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EXISTENCE AND STABILITY FOR A SYSTEM OF HIGH ORDER NONLINEAR BOUNDARY VALUE PROBLEM WITH NONLINEAR ϕ_p OPERATOR

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Abstract. We deal with stability investigation and existence results for a system of nonlinear fractional differential equations (FDEs) with ϕ_p p-Laplacian operator where p is satisfying the identity $\frac{1}{p}+\frac{1}{q}=1$. Our system involves Caputo's fractional derivative and is more general and complex than many nonlinear systems in the literature. In this study, we take help from the topological degree theory and very easily the task is accomplished for the existence and stability.

1. Introduction

The fractional calculus have taken start from a letter of L'Hospital to Leibniz to know how can they determine $\frac{d^n}{dx^n}=x$ for $n=1/2$. In his answer, Leibniz given his view that "An apparent paradox, from which one day useful consequences will be drawn". The existence of the fractional calculus has produced so many consequences leading to important theories in mathematics, physics, engineering and other areas. Nowadays, we have a lot of definitions of fractional derivative and integrals including it were possible to define several fractional integrals, for example; Riemann-Liouville, Katugampola, conformable, Hadamard, Saigo, Erdelyi-Kober, Liouville, local and Weyl types and many others. These integrals and differential operators are having a lot of importance and applications in image processing, viscoelastic theory, fluid, dynamics, biology, hydrodynamics, control theory and many other fields. For a related study about history, development and applications of fractional calculus, we refer the readers to the well-known published books [11, 24, 25].

In order to highlight some recently published contributions of authors for the existence results, we present some example here. Baleanu et al. [5] proved existence of solution for a nonlinear FDE on partially ordered Banach spaces and provided applications. Mahmudov and Unul [21] studied a FDE of order $\alpha \in (2,3]$ with integral conditions, an impulsive FDE [22] and FDE with nonlinear operator ϕ_p [23], for the existence of solutions. Hu et al. [15] studied a system of nonlinear FDEs with operator ϕ_p at resonance.

Recently, Li [20] considered a class of FDEs with nonlinear operator ϕ_p, for positive solutions of

$$D_0^\beta(\phi_p(cD_0^\alpha v(t)) = -\psi(t,u(t)),$$
$$(\phi_p(cD_0^\alpha v(t)))_{t=0} = (\phi_p(cD_0^\alpha v(t)))'_{t=0} = (\phi_p(cD_0^\alpha v(t)))_{t=1},$$

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$$v''(t)_{t=0} = v'(t)_{t=1} = 0,\ av(0) + v'(0) = \int_0^1 g(t)v(t)dt,$$

where $\alpha + \beta \in (5, 6]$ and the fractional derivatives cD_0^{α} are in the Caputo's sense while D_0^{β} is in Riemann-Liouville. Our goal is to utilize topological degree theory for the study of existence and stability and existence

$$\begin{aligned}
& D_0^{\beta_1}(\phi_p(D_0^{\alpha_1}v(t))) = -\psi_1(t, u(t)), D_0^{\beta_2}(\phi_p(D_0^{\alpha_2}u(t))) = -\psi_2(t, v(t)), \\
& (\phi_p(D_0^{\alpha_1}v(t)))^{(i)}_{t=0} = (\phi_p(D_0^{\alpha_1}v(t)))^{(n-1)}_{t=1} = 0 \\
& (\phi_p(D_0^{\alpha_2}u(t)))^{(i)}_{t=0} = (\phi_p(D_0^{\alpha_2}u(t)))^{(n-1)}_{t=1} = 0, \\
& v(1) - \frac{v^{(n-1)}(0)}{(n-1)!} = 0, u(1) - \frac{u^{(n-1)}(0)}{(n-1)!} = 0 \\
& v^{(i)}(t)_{t=0} = v^{(n-1)}(t)_{t=1} = 0, \\
& u^{(i)}(t)_{t=0} = u^{(n-1)}(t)_{t=1} = 0,\ \text{for } i = 1, 2 \ldots, n-2,
\end{aligned} \tag{1.1}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in (n-1, n]$ for $n \geq 3$. The fractional order derivatives $D_0^{\alpha_1}, D_0^{\alpha_2}$, $D_0^{\beta_1}, D_0^{\beta_2}$ are in the Caputo's sense. The functions ψ_1, ψ_2 are fractional integrable. ϕ_p is nonlinear p-Laplacian operator satisfying the identity $1/p + 1/q = 1$, ϕ_q is inverse of operator ϕ_p.

In utmost of the formerly considered cases the authors would need to the assumption of compactness of the integral operators which would restrict the impact of the problem and mathematical method at large. In this paper, three important aspects of the FDE with nonlinear p -Laplacian operator (1.1) are considered including existence of solution, uniqueness of solution and Hyers-Ulam stability of the suggested problem. For these objectives, we are going to convert the problem (1.1) to an integral equation by Green functions. After this, we will prove results for existence and uniqueness by topological degree method. By the use of this technique, we do not need to the assumption of the compactness of the operator. Then after, Hyers-Ulam stability will be investigated. In literature, we could not found published work on Hyers-Ulam stability of FDEs with nonlinear operator ϕ_p and boundary condition. Therefore, this work may get the attention of researchers to the study of Hyers-Ulam stability as well many other types of stability for more complex problems. We also suggest the readers that the problem (1.1) has potentials to be studied for further aims including multiplicity results.

2. Auxiliary results

Definition 2.1. For a function $g(\tau)$ the fractional Caputo's derivative of β order is defined by the following integral form (provided it exist)

$$D_0^{\beta}g(\tau) = \frac{1}{\Gamma([\beta] + 1 - \beta)} \int_0^{\tau} (\tau - s)^{[\beta]-\beta} g^{([\beta]+1)}(s)ds,$$

where $[\beta]$ is the integer part of β.

Definition 2.2. Let $\beta > 0$ and $\gamma \in C(0, 1) \cap L^1(0, 1)$. Then the general solution of the FDE

$$D_0^{\beta}\gamma(\tau) = h(\tau),$$

is given by

$$\gamma(\tau) = h(\tau) + c_1 + c_2\tau + c_3\tau^2 + \cdots + c_k\tau^{k-1},$$

where, for some $c_i \in R, i = 1, 2, 3, \ldots, k$, when $k = [\beta] + 1$.

Definition 2.3. For a function $h(t)$, fractional integral in the Riemann-Liouville sense of order $\beta > 0$ is defined as

$$I_0^\beta h(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} h(s) ds,$$

provided the integral exist on $(0, \infty)$.

Lemma 2.4. *Let $\beta \in (n, n-1], \psi \in BC^{n-1}$. Then*

$$I_0^\beta D_0^\beta \psi(\tau) = \psi(\tau) + b_1 + b_2\tau + b_3\tau^2 + \cdots + b_n\tau^{n-1},$$

for $b_i \in R$, $i = 1, 2, 3, \ldots, n$.

Consider a set containing continuous functions $\mathfrak{B} = C((0, 1], R)$, is a Banach space with a convergence topology and a norm $\|u(\tau)\| = sup\{|u(\tau)| : \tau \in [0, 1]\}$. The product space denoted by $L = \mathfrak{B} \times \mathfrak{B}$ under the norms $\|(u, v)(\tau)\| = \|u(\tau)\| + \|v(\tau)\|$ is also Banach space.

Definition 2.5. Let us consider a sets $C(\mathfrak{B})$ to be a class of all bounded sets denoted by R Then, for the mapping $F : R \to (0, \infty)$ Kuratowski measure of noncompactness is

$$F(z) = inf\{r > 0 : \text{z is the finite cover for sets of diameter} \leq r\},$$

where $z \in R$.

Proposition 2.6. *For the measure F, the following are holding true:*

(1) *For relative compact B, the Kuratowski measure $F(B) = 0$;*
(2) *Semi-norm F, that is $F(\mu B) = |\mu|F(B), \mu \in R$ and* $F(B_1 + B_2) \leq F(B_1) + F(B_2)$;
(3) $B_1 \subset B_2$ *yields* $F(B_1)$; $F(B_1 \cup B_2) = sup\{F(B_1), F(B_2)\}$;
(4) $F(convB) = F(B)$;
(5) $F(\bar{B}) = F(B)$.

Definition 2.7. Let $h : S \to \mathfrak{B}$ be bounded and continuous mapping with $S \subset \mathfrak{B}$. Then h is an F-Lipschitz, where $\zeta \geq 0$ if

$$F(h(B)) \leq \zeta F(B) \text{ for all bounded } B \subset S.$$

And h is a strict F-contraction with $\zeta < 1$.

Definition 2.8. The function h is F-contraction if

$$F(h(B)) \leq F(B), \text{ for all bounded } B \subset S \text{ such that } F(B) > 0.$$

Therefore $F(h(B)) \geq F(B)$ yields $F(B) = 0$.

Further we have $h : S \to \mathfrak{B}$ is Lipschitz for $\zeta > 0$, such that

$$\|h(\nu) - h(\bar{\nu})\| \leq \zeta\|\nu - \bar{\nu}\|,$$

for all $\nu, \bar{\nu} \in S$.

The condition $\zeta < 1$ causes h to be strict contraction.

Proposition 2.9. *The h is said to be F-Lipschitz with $\zeta = 0$ if and only if $h : S \to \mathfrak{B}$ is said to be compact.*

Proposition 2.10. *The h is said to be F-Lipschitz for constant value ζ if and only if $h : S \to \mathfrak{B}$ is Lipschitz with a constant ζ.*

Theorem 2.11. *Let $h : \mathfrak{B} \to \mathfrak{B}$ be a F-contraction and*

$$A = \{x \in \mathfrak{B} : \text{ there exist } 0 \leq \Upsilon \leq 1 \text{ such that } x = \Upsilon h(x)\}.$$

If A is bounded in $\mathfrak{B}$, there exist $r > 0$ and $A \subset x_r(0)$, with the degree

$$deg(I - \Upsilon_h, x_r(0), 0) = 1, \quad \text{for every } \Upsilon \in [0,1].$$

Consequently, h has at least one fixed point and collection of all fixed points of h are contained in $x_r(0)$.

Lemma 2.12. *For the nonlinear operator ϕ_p, we have*

(1) *If $1 < p \leq 2, M_1, M_2 > 0$, and $|M_1|, |M_2| \geq \rho > 0$, then*

$$|\phi_p(M_1) - \phi_p(M_2)| \leq (p-1)\rho^{p-2}|M_1 - M_2|;$$

(2) *If $p > 2$, and $|M_1|, |M_2| \leq \rho$, then*

$$|\phi_p(M_1) - \phi_p(M_2)| \leq (p-1)\rho^{p-2}|M_1 - M_2|.$$

3. Main results

Theorem 3.1. *For an integrable function $\psi \in C[0,1]$ the solution of*

$$\begin{aligned}
&D_0^{\beta_1}(\phi_p(D_0^{\alpha_1}v(t))) = -\psi_1(t, v(t)),\\
&(\phi_p(D_0^{\alpha_1}v(t)))_{t=0}^{(i)} = (\phi_p(D_0^{\alpha_1}v(t)))_{t=1}^{(n-1)} = 0, i = 0,1,2,\dots,n-2,\\
&v^{(i)}(t)_{t=0} = v^{(n-1)}(t)_{t=1} = 0, i = 1,2\dots,n-2, v(1) - \frac{v^{(n-1)}(0)}{(n-1)!} = 0,
\end{aligned} \tag{3.1}$$

is given by the integral equation

$$v(t) = \int_0^1 G^{\alpha_1}(t,s)\phi_q(\int_0^1 G^{\beta_1}(s,\tau)\psi_1(\tau)d\tau)ds,$$

where $G^{\alpha_1}(t,s)$ and $G^{\beta_1}(t,s)$ are Green functions defined by

$$G^{\beta_1}(t,s) = \begin{cases} \dfrac{-(t-s)^{\beta_1-1}}{\Gamma(\beta_1)} + \dfrac{t^{n-1}(1-s)^{\beta_1-n}}{\Gamma(n)\Gamma(\beta_1-n+1)}, 0 \leq s < t \leq 1,\\[2mm] \dfrac{t^{n-1}(1-s)^{\beta_1-n}}{\Gamma(n)\Gamma(\beta_1-n+1)}, 0 \leq t < s \leq 1, \end{cases}$$

$$G^{\alpha_1}(t,s) = \begin{cases} \dfrac{(t-s)^{\alpha_1-1} - (1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} - \dfrac{t^{n-1}(1-s)^{\alpha_1-n}}{\Gamma(n)\Gamma(\alpha_1-n+1)}, 0 \leq s \leq t \leq 1,\\[2mm] -\dfrac{(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} - \dfrac{t^{n-1}(1-s)^{\alpha_1-n}}{\Gamma(n)\Gamma(\alpha_1-n+1)}, 0 \leq t \leq s \leq 1. \end{cases}$$

Proof. Applying operator $I_0^{\beta_1}$ on (3.1) and Lemma (2.4), we have

$$\phi_p(D_0^{\alpha_1}v(t)) = I_0^{\beta_1}[-\psi_1(t,v(t))] + c_1 + c_2t + \cdots + c_{n-1}t^{n-2} + c_nt^{n-1}. \tag{3.2}$$

Taking the (n-2)th order derivative for (3.2) and using the conditions $(\phi_p(D_0^{\alpha_1}v(t)))^{(i)}_{t=0} = 0, i = 0,1,2\dots,n-2$, we get $c_1 = c_2 = \cdots = c_{n-1} = 0$. Substituting the values of c_i for , $i = 0,1,2\dots,n-2$in (3.2), we get

$$\phi_p(D_0^{\alpha_1}v(t)) = I_0^{\beta_1}[-\psi_1(t,v(t))] + c_nt^{n-1}. \tag{3.3}$$

Using the condition $(\phi_p(D_0^{\alpha_1}v(t)))^{(n-1)}_{t=1} = 0$ in (3.3), we get

$$c_n = \frac{1}{\Gamma(n)\Gamma(\beta_1-n+1)}\int_0^1 (1-s)^{\beta_1-n}\psi_1(s)ds. \tag{3.4}$$

Using (3.4) in (3.3), we get

$$\begin{aligned}\phi_p(D_0^{\alpha_1}v(t)) =& \frac{-1}{\Gamma(\beta_1)}\int_0^t (t-s)^{\beta_1-1}\psi_1(s,v(s))ds\\ &+ \frac{t^{n-1}}{\Gamma(n)\Gamma(\beta_1-n+1)}\int_0^1 (1-s)^{\beta_1-n}\psi_1(s)ds.\end{aligned}$$

This implies

$$\phi_p(D_0^{\alpha_1}v(t)) = \int_0^1 G^{\beta_1}(t,s)\psi_1(s)ds. \tag{3.5}$$

Applying $\phi_q = \phi_p^{-1}$ in (3.5), we get

$$D_0^{\alpha_1}v(t) = \phi_q(\int_0^1 G^{\beta_1}(t,s)\psi_1(s)ds). \tag{3.6}$$

Applying operator $I_0^{\alpha_1}$ on (3.6), we get

$$v(t) = I_0^{\alpha_1}(\phi_q(\int_0^1 G^{\beta_1}(t,s)\psi_1(s)ds)) + k_1 + k_2t + \cdots + k_{n-1}t^{n-2} + k_nt^{n-1}. \tag{3.7}$$

Using the conditions $(v(t))^{(i)}_{t=0} = 0, i = 1,2\dots,n-2$, we get $k_2 = k_3 = \cdots = k_{n-1} = 0$. Substituting in (3.7), we get

$$v(t) = I_0^{\alpha_1}(\phi_q(\int_0^1 G^{\beta_1}(t,s)\psi_1(s)ds)) + k_1 + k_nt^{n-1}. \tag{3.8}$$

Since $v^{(n-1)}(t)|_{t=1} = 0$, then

$$k_n = \frac{-1}{\Gamma(n)}I_0^{\alpha_1-n+1}(\phi_q(\int_0^1 G^{\beta_1}(t,s)\psi_1(s)ds))_{t=1}.$$

Also, $v(1) = \frac{-1}{\Gamma(n)}I^{\alpha_1-(n-1)}[\phi_q(\int_0^1 G^{\beta_1}(t,s)\psi_1(s)ds)]_{t=1}$, substituting in (3.8), we get

$$k_1 = -\frac{1}{\Gamma(\alpha_1)}\int_0^1 (1-s)^{\alpha_1-1}(\phi_q(\int_0^1 G^{\beta_1}(t,s)\psi_1(s)ds))ds.$$

Substituting k_1 and k_n in (3.8), we get

$$v(t) = \int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}\phi_q(\int_0^1 G^{\beta_1}(s,\tau)\psi_1(\tau)d\tau)ds$$

$$- \frac{1}{\Gamma(\alpha_1)} \int_0^1 (1-s)^{\alpha_1 - 1} (\int_0^1 G^{\beta_1}(s,\tau)\psi_1(\tau)d\tau)ds$$
$$- \frac{t^{n-1}}{\Gamma(n)\Gamma(\alpha_1 - n + 1)} \int_0^1 (1-s)^{\alpha_1 - n}\phi_q(\int_0^1 G^{\beta_1}(s,\tau)\psi_1(\tau)d\tau)ds$$
$$v(t) = \int_0^1 G^{\alpha_1}(t,s)\phi_q(\int_0^1 G^{\beta_1}(s,\tau)\psi_1(u(\tau))d\tau)ds.$$

Following Theorem 3.1, we may write our problem in the following system

$$\begin{aligned} v(t) &= \int_0^1 G^{\alpha_1}(t,s)\phi_q(\int_0^1 G^{\beta_1}(s,\tau)\psi_1(u(\tau))d\tau)ds, \\ u(t) &= \int_0^1 G^{\alpha_2}(t,s)\phi_q(\int_0^1 G^{\beta_2}(s,\tau)\psi_2(v(\tau))d\tau)ds, \end{aligned} \tag{3.9}$$

where $G^{\beta_2}(t,s)$ and $G^{\alpha_2}(t,s)$ are Green functions defined by

$$G^{\beta_2}(t,s) = \begin{cases} \dfrac{-(t-s)^{\beta_2 - 1}}{\Gamma(\beta_2)} + \dfrac{t^{n-1}(1-s)^{\beta_2 - n}}{\Gamma(n)\Gamma(\beta_2 - n + 1)}, 0 \le s < t \le 1, \\ \dfrac{t^{n-1}(1-s)^{\beta_2 - n}}{\Gamma(n)\Gamma(\beta_2 - n + 1)}, 0 \le t < s \le 1. \end{cases}$$

$$G^{\alpha_2}(t,s) = \begin{cases} \dfrac{(t-s)^{\alpha_2 - 1} - (1-s)^{\alpha_2 - 1}}{\Gamma(\alpha_2)} - \dfrac{t^{n-1}(1-s)^{\alpha_2 - n}}{\Gamma(n)\Gamma(\alpha_2 - n + 1)}, 0 \le s \le t \le 1, \\ -\dfrac{(1-s)^{\alpha_2 - 1}}{\Gamma(\alpha_2)} - \dfrac{t^{n-1}(1-s)^{\alpha_2 - n}}{\Gamma(n)\Gamma(\alpha_2 - n + 1)}, 0 \le t \le s \le 1. \end{cases}$$

Define $H_j : \mathfrak{B} \to \mathfrak{B}$ for $j = 1, 2$ by

$$\begin{aligned} H_1 v(t) &= \int_0^1 G^{\alpha_1}(t,s)\phi_q(\int_0^1 G^{\beta_1}(s,\tau)\psi_1(u(\tau))d\tau)ds. \\ H_2 u(t) &= \int_0^1 G^{\alpha_2}(t,s)\phi_q(\int_0^1 G^{\beta_2}(s,\tau)\psi_2(v(\tau))d\tau)ds. \end{aligned} \tag{3.10}$$

With the help of Theorem 3.1, solution of the system (3.12) is equivalent to fixed point of operator H, say (v, u), with

$$(v, u) = H(v, u) = (H_1 v(t), H_2 u(t)), \tag{3.11}$$

for $H = (H_1, H_2)$. □

We need the following assumptions in the main results of the paper.

$(\mathfrak{L}_1)$ The functions ψ_1, ψ_2 satisfy the following growth conditions for the constant c, d, Λ_{ψ_1} and Λ_{ψ_2}

$$|\psi_1(x,u)| \le \phi_p(c|u| + \Lambda_{\psi_1}),$$
$$|\psi_2(x,v)| \le \phi_p(d|v| + \Lambda_{\psi_2}).$$

$(\mathfrak{L}_2)$ For ψ_1, ψ_2, there exist real valued constants $\Upsilon_{\psi_1}, \Upsilon_{\psi_2}$such that, for all $v, u, x, y \in \mathfrak{B}$, we have

$$|\psi_1(t,\delta) - \psi_1(t,x)| \le \Upsilon_{\psi_1}|\delta - x|,$$
$$|\psi_2(t,\xi) - \psi_2(t,y)| \le \Upsilon_{\psi_2}|\xi - y|.$$

Theorem 3.2. *With assumption $(\mathfrak{L}_1)$, the operator $H : L \to L$ is continuous and satisfies the following growth condition:*

$$\|H(v,u)\| \le K\|(v,u)\| + M,$$

where $K = \Omega(c+d), M = \Omega(W_1 + W_2)$, and

$$\Omega = \max\left\{\begin{array}{l} \frac{1}{\Gamma(n)\Gamma(\alpha_1 - n+2)}(\frac{1}{\Gamma(\beta_1+1)} + \frac{1}{\Gamma(n)\Gamma(\beta_1 - n+2)})^{q-1}, \\ \frac{1}{\Gamma(n)\Gamma(\alpha_2 - n+2)}(\frac{1}{\Gamma(\beta_2+1)} + \frac{1}{\Gamma(n)\Gamma(\beta_2 - n+2)})^{q-1} \end{array}\right\},$$

for each $(v,u) \in g_r \subset L$.

Proof. Consider the bounded set $g_r = \{(v,u) \in L : \|(v,u)\| \le r\}$ with sequence $\{(v_n,u_n)\}$ converging to (v,u) in g_r. To show that $\|H(v_n,u_n) - H(v,u)\| \to 0$ as $n \to \infty$, let us consider

$$\begin{aligned}
&|H_1v_n(t) - H_1v(t)| \\
&= |\int_0^1 G^{\alpha_1}(t,s)\phi_q(\int_0^1 G^{\beta_1}(s,\tau)\psi_1(u_n(\tau))d\tau)ds \\
&- \int_0^1 G^{\alpha_1}(t,s)\phi_q(\int_0^1 G^{\beta_1}(s,\tau)\psi_1(u(\tau))d\tau)ds| \\
&= |\int_0^1 G^{\alpha_1}(t,s)[\phi_q(\int_0^1 G^{\beta_1}(s,\tau)\psi_1(u_n(\tau))d\tau) \\
&- \phi_q(\int_0^1 G^{\beta_1}(s,\tau)\psi_1(u(\tau))d\tau)]ds| \\
&\le \int_0^1 |G^{\alpha_1}(t,s)||\phi_q(\int_0^1 G^{\beta_1}(s,\tau)\psi_1(u_n(\tau))d\tau) \\
&- \phi_q(\int_0^1 G^{\beta_1}(s,\tau)\psi_1(u(\tau))d\tau)|ds \\
&\le (q-1)\delta_1^{q-2}\int_0^1 |G^{\alpha_1}(t,s)|\int_0^1 |G^{\beta_1}(s,\tau)||\psi_1(u_n(\tau)) - \psi_1(u(\tau))|d\tau ds,
\end{aligned}$$

$$\begin{aligned}
&|H_2u_n(t) - H_2u(t)| \\
&= |\int_0^1 G^{\alpha_2}(t,s)\phi_q(\int_0^1 G^{\beta_2}(s,\tau)\psi_2(v_n(\tau))d\tau)ds \\
&- \int_0^1 G^{\alpha_2}(t,s)\phi_q(\int_0^1 G^{\beta_2}(s,\tau)\psi_2(v(\tau))d\tau)ds| \\
&= |\int_0^1 G^{\alpha_2}(t,s)[\phi_q(\int_0^1 G^{\beta_2}(s,\tau)\psi_2(v_n(\tau))d\tau) \\
&- \phi_q(\int_0^1 G^{\beta_2}(s,\tau)\psi_2(v(\tau))d\tau)]ds| \\
&\le \int_0^1 |G^{\alpha_2}(t,s)||\phi_q(\int_0^1 G^{\beta_2}(s,\tau)\psi_2(v_n(\tau))d\tau)
\end{aligned} \tag{3.12}$$

$$
\begin{aligned}
&- \phi_q(\int_0^1 G^{\beta_1}(s,\tau)\psi_1(v(\tau))d\tau)|ds \\
&\le (q-1)\delta_1^{q-2}\int_0^1 |G^{\alpha_2}(t,s)|\int_0^1 |G^{\beta_2}(s,\tau)||\psi_2(v_n(\tau))-\psi_2(v(\tau))|d\tau ds. \qquad (3.13)
\end{aligned}
$$

Since ψ_1, ψ_2 are continuous functions, therefore, $|\psi_1(u_n(\tau)) - \psi_1(u(\tau))| \to 0$ and $|\psi_2(v_n(\tau)) - \psi_2(v(\tau))| \to 0$, as $n \to \infty$. Using (3.21) and (3.22) to conclude that $|H_1 v_n(t) - H_1 v(t)| \to 0$ and $|H_2 u_n(t) - H_2 u(t)| \to 0$, as $n \to \infty$. This implies H_1 and H_2 are continuous functions. Thus, the operator $H = (H_1, H_2)$ is continuous. Furthermore, using (3.16) and (3.17), we continue as follows

$$
\begin{aligned}
|H_1 v(t)| =& |\int_0^1 G^{\alpha_1}(t,s)\phi_q(\int_0^1 G^{\beta_1}(s,\tau)\psi_1(u(\tau))d\tau)ds| \\
\le& \int_0^1 |G^{\alpha_1}(t,s)||\phi_q(\int_0^1 G^{\beta_1}(s,\tau)\psi_1(u(\tau))d\tau)|ds \\
\le& \int_0^1 |G^{\alpha_1}(t,s)|\phi_q(\int_0^1 |G^{\beta_1}(s,\tau)||\psi_1 u(\tau)|d\tau)ds \\
\le& \int_0^1 |G^{\alpha_1}(t,s)|\phi_q(\int_0^1 |G^{\beta_1}(s,\tau)|\phi_p(c|u|+\Lambda_{\psi_1})d\tau)ds \qquad (3.14)\\
=& \int_0^1 [|G^{\alpha_1}(t,s)|\phi_q(\int_0^1 |G^{\beta_1}(s,\tau)|d\tau)]ds(c|u|+\Lambda_{\psi_1}) \\
\le& \int_0^1 [|G^{\alpha_1}(t,s)|\phi_q(\frac{1}{\Gamma(\beta_1+1)}+\frac{1}{\Gamma(n)\Gamma(\beta_1-n+2)})]ds(c|u|+\Lambda_{\psi_1}) \\
=& \int_0^1 [|G^{\alpha_1}(t,s)|(\frac{1}{\Gamma(\beta_1+1)}+\frac{1}{\Gamma(n)\Gamma(\beta_1-n+2)})^{q-1}]ds(c|u|+\Lambda_{\psi_1}) \\
\le& (\frac{2}{\Gamma(\alpha_1+1)}+\frac{1}{\Gamma(n)\Gamma(\alpha_1-n+2)})(\frac{1}{\Gamma(\beta_1+1)}+\frac{1}{\Gamma(n)\Gamma(\beta_1-n+2)})^{q-1} \\
&.(c|u|+\Lambda_{\psi_1}) \\
|H_2 u(t)| =& |\int_0^1 G^{\alpha_2}(t,s)\phi_q(\int_0^1 G^{\beta_2}(s,\tau)\psi_2(v(\tau))d\tau)ds| \\
\le& \int_0^1 |G^{\alpha_2}(t,s)||\phi_q(\int_0^1 G^{\beta_2}(s,\tau)\psi_2(v(\tau))d\tau)|ds \\
\le& \int_0^1 |G^{\alpha_2}(t,s)|\phi_q(\int_0^1 |G^{\beta_2}(s,\tau)||\psi_2 v(\tau)|d\tau)ds \\
\le& \int_0^1 |G^{\alpha_2}(t,s)|\phi_q(\int_0^1 |G^{\beta_2}(s,\tau)|\phi_p(d|v|+\Lambda_{\psi_2})d\tau)ds \\
=& \int_0^1 [|G^{\alpha_2}(t,s)|\phi_q(\int_0^1 |G^{\beta_2}(s,\tau)|d\tau)]ds(d|v|+\Lambda_{\psi_2}) \qquad (3.15)\\
\le& \int_0^1 [|G^{\alpha_2}(t,s)|\phi_q(\frac{1}{\Gamma(\beta_2+1)}+\frac{1}{\Gamma(n)\Gamma(\beta_2-n+2)})]ds(d|v|+\Lambda_{\psi_2})
\end{aligned}
$$

$$=\int_0^1 [|G^{\alpha_2}(t,s)|(\frac{1}{\Gamma(\beta_2+1)}+\frac{1}{\Gamma(n)\Gamma(\beta_2-n+2)})^{q-1}]ds(d|v|+\Lambda_{\psi_2})$$
$$\leq(\frac{2}{\Gamma(\alpha_2+1)}+\frac{1}{\Gamma(n)\Gamma(\alpha_2-n+2)})(\frac{1}{\Gamma(\beta_2+1)}+\frac{1}{\Gamma(n)\Gamma(\beta_2-n+2)})^{q-1}$$
$$.(d|v|+\Lambda_{\psi_2}).$$

From (3.23) and (3.24), we have

$$|H(v,u)|=|H_1v(t)+H_2u(t)|\leq|H_1v(t)|+|H_2u(t)|$$
$$\leq\Omega(c|u|+\Lambda_{\psi_1})+\Omega(d|v|+\Lambda_{\psi_2})\leq K\|(v,u)\|+M.$$

□

Theorem 3.3. *Assume that* $(\mathfrak{L}_1)$ *holds true. Then the operator* $H:L\to L$ *is compact and* $\xi-$*Lipschits with constant zero.*

Proof. From Theorem 3.2, we have the operator $H:L\to L$ is bounded. Next, using the assumption $(\mathfrak{L}_1)$, Lemma 2.12 and equation (3.12), for any $t_1,t_2\in[0,1]$, we have

$$|H_1v(t_1)-H_1v(t_2)|$$
$$=|\int_0^1 G^{\alpha_1}(t_1,s)\phi_q(\int_0^1 G^{\beta_1}(s,\tau)\psi_1(u(\tau))d\tau)ds$$
$$-\int_0^1 G^{\alpha_1}(t_2,s)\phi_q(\int_0^1 G^{\beta_1}(s,\tau)\psi_1(u(\tau))d\tau)ds|$$
$$=|\int_0^1 (G^{\alpha_1}(t_1,s)-G^{\alpha_1}(t_2,s))\phi_q(\int_0^1 G^{\beta_1}(s,\tau)\psi_1(u(\tau))d\tau)ds|$$
$$\leq\int_0^1 |G^{\alpha_1}(t_1,s)-G^{\alpha_1}(t_2,s)|\phi_q(\int_0^1 |G^{\beta_1}(s,\tau)||\psi_1(u(\tau))|d\tau)ds$$

(3.16)

$$\leq\int_0^1 |G^{\alpha_1}(t_1,s)-G^{\alpha_1}(t_2,s)|\phi_q(\int_0^1 |G^{\beta_1}(s,\tau)|\phi_p(c|u|+\Lambda_{\psi_1})d\tau)ds$$
$$=(c|u|+\Lambda_{\psi_1})\int_0^1 |G^{\alpha_1}(t_1,s)-G^{\alpha_1}(t_2,s)|\phi_q(\int_0^1 |G^{\beta_1}(s,\tau)|d\tau)ds$$
$$\leq(c|u|+\Lambda_{\psi_1})\int_0^1 |G^{\alpha_1}(t_1,s)-G^{\alpha_1}(t_2,s)|\phi_q(\frac{1}{\Gamma(\beta_1+1)}+\frac{1}{\Gamma(n)\Gamma(\beta_1-n+2)})ds$$
$$=(c|u|+\Lambda_{\psi_1})(\frac{1}{\Gamma(\beta_1+1)}+\frac{1}{\Gamma(n)\Gamma(\beta_1-n+2)})^{q-1}\int_0^1 |G^{\alpha_1}(t_1,s)-G^{\alpha_1}(t_2,s)|ds$$

and

$$|H_2u(t_1)-H_2u(t_2)|$$
$$=|\int_0^1 G^{\alpha_2}(t_1,s)\phi_q(\int_0^1 G^{\beta_2}(s,\tau)\psi_2(v(\tau))d\tau)ds$$
$$-\int_0^1 G^{\alpha_2}(t_2,s)\phi_q(\int_0^1 G^{\beta_2}(s,\tau)\psi_2(v(\tau))d\tau)ds|$$

$$
\begin{aligned}
&=\left|\int_0^1 (G^{\alpha_2}(t_1,s)-G^{\alpha_2}(t_2,s))\phi_q(\int_0^1 G^{\beta_2}(s,\tau)\psi_2(v(\tau))d\tau)ds\right| \\
&\leq \int_0^1 |G^{\alpha_2}(t_1,s)-G^{\alpha_2}(t_2,s)|\phi_q(\int_0^1 |G^{\beta_2}(s,\tau)||\psi_2(v(\tau))|d\tau)ds \\
&\leq \int_0^1 |G^{\alpha_2}(t_1,s)-G^{\alpha_2}(t_2,s)|\phi_q(\int_0^1 |G^{\beta_2}(s,\tau)|\phi_p(d|v|+\Lambda_{\psi_2})d\tau)ds \quad (3.17)\\
&\leq (d|v|+\Lambda_{\psi_2})\int_0^1 |G^{\alpha_2}(t_1,s)-G^{\alpha_2}(t_2,s)|\phi_q(\int_0^1 |G^{\beta_2}(s,\tau)|d\tau)ds \\
&=(d|v|+\Lambda_{\psi_2})\int_0^1 |G^{\alpha_2}(t_1,s)-G^{\alpha_2}(t_2,s)|\phi_q(\frac{1}{\Gamma(\beta_2+1)}+\frac{1}{\Gamma(n)\Gamma(\beta_2-n+2)})ds \\
&=(d|v|+\Lambda_{\psi_2})(\frac{1}{\Gamma(\beta_2+1)}+\frac{1}{\Gamma(n)\Gamma(\beta_2-n+2)})^{q-1}\int_0^1 |G^{\alpha_2}(t_1,s)-G^{\alpha_2}(t_2,s)|ds \\
&=(d|v|+\Lambda_{\psi_2})(\frac{1}{\Gamma(\beta_2+1)}+\frac{1}{\Gamma(n)\Gamma(\beta_2-n+2)})^{q-1} \\
&\qquad [\frac{|t_1^{\alpha_2}-t_2^{\alpha_2}|}{\Gamma(\alpha_2+1)}-\frac{|t_1^{n-1}-t_2^{n-1}|}{\Gamma(n)\Gamma(\alpha_2-n+2)}].
\end{aligned}
$$

Using (3.25), (3.26), we get

$$
\begin{aligned}
&|H(v,u)(t_1)-H(v,u)(t_2)| \leq |H_1v(t_1)-H_1v(t_2)|+|H_2u(t_1)-H_2u(t_2)| \\
&\leq (c|u|+\Lambda_{\psi_1})(\frac{1}{\Gamma(\beta_1+1)}+\frac{1}{\Gamma(n)\Gamma(\beta_1-n+2)})^{q-1} \quad (3.18)\\
&\qquad [\frac{|t_1^{\alpha_1}-t_2^{\alpha_1}|}{\Gamma(\alpha_1+1)}-\frac{|t_1^{n-1}-t_2^{n-1}|}{\Gamma(n)\Gamma(\alpha_1-n+2)}] \\
&+(d|v|+\Lambda_{\psi_2})(\frac{1}{\Gamma(\beta_2+1)}+\frac{1}{\Gamma(n)\Gamma(\beta_2-n+2)})^{q-1} \\
&\qquad [\frac{|t_1^{\alpha_2}-t_2^{\alpha_2}|}{\Gamma(\alpha_2+1)}-\frac{|t_1^{n-1}-t_2^{n-1}|}{\Gamma(n)\Gamma(\alpha_2-n+2)}].
\end{aligned}
$$

As $t_1 \to t_2$, the right side of (3.27) becoming zero. Thus, the operator $H=(H_1,H_2)$, is equicontinuous on the space L. Thus, by Arzela-Ascoli's theorem, $H(L)$ is compact. Ultimately, Ł is ξ-Lipschitz with constant zero. □

Theorem 3.4. *Assume that* $(\mathfrak{L}_1),(\mathfrak{L}_2)$ *are holding true and* $K=\Omega(c+d)<1$. *Then, the system of nonlinear FDEs with* ϕ_p *(1.1) has a solution and the set containing all the solutions is bounded in* L.

Proof. For existence of solution of the system (1.1), we take a help from Theorem (2.11). Let us consider the following set

$$X=\{(v,u)\in L: \text{ there exist } 0\leq\sigma\leq 1, \text{such that } \sigma H(v,u)=(v,u)\}.$$

Here, we show that $X$is a bounded set. For this, we assume a contrary path. Let for some $(v,u)\in X$, we have $\|(v,u)\|=B\to\infty$. From Theorem 3.2, we have

$$\begin{aligned}\|(v,u)\| =&\|\sigma H(v,u)\|\leq\|H(v,u)\|\\ \leq& K\|(v,u)\|+M.\end{aligned}\tag{3.19}$$

Since $\|(v,u)\|=B$, then (3.28) implies

$$\begin{aligned}\|(v,u)\| \leq& K\|(v,u)\|+M,\\ 1\leq& K+\frac{M}{\|(v,u)\|},\\ 1\leq& K+\frac{M}{B}.\end{aligned}$$

This implies that $1\leq K$ as $B\to\infty$, but this is a contradiction of our assumption. Ultimately $\|(v,u)\|<\infty$ and hence the set X is bounded and therefore, by Theorem 2.11, the operator H has at least one fixed point which is the solution of the coupled system of FDEs (1.1). And the set containing all the solutions of the system of FDEs (1.1) is bounded in L. □

Theorem 3.5. *Let* $(\mathfrak{L}_1)$, $(\mathfrak{L}_2)$ *are holding true. Then the system of FDEs with nonlinear operator of p-Laplacian (1.1) has a unique solution with the assumption* $\Delta<1$, *where*

$$\begin{aligned}\Delta=&[(q-1)\rho_1^{q-2}\Upsilon_{\psi_1}(\frac{2}{\Gamma(\alpha_1+1)}+\frac{1}{\Gamma(n)\Gamma(\alpha_1-n+2)})\\ &.(\frac{1}{\Gamma(\beta_1+1)}+\frac{1}{\Gamma(n)\Gamma(\beta_1-n+2)})\\ &+(q-1)\rho_2^{q-2}\Upsilon_{\psi_1}(\frac{2}{\Gamma(\alpha_2+1)}+\frac{1}{\Gamma(n)\Gamma(\alpha_2-n+2)})\\ &.(\frac{1}{\Gamma(\beta_2+1)}+\frac{1}{\Gamma(n)\Gamma(\beta_2-n+2)})].\end{aligned}$$

Proof. Using (3.16), (3.17), and assumptions $(\mathfrak{L}_1)$, $(\mathfrak{L}_2)$, we have

$$\begin{aligned}&|H_1v(t)-H_1v(t)|\\ =&|\int_0^1 G^{\alpha_1}(t,s)\phi_q(\int_0^1 G^{\beta_1}(s,\tau)\psi_1(u(\tau))d\tau)ds\\ &-\int_0^1 G^{\alpha_1}(t,s)\phi_q(\int_0^1 G^{\beta_1}(s,\tau)\psi_1(u(\tau))d\tau)ds|\\ =&|\int_0^1 G^{\alpha_1}(t,s)[\phi_q(\int_0^1 G^{\beta_1}(s,\tau)\psi_1(u(\tau))d\tau)\\ &-\phi_q(\int_0^1 G^{\beta_1}(s,\tau)\psi_1(u(\tau))d\tau)]ds|\\ \leq&\int_0^1|G^{\alpha_1}(t,s)||\phi_q(\int_0^1 G^{\beta_1}(s,\tau)\psi_1(u(\tau))d\tau)\end{aligned}$$

$$
\begin{aligned}
&\quad - \phi_q(\int_0^1 G^{\beta_1}(s,\tau)\psi_1(u(\tau))d\tau)|ds \\
&\leq (q-1)\rho_1^{q-2}\int_0^1 |G^{\alpha_1}(t,s)||\int_0^1 G^{\beta_1}(s,\tau)\psi_1(u(\tau))d\tau \\
&\quad - \int_0^1 G^{\beta_1}(s,\tau)\psi_1(u(\tau))d\tau|ds \\
&\leq (q-1)\rho_1^{q-2}\int_0^1 |G^{\alpha_1}(t,s)|\int_0^1 |G^{\beta_1}(s,\tau)||\psi_1(\tau,u(\tau)) \\
&\quad - \psi_1(\tau,u(\tau))|d\tau ds \\
&\leq (q-1)\rho_1^{q-2}\Upsilon_{\psi_1}|u(\tau)-u(\tau)|\int_0^1 |G^{\alpha_1}(t,s)|\int_0^1 |G^{\beta_1}(s,\tau)|d\tau ds \\
&\leq (q-1)\rho_1^{q-2}\Upsilon_{\psi_1}|u(\tau)-u(\tau)| \\
&\quad .(\frac{1}{\Gamma(\beta_1+1)}+\frac{1}{\Gamma(n)\Gamma(\beta_1-n+2)})\int_0^1 |G^{\alpha_1}(t,s)|ds \\
&\leq (q-1)\rho_1^{q-2}\Upsilon_{\psi_1}(\frac{2}{\Gamma(\alpha_1+1)}+\frac{1}{\Gamma(n)\Gamma(\alpha_1-n+2)}) \\
&\quad .(\frac{1}{\Gamma(\beta_1+1)}+\frac{1}{\Gamma(n)\Gamma(\beta_1-n+2)}) \\
&\quad .|u(\tau)-u(\tau)|.
\end{aligned}
\tag{3.20}
$$

Also, we have

$$
\begin{aligned}
&|H_2u(t)-H_2u(t)| \\
&\quad =|\int_0^1 G^{\alpha_2}(t,s)\phi_q(\int_0^1 G^{\beta_2}(s,\tau)\psi_2(v(\tau))d\tau)ds \\
&\qquad - \int_0^1 G^{\alpha_2}(t,s)\phi_q(\int_0^1 G^{\beta_2}(s,\tau)\psi_2(v(\tau))d\tau)ds| \\
&\quad =|\int_0^1 G^{\alpha_2}(t,s)[\phi_q(\int_0^1 G^{\beta_2}(s,\tau)\psi_2(v(\tau))d\tau) \\
&\qquad - \phi_q(\int_0^1 G^{\beta_2}(s,\tau)\psi_2(v(\tau))d\tau)]ds| \\
&\quad \leq \int_0^1 |G^{\alpha_2}(t,s)||\phi_q(\int_0^1 G^{\beta_2}(s,\tau)\psi_2(v(\tau))d\tau) \\
&\qquad - \phi_q(\int_0^1 G^{\beta_2}(s,\tau)\psi_2(v(\tau))d\tau)|ds \\
&\quad \leq (q-1)\rho_2^{q-2}\int_0^1 |G^{\alpha_2}(t,s)||\int_0^1 G^{\beta_2}(s,\tau)\psi_2(v(\tau))d\tau \\
&\qquad - \int_0^1 G^{\beta_2}(s,\tau)\psi_2(v(\tau))d\tau|ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq (q-1)\rho_2^{q-2}\int_0^1 |G^{\alpha_2}(t,s)| \int_0^1 |G^{\beta_2}(s,\tau)||\psi_2(\tau,v(\tau))\\
&\quad -\psi_2(\tau,v(\tau))|d\tau ds \qquad (3.21)\\
&\leq (q-1)\rho_2^{q-2}\Upsilon_{\psi_2}|v(\tau)-v(\tau)|\\
&\quad .\int_0^1 |G^{\alpha_2}(t,s)| \int_0^1 |G^{\beta_2}(s,\tau)|d\tau ds\\
&\leq (q-1)\rho_2^{q-2}\Upsilon_{\psi_2}|v(\tau)-v(\tau)|\\
&\quad .(\frac{1}{\Gamma(\beta_2+1)}+\frac{1}{\Gamma(n)\Gamma(\beta_2-n+2)})\int_0^1 |G^{\alpha_2}(t,s)|ds\\
&\leq (q-1)\rho_2^{q-2}\Upsilon_{\psi_1}(\frac{2}{\Gamma(\alpha_2+1)}+\frac{1}{\Gamma(n)\Gamma(\alpha_2-n+2)})\\
&\quad .(\frac{1}{\Gamma(\beta_2+1)}+\frac{1}{\Gamma(n)\Gamma(\beta_2-n+2)})\\
&\quad .|v(\tau)-v(\tau)|.
\end{aligned}
$$

Using (3.29) and (3.30), we get

$$
\begin{aligned}
&|H(v,u)-H(v,u)|\\
&\quad \leq (q-1)\rho_1^{q-2}\Upsilon_{\psi_1}(\frac{2}{\Gamma(\alpha_1+1)}+\frac{1}{\Gamma(n)\Gamma(\alpha_1-n+2)})\\
&\qquad .(\frac{1}{\Gamma(\beta_1+1)}+\frac{1}{\Gamma(n)\Gamma(\beta_1-n+2)})|u(\tau)-u(\tau)|\\
&\qquad +(q-1)\rho_2^{q-2}\Upsilon_{\psi_1}(\frac{2}{\Gamma(\alpha_2+1)}+\frac{1}{\Gamma(n)\Gamma(\alpha_2-n+2)})\\
&\qquad .(\frac{1}{\Gamma(\beta_2+1)}+\frac{1}{\Gamma(n)\Gamma(\beta_2-n+2)})\\
&\qquad .|v(\tau)-v(\tau)|\\
&\quad \leq [(q-1)\rho_1^{q-2}\Upsilon_{\psi_1}(\frac{2}{\Gamma(\alpha_1+1)}+\frac{1}{\Gamma(n)\Gamma(\alpha_1-n+2)})\\
&\qquad .(\frac{1}{\Gamma(\beta_1+1)}+\frac{1}{\Gamma(n)\Gamma(\beta_1-n+2)})+(q-1)\rho_2^{q-2}\Upsilon_{\psi_1}\\
&\qquad .(\frac{2}{\Gamma(\alpha_2+1)}+\frac{1}{\Gamma(n)\Gamma(\alpha_2-n+2)})\\
&\qquad .(\frac{1}{\Gamma(\beta_2+1)}+\frac{1}{\Gamma(n)\Gamma(\beta_2-n+2)})]\|(v,u)(\tau)-(v,u)(\tau)\|\\
&\quad =\Delta\|(v,u)(\tau)-(v,u)(\tau)\|. \qquad (3.22)
\end{aligned}
$$

By Banach's FPT and with the assumption$\Delta<1$, (3.30) implies that the contraction H has a unique fixed point. Consequently, solution of the system of nonlinear FDEs with ϕ_p (1.1) is unique. □

4. Hyers-Ulam stability

This section is reserved for studying Hyers-Ulam stability of nonlinear system of FDEs with ϕ_p (1.1). In literature, we could not find published work describing Hyers-Ulam stability for high order nonlinear system of FDEs with ϕ_p operator. Therefore, this may get the attention of researchers to the study of Hyers-Ulam stability as well many other types of stabilities for more complex problems. Following the definitions of Hyers-Ulam stability as suggested in [3, 16, 33], we propose a definition below:

Definition 4.1. The coupled system of integral equations given by (3.12), is said to be Hyers-Ulam stable provided that there are some positive constants A_1, A_2 which satisfy the conditions

$$|v(t) - \int_0^1 G^{\alpha_1}(t,s)\phi_q(\int_0^1 G^{\beta_1}(s,\tau)\psi_1(u(\tau))d\tau)ds| \leq \zeta_1,$$
$$|u(t) - \int_0^1 G^{\alpha_2}(t,s)\phi_q(\int_0^1 G^{\beta_2}(s,\tau)\psi_2(v(\tau))d\tau)ds| \leq \zeta_2,$$

there exists a pair, say $(\bar{v}(t), \bar{u}(t))$, satisfying

$$\bar{v}(t) = \int_0^1 G^{\alpha_1}(t,s)\phi_q(\int_0^1 G^{\beta_1}(s,\tau)\psi_1(\bar{u}(\tau))d\tau)ds,$$
$$\bar{u}(t) = \int_0^1 G^{\alpha_2}(t,s)\phi_q(\int_0^1 G^{\beta_2}(s,\tau)\psi_2(\bar{v}(\tau))d\tau)ds.$$

Such that

$$|v(t) - \bar{v}(t)| \leq A_1\zeta_2, |u(t) - \bar{u}(t)| \leq A_2\zeta_1.$$

We define the following terms for the simplicity in our calculation:

$$\begin{aligned} A_1 =&(q-1)\rho_1^{q-2}\Upsilon_{\psi_1}(\frac{2}{\Gamma(\alpha_1+1)} + \frac{1}{\Gamma(n)\Gamma(\alpha_1-n+2)}) \\ &.(\frac{1}{\Gamma(\beta_1+1)} + \frac{1}{\Gamma(n)\Gamma(\beta_1-n+2)}), \\ A_2 =&(q-1)\rho_2^{q-2}\Upsilon_{\psi_1}(\frac{2}{\Gamma(\alpha_2+1)} + \frac{1}{\Gamma(n)\Gamma(\alpha_2-n+2)}) \\ &.(\frac{1}{\Gamma(\beta_2+1)} + \frac{1}{\Gamma(n)\Gamma(\beta_2-n+2)}). \end{aligned}$$

Theorem 4.2. *With the assumptions $(\mathfrak{L}_1), (\mathfrak{L}_2)$, solution of the nonlinear system of FDEs with ϕ_p (1.1), is Hyers-Ulam stable.*

Proof. From Theorem 3.4 and Definition 4.1, let $(v(t), u(t))$ be a solution of the system (3.12). Let $(\bar{v}(t), \bar{u}(t))$ be any other approximation satisfying (4.1, 4.2). Then, we have

$$|v(t) - \bar{v}(t)| = \left| \begin{array}{l} \int_0^1 G^{\alpha_1}(t,s)\phi_q(\int_0^1 G^{\beta_1}(s,\tau)\psi_1(u(\tau))d\tau)ds \\ -\int_0^1 G^{\alpha_1}(t,s)\phi_q(\int_0^1 G^{\beta_1}(s,\tau)\psi_1(\bar{u}(\tau))d\tau)ds \end{array} \right|$$

$$
\begin{aligned}
&\le \int_0^1 |G^{\alpha_1}(t,s)||\phi_q(\int_0^1 G^{\beta_1}(s,\tau)\psi_1(u(\tau))d\tau) \\
&\quad - \phi_q(\int_0^1 G^{\beta_1}(s,\tau)\psi_1(\bar{u}(\tau))d\tau)|ds \\
&\le (q-1)\rho_1^{q-2}\int_0^1 |G^{\alpha_1}(t,s)|\int_0^1 |G^{\beta_1}(s,\tau)||\psi_1(u(\tau)) \\
&\quad - \psi_1(\bar{u}(\tau))|d\tau ds \\
&\le (q-1)\rho_1^{q-2}\Upsilon_{\psi_1}|u(t)-\bar{u}(t)| \\
&\quad .\int_0^1 |G^{\alpha_1}(t,s)|\int_0^1 |G^{\beta_1}(s,\tau)|d\tau ds \\
&\le (q-1)\rho_1^{q-2}\Upsilon_{\psi_1}(\frac{2}{\Gamma(\alpha_1+1)}+\frac{1}{\Gamma(n)\Gamma(\alpha_1-n+2)}) \\
&\quad .(\frac{1}{\Gamma(\beta_1+1)}+\frac{1}{\Gamma(n)\Gamma(\beta_1-n+2)})|u(t)-\bar{u}(t)| \\
&\le A_1\zeta_2.
\end{aligned} \tag{4.1}
$$

Also, we have

$$
\begin{aligned}
|u(t)-\bar{u}(t)| =& |\int_0^1 G^{\alpha_2}(t,s)\phi_q(\int_0^1 G^{\beta_2}(s,\tau)\psi_2(v(\tau))d\tau)ds \\
&- \int_0^1 G^{\alpha_2}(t,s)\phi_q(\int_0^1 G^{\beta_2}(s,\tau)\psi_2(\bar{v}(\tau))d\tau)ds| \\
\le& \int_0^1 |G^{\alpha_2}(t,s)||\phi_q(\int_0^1 G^{\beta_2}(s,\tau)\psi_2(v(\tau))d\tau) \\
&- \phi_q(\int_0^1 G^{\beta_2}(s,\tau)\psi_2(\bar{v}(\tau))d\tau)|ds \\
\le& (q-1)\rho_2^{q-2}\int_0^1 |G^{\alpha_2}(t,s)| \\
&.\int_0^1 |G^{\beta_2}(s,\tau)||\psi_2(v(\tau))-\psi_2(\bar{v}(\tau))|d\tau ds \\
\le& (q-1)\rho_2^{q-2}\Upsilon_{\psi_2}|v(t)-\bar{v}(t)|\int_0^1 |G^{\alpha_2}(t,s)|\int_0^1 |G^{\beta_2}(s,\tau)|d\tau ds \\
\le& (q-1)\rho_2^{q-2}\Upsilon_{\psi_1}(\frac{2}{\Gamma(\alpha_2+1)}+\frac{1}{\Gamma(n)\Gamma(\alpha_2-n+2)}) \\
&.(\frac{1}{\Gamma(\beta_2+1)}+\frac{1}{\Gamma(n)\Gamma(\beta_2-n+2)})|v(t)-\bar{v}(t)| \\
\le& A_2\zeta_1.
\end{aligned} \tag{4.2}
$$

From (4.6) and (4.7), the system (3.12) is Hyers-Ulam stable. Ultimately, (1.1) is Hyers-Ulam stable. This completed the proof. □

5. Example

Consider the following High order FDE system with P-Laplacian operator when $n = 3$

(5.1)
$$\begin{cases} {}^cD_0^{\frac{4}{3}}(\phi_4({}^cD_0^{\frac{5}{2}}(u(t) - \psi_2(t, v(t))))) = -\psi_1(t, v(t)), \\ {}^cD_0^{\frac{2}{3}}(\phi_4({}^cD_0^{\frac{7}{2}}(v(t) - \psi_4(t, u(t))))) = -\psi_3(t, u(t)), \\ (\phi_4({}^cD_0^{\frac{5}{2}}(u(t) - \psi_2(t, v(t)))))_{t=0}^{(i)} = (\phi_4({}^cD_0^{\frac{5}{2}}(u(t) - \psi_2(t, v(t)))))_{t=0.5}' = 0, \\ (\phi_4({}^cD_0^{\frac{7}{2}}(v(t) - \psi_4(t, u(t)))))_{t=0}^{(i)} = (\phi_4({}^cD_0^{\frac{7}{2}}(v(t) - \psi_4(t, u(t)))))_{t=0.5}' = 0, \\ \text{for } i = 0, 2, \\ (\psi_2(t, v(t)))_{t=0}^{(i)} = (\psi_4(t, u(t)))_{t=0}^{(i)} = 0, \\ u^{(i)}(t)_{t=0} = u^{(2)}(t)_{t=1} = 0, v^{(i)}(t)_{t=0} = v^{(2)}(t)_{t=1} = 0. \\ u(1) - \frac{u^{(2)}(0)}{(2)!} = 0, v(1) - \frac{v^{(2)}(0)}{(2)!} = 0, for i = 1, 2, \end{cases}$$

where $t \in [0, 1], a_1 = a_2 = d_1 = d_2 = 0.3, p = 4, q = \frac{4}{3}, \lambda = 0.5, \alpha_1 = \frac{5}{2}, \alpha_2 = \frac{8}{3}, \beta_1 = \frac{9}{4}, \beta_2 = \frac{11}{4}, \rho_1 = \rho_2 = 1, \psi_1 = \frac{5}{16}t + \sin(v(t)), \psi_2 = \sqrt{t}(\frac{17}{26} + 5cos(v(t)))$, $\psi_3 = \frac{4}{13}t^2 + \cos(u(t)), \psi_4 = \sqrt[3]{t}(\frac{5}{16}t + \sin(u(t)))$, $\varepsilon_{\psi_1} = \varepsilon_{\psi_3} = \frac{3}{8}, \varepsilon_{\psi_2} = \varepsilon_{\psi_4} = \frac{1}{8}$. By doing a simple calculation, we get $A_3 = 0.111259, A_4 = 0.0540563$. So, $Z_1 = 0.1653153, Z_2 = 0.25$. Then, we have $Z_1 + Z_2 = 0.4153153 < 1$. By Theorem 3.5, we conclude that the problem (5.1) has unique solution. With similar fashion, the satisfaction of the conditions of Theorem 4.2 can be checked easily and consequently the problem (5.1) is Hyers-Ulam stable.

6. Conclusion

We have considered the study about stability and existence results for a nonlinear system of FDEs in Caputo's sense which is more general and complex than many nonlinear problems in the literature. In this paper, three important aspects of the system of nonlinear FDEs with ϕ_p (1.1) were considered including existence of solution, uniqueness of solution and Hyers-Ulam stability of the proposed problem. For these objectives, we converted the problem (1.1) to a system of integral equations by means of Green functions. After this, we proved results for existence and uniqueness by topological degree method. By the use of this technique, we do not need to the assumption of the compactness of the operator. Then after, Hyers-Ulam stability was investigated. In literature, we could not find published work describing the Hyers-Ulam stability of high order system of nonlinear FDEs with ϕ_p operator. Therefore, this work may get the attention of researchers to the study of Hyers-Ulam stability as well many other types of stabilities for more complex problems. We also suggest the readers that the problem (1.1) has potentials to be studied for further aims including multiplicity results.

References

[1] H. Aktuglu and M. Ozarslan, *Solvability of differential equations of order* $2 < \varepsilon \leq 3$ *involving the p-Laplacian operator with boundary conditions* Adv. Differ. Equ. **358** (1985), 1–13.

[2] A. Ali, B. Samet, K. Shah and R. A. Khan, *Existence and stability of solution to a toppled systems of differential equations of non-integer order*, Bound. Value Prob. (2017), 13pp.

[3] G. A. Anastassiou, *On right fractional calculus*, Chaos Solitons Fractals **42** (2009), 365–376.
[4] D. Baleanu, R. P. Agarwal, H. Khan, R. A. Khan and H. Jafari, *On the existence of solution for fractional differential equations of order* $3 < \delta_1 \leq 4$, Adv. Difference Equ. (2015), 9 pp.
[5] D. Baleanu, R. P. Agarwal, H. Mohammadi and S. Rezapour, *Some existence results for a nonlinear fractional differential equation on partially ordered Banach spaces*, Bound. Value Probl. (2013), 8 pp.
[6] D. Baleanu, H. Jafari, H. Khan and S. J. Johnston, *Results for Mild solution of fractional coupled hybrid boundary value problems*, Open Math. **13** (2015), 601–608.
[7] D. Baleanu, H. Khan, H. Jafari, R. A. Khan and M. Alipour, *On existence results for solutions of a coupled system of hybrid boundary value problems with hybrid conditions*, Adv. Difference Equ. **318** (2015), 14 pp.
[8] D. Baleanua, O. G. Mustafa and R. P. Agarwal, *An existence result for a superlinear fractional differential equation*, Appl. Math. Lett. **23** (2010), 1129–1132.
[9] D. Baleanu, O. G. Mustafa and R. P. Agarwal, *On the solution set for a class of sequential fractional differential equations*, J. Phys. A **43** (2010), 7 pp.
[10] V. L. Chinchane and D. B. Pachpatte, *Certain inequalities using Saigo fractional intregral operator*, Ser. Math. Inform. **29**, (2014), 343–350
[11] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.
[12] B. Dhage and N. Jadhav, *Basic results in the theory of hybrid differential equations with linear perturbations of second type*, Tamkang J. Math. **44** (2013), 171–186.
[13] P. Gavruta, S. M. Jung and Y. Li, *Hyers-Ulam stability for second-order linear differential equations with boundary conditions*, Electron. J. Differential Equations (2011), 5 pp.
[14] L. Hu and S. Zhang, *Existence results for a coupled system of fractional differential equations with p-Laplacian operator and infinite-point boundary conditions*, Bound. Value Prob. (2017), 16 pp.
[15] Z. Hu, W. Liu and J. Liu, *Existence of solutions for a coupled system of fractional p-Laplacian equations at resonance*, Adv. Difference Equ. (2013), 14 pp.
[16] R. W. Ibrahim and H. A. Jalab, *Existence of Ulam Stability for Iterative Fractional Differential Equations Based on Fractional Entropy*, Entropy **17** (2017), 3172–3181.
[17] F. Isaia, *On a nonlinear integral equation without compactness*, Acta Math. Univ. Comenian. (N.S.) **75** (2006), 233–240.
[18] H. Jafari, D. Baleanu, H. Khan, R. A. Khan and A. Khan, *Existence criterion for the solution of fractional order p-Laplacian boundary value problem*, Bound. Value Probl. (2015), 10 pp.
[19] R. A. Khan and A. Khan, *Existence and uniqueness of solutions for p-Laplacian fractional order boundary value problems*, Comput. Methods Differ. Equ. **2** (2014), 205–215.
[20] B. L. Li and H. D. Gou, *Existence results of mild solutions for Impulsive fractional evolutions with periodic boundary condition*, Int. J. Nonlinear Sci. Numer. Simul. **18** (2017), 585–598.
[21] Y. Li, *Existence of positive solutions for fractional differential equations involving integral boundary conditions with p-Laplacian operator*, Adv. Difference Equ. (2017), 11 pp.
[22] N. I. Mahmudov and S. Unul, *Existence of solutions of* $\alpha \in (2, 3]$ *order fractional three-point boundary value problems with integral conditions*, Abstr. Appl. Anal. (2014), 12 pp.
[23] N. I. Mahmudov and S. Unul, *Existence of solutions of fractional boundary value problems with p-Laplacian operator*, Bound. Value Probl. (2015), 16 pp.
[24] N. I. Mahmudov and S. Unul, *On existence of BVPs for impulsive fractional differential equations*, Adv. Difference Equ. (2017), 16 pp.
[25] I. Podlubny, *Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, Mathematics in Science and Engineering, 198. Academic Press, Inc., San Diego, CA, 1999.
[26] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science, Yverdon, Switzerland, 1993.
[27] T. Shen, W. Liu and X. Shen, *Existence and uniqueness of solutions for several BVPs of fractional differential equations with p-Laplacian operator*, Mediterr. J. Math. **13** (2016), 4623–4637.

[28] S. Sitho, S. K. Ntouyas and J. Tariboon, *Existence results for hybrid fractional integro-differential equations*, Bound. Value Probl. (2015), 13 pp.

[29] H. M. Srivastava, *An application of the fractional derivative*, Math. Japon. **29** (1984), 383–389.

[30] H. M. Srivastava and V. Gupta, *A certain family of summation-integral type operators*, Math. Comput. Modelling **42** (2005), 181–191.

[31] H. M. Srivastava and A. K. Mishra, *Applications of fractional calculus to parabolic starlike and uniformly convex functions*, Comput. Math. Appl. **39** (2000), 57–69.

[32] C. Urs, *Coupled fixed point theorems and applications to perodic boundary value problems*, Miskolc Math. Notes **14** (2013), 323–333.

[33] S. Xie and Y. Xie, *Positive solutions of higher-order nonlinear fractional differential system with nonlocal boundary conditions*, J. Appl. Anal. Comput. **6** (2016), 1211–1227.

[34] A. Zada, S. Faisal and Y. Li, *Hyers-Ulam-Rassias stability of non-linear delay differential equations*, J. Nonlinear Sci. Appl. **10** (2017), 504–510.

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