependence

4.1. **Continuous dependence on the set of solutions of the constraint.** Consider firstly the following assumptions

1*: $f_1 : [0,T] \times \mathbb{R} \to \mathbb{R}$ is measurable in $t \in [0,T]$ and satisfies the Lipschitz condition

$$|f_1(t,x) - f_1(t,y)| \leq b_1|x-y|, \tag{4.1}$$

2*: $f_2 : [0,T] \times [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is measurable in $t, s \in [0,T]$ and satisfies the Lipschitz condition

$$|f_2(t,s,x,u) - f_2(t,s,x_1,u_1)| \leq b_2(|x-x_1| + |u-u_1|).$$

Theorem 4.1. *Let the assumptions* 1^*–2^* *be satisfied, then the solution of the functional integral equation* (1.1) *is unique. Moreover, this solution depends continuously on the set of solutions of the (constraint) functional equations* (1.2) *in the sense that, if*

$$\forall\epsilon > 0, \quad \exists \quad \delta(\epsilon) \quad s.t \quad \|u - u^*\|_{L_1} < \delta \Rightarrow ||x - x^*||_{L_1} < \epsilon,$$

where $x^*(t)$ *is the solution of*

$$x^*(t) = f_1(t, \int_0^t f_2(t,s,x^*(s),u^*(s))ds), \quad t \in (0,T], \tag{4.2}$$

and u, u^* *are any two solutions of the (constraint) functional equations* (1.2).

Proof. From assumption 1^*, we obtain

$$|f_1(t,x)| - |f_1(t,0)| \leq |f_1(t,x) - f_1(t,0)| \leq b_1|x|$$

and

$$|f_1(t,x)| \leq b_1|x| + |f_1(t,0)| = b_1|x| + a_1(t),\ a_1(t) = |f_1(t,0)|.$$

Therefor assumption 1 is satisfied, also by the same way we can show that assumption 2 is satisfied. Then the assumptions of Theorem 2.2 are satisfied and the solutions of the functional integral equation (1.1) exist.

Let x, y be two the solution of (1.1), then

$$|x(t) - y(t)| \quad = \quad |f_1(t, \int_0^t f_2(t,s,x(s),u(s))ds)$$

$$
\begin{aligned}
& -f_1(t, \int_0^t f_2(t,s,y(s),u(s))ds)| \\
\leq \;& b_1 \int_0^t |f_2(t,s,x(s),u(s)) - f_2(t,s,y(s),u(s))|ds \\
\leq \;& b_1 b_2 \int_0^t |x(s) - y(s)|ds
\end{aligned}
$$

Integrating the above inequality from 0 to T and making the change of variable we have

$$
\begin{aligned}
\int_0^T |x(t) - y(t)|dt \;&\leq\; \int_0^T (b_1 b_2 \int_0^t |x(s) - y(s)|ds|)dt \\
&\leq\; b_1 b_2 T \|x - y\|_{L_1}.
\end{aligned}
$$

Hence

$$(1 - b_1 b_2 T)\|x - y\|_{L_1} \leq 0.$$

Since $b_1 b_2 T < 1$, then $x(t) = y(t)$ and the solution of the functional integral equation (1.1) is unique.

Now, Let x, x^* be two solutions of the functional integral equations (1.1) and (4.2) respectively corresponding to the two solutions u, u^* of the (constraint) functional equations (1.2). Then

$$
\begin{aligned}
|x(t) - x^*(t)| \;&=\; |f_1(t, \int_0^t f_2(t,s,x(s),u(s))ds) \\
&\quad -f_1(t, \int_0^t f_2(t,s,x^*(s),u^*(s))ds)| \\
&\leq\; b_1 \int_0^t |f_2(t,s,x(s),u(s)) - f_2(t,s,x^*(s),u^*(s))|ds \\
&\leq\; b_1 b_2 \int_0^t |x(s) - x^*(s)|ds + b_1 b_2 \int_0^t |u(s) - u^*(s)|ds
\end{aligned}
$$

Integrating the above inequality from 0 to T and making the change of variable we have

$$
\begin{aligned}
\int_0^T |x(t) - x^*(t)|dt \;&\leq\; \int_0^T [b_1 b_2 \int_0^t |x(s) - x^*(s)|ds \\
&\quad +b_1 b_2 \int_0^t |u(s) - u^*(s)|ds]dt \\
&\leq\; b_1 b_2 T \|x - x^*\|_{L_1} + b_1 b_2 T \delta
\end{aligned}
$$

Hence

$$\|x - x^*\|_{L_1} \leq \frac{b_1 b_2 T \delta}{1 - b_1 b_2 T}.$$

Then the solution of the functional integral equation (1.1) depends continuously on the set of solutions $u \in L_1[0,T]$ of the (constraint) functional equation (1.1).. □

4.2. **Continuous dependence on the parameter λ and the functional g.** Consider firstly the following assumptions

3^*: $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ is measurable in $t \in [0,T]$ and satisfies the Lipschitz condition

$$|g(t,x) - g(t,y)| \le b_3|x-y|. \tag{4.3}$$

Theorem 4.2. *Let the assumption 3^* be satisfied, then the solution of the functional equation (1.2) is unique. Moreover, this solution depends continuously on the parameters λ, μ and the functional g, if*

$$\forall \epsilon > 0,\ \exists\ \delta(\epsilon)\ s.t\ |\lambda - \lambda^*| < \delta_1, \|g - g^*\|_{L_1} < \delta_3 \Rightarrow ||u - u^*||_{L_1} < \epsilon_1,$$

where $u^*(t)$ is the solution of

$$u^*(t) = g(t, \lambda^* u^*(t)), \quad t \in (0,T], \tag{4.4}$$

Proof. From assumption 3^*, we obtain

$$|g(t,x)| - |g(t,0)| \le |g(t,x) - g(t,0)| \le b_3|x|$$

and

$$|g(t,x)| \le b_3|x| + |g(t,0)| = b_3|x| + a_2(t),\ \ a_2(t) = |g(t,0)|.$$

Therefor assumption 4 is satisfied. Then the assumptions of Theorem 2.2 are satisfied and the solutions of the functional equation (1.2) exist.

Let u, v be two the solution of equation (1.2), then

$$\begin{aligned} |u(t) - v(t)| &= |g(t, \lambda u(t)) - g(t, \lambda v(t))| \\ &\le b_3|\lambda u(t) - \lambda v(t)| \end{aligned}$$

Integrating the above inequality from 0 to T and making the change of variable we have

$$\begin{aligned} \int_0^T |u(t) - v(t)|dt &\le \int_0^T (b_3|\lambda u(t) - \lambda v(t)|)dt \\ &\le b_3|\lambda| \int_0^T |u(t) - v(t)|dt. \end{aligned}$$

Hence

$$\|u - v\|_{L_1} \le b_3|\lambda| \|u - v\|_{L_1}.$$

Since $b_3|\lambda| < 1$, then $u(t) = v(t)$ and the solution of the functional equation (1.2) is unique.

Let u, u^* be two solutions of the functional equations (1.2) and (4.4) respectively. Then

$$\begin{aligned} |u(t) - u^*(t)| &= |g(t, \lambda u(t)) - g^*(t, \lambda^* u^*(t))| \\ &\le |g(t, \lambda u(t)) - g(t, \lambda^* u(t))| \\ &\quad + |g(t, \lambda^* u(t)) - g(t, \lambda^* u^*(t))| \\ &\quad + |g(t, \lambda^* u^*(t)) - g^*(t, \lambda^* u^*(t))| \end{aligned}$$

$$\begin{aligned}&\leq b_3|\lambda u(t)-\lambda^* u(t)|+b_3|\lambda^* u(t)-\lambda^* u^*(t)|\\&+|g(t,\lambda^* u^*(t))-g^*(t,\lambda^* u^*(t))|.\end{aligned}$$

Integrating the above inequality from 0 to T and making the change of variable we have

$$\begin{aligned}\int_0^T |u(t)-u^*(t)|dt &\leq \int_0^T [b_3|\lambda u(t)-\lambda^* u(t)|+b_3|\lambda^* u(t)\\&-\lambda^* u^*(t)|]dt\\&+\int_0^T |g(t,\lambda^* u^*(t))-g^*(t,\lambda^* u^*(t))|dt\\&\leq b_3 T\delta_1+b_3\delta_2\|u\|_{L_1}+b_3|\lambda^*|\|u-u^*\|_{L_1}+\delta_3.\end{aligned}$$

Hence

$$\|u-u^*\|_{L_1}\leq \frac{b_3\delta_1\|u\|_{L_1}+\delta_3}{1-b_3|\lambda^*|}=\epsilon_1. \tag{4.5}$$

Then the solution of the functional integral equation (1.2) depends continuously on the parameter λ and the functional g. □

Definition 4.3. The solution $x\in L_1[0,T]$ of the functional integral equation (1.1) depends continuously on the parameter λ and the functional g, if

$$\forall\epsilon>0,\ \exists\ \delta(\epsilon)\ s.t\ |\lambda-\lambda^*|<\delta_1, \|g-g^*\|_{L_1}<\delta_3 \Rightarrow ||x-x^*||_{L_1}<\epsilon_2,$$

where $x^*(t)$ is the solution of equation (4.2).

Theorem 4.4. *Let the assumptions of Theorem 4.1 and 4.2 be satisfied, then the solution of the functional integral equation* (1.1) *depends continuously on the parameters λ, μ and the functional g.*

Proof. Let x, x^* be the two solutions of the functional integral equations (1.1) and (4.2) respectively. Then

$$\begin{aligned}|x(t)-x^*(t)| &= |f_1(t,\int_0^t f_2(t,s,x(s),u(s))ds)\\&-f_1(t,\int_0^t f_2(t,s,x^*(s),u^*(s))ds)|\\&\leq b_1\int_0^t |f_2(t,s,x(s),u(s))-f_2(t,s,x^*(s),u^*(s))|ds\\&\leq b_1b_2\int_0^t |x(s)-x^*(s)|ds+b_1b_2\int_0^t |u(s)-u^*(s)|ds\end{aligned}$$

Integrating the above inequality from 0 to T and making the change of variable we have

$$\int_0^T |x(t)-x^*(t)|dt \leq \int_0^T [b_1b_2\int_0^t |x(s)-x^*(s)|ds$$

$$+b_1b_2\int_0^t |u(s)-u^*(s)|ds]dt$$
$$\leq b_1b_2T\|x-x^*\|_{L_1}+b_1b_2T\|u-u^*\|_{L_1}, \tag{4.6}$$

from (4.5) and (4.6), we obtain

$$\|x-x^*\|_{L_1} \leq \frac{b_1b_2T(b_3\delta_1\|u\|_{L_1}+T\delta_3)}{(1-b_1b_2T)(1-b_3|\lambda^*|)} = \epsilon_2.$$

Then the solution of the functional integral equation (1.1) depends continuously on the parameters λ, μ and the functional g. □

5. Examples:

Example 5.1. Consider the nonlinear integro–differential equation

$$x(t) = t^4e^{-t} + \int_0^t \frac{1}{2}(\sin(3s+3t) + \frac{\ln(1+|x(s)|)}{4+s^3} + \frac{s^4\cos u(s)}{e^{|u(s)|}})dt, \quad t\in[0,T], \tag{5.1}$$

where

$$u(t) = t^5+t^2+1+\frac{|\lambda u(t)|}{\sqrt{|\lambda u(t)|+t+9}} \quad t\in[0,T], \tag{5.2}$$

Set

$$f_1(t,\int_0^t f_2(t,s,x(s),u(s))ds) = t^4e^{-t} + \int_0^t \frac{1}{2}(\sin(3s+3t) + \frac{\ln(1+x(s))}{4+s^3} + \frac{s^4\cos u(s)}{e^{|u(s)|}})ds.$$

and

$$g(t,\lambda u(t)) = t^5+t^2+1+\frac{|\lambda u(t)|}{\sqrt{|\lambda u(t)|+t+9}}$$

Then

$$f_2(t,s,x(s),u(s)))| \leq \frac{1}{2}(\sin(3s+3t)) + \frac{1}{8}|x(s)| + \frac{1}{8}\frac{s^4\cos u(s)}{e^{|u(s)|}},$$

and

$$|g(t,\lambda u(t))| = |t^5+t^2+1| + \frac{1}{3}(|\lambda u(t)|)$$

The assumptions 1–5 of Theorem 2.2 are satisfied with $a_1(t) = t^4e^{-t} \in L_1[0,1]$, $k(t,s) = \frac{1}{2}\sin(3s+3t) \in L^1[0,1]$, $a_2(t) = t^5+t^2+1 \in L_1[0,1]$, $b_1 = \frac{1}{2}$, $b_2 = \frac{1}{8}, b_3 = \frac{1}{3}$, $b_3|\lambda| = \frac{1}{3}|\lambda| < 1$, $b_1b_2T = \frac{1}{16} < 1$ Therefore, by applying to Theorem 2.2, the given the control problem of the functional integral equation (5.1)-(5.2) has a solution $x \in L_1[0,T]$.

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Manuscript received July 15 2021
revised October 13 2021

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