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ON AN INITIAL VALUE PROBLEM OF A DELAY-REFEREED DIFFERENTIAL EQUATION

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Abstract. In this paper we study the existence of solutions for an initial value problem of a delay refereed differential equation. The continuous dependence of the unique solution on the initial data will be proved. Some especial cases and examples will be given.

1. Introduction

For the differential and integral equations with deviating arguments, the deviation of the argument usually involves only the time itself, however, the case of the deviating arguments depend on both the state variable x and the time t is very important in theory and practice. Several papers that are devoted to such kind of differential equations, (see for example [2]-[4], [6]-[13], [14], [15], [18] and [19]). These kind of equations play an important role in nonlinear analysis, and have many applications in most fields especially in the class of problems that have past memories for example in hereditary phenomena (see for example [1], [20], [21], [22] and [23]). Now consider the initial value problem

$$\frac{dx(t)}{dt} = f(t, x(\phi(x(t)))),\ t \in (0, T] \tag{1.1}$$

$$x(0) = x_0. \tag{1.2}$$

Our aim in this work is to study the initial value problem (1.1), (1.2) which is equivalent to the integral equation

$$x(t) = x_0 + \int_0^t f(s, x(\phi(x(s)))),\ t \in [0, T], \tag{1.3}$$

we prove the existence and the uniqueness of the solution and the continuous dependence on the initial data under two different groups of conditions.

2. Group (1) of conditions

(1) $f : [0, T] \times R \to R$ satisfies Carathéodory condition i.e. f is measurable in t for all $x \in C[0, T]$ and continuous in x for almost all $t \in [0, T]$.
(2) There exists a measurable bounded function $m(t)$ and a constant $b \geq 0$ such that $|f(t, x)| \leq m(t) + b|x|$.
(3) $\phi :\ R \to [0, T]$ is continuous such that $\phi(x(t)) \leq t$.
(4) $b\,T < 1$.

Some examples for the function ϕ

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(1) $\phi(x(t)) = \frac{t\ \sin^2 x(t)}{1+x^2(t)}$,
(2) $\phi(x(t)) = \frac{t\ e^{-x^2(t)}}{1+x^2(t)}$,
(3) $\phi(x(t)) = \frac{t}{1+e^{-x^2(t)}}$.

2.1. **Existence of solutions.**

Theorem 2.1. *Let the assumptions* (1) − (4) *be satisfied, then the initial value problem (1.1), (1.2) has at least one solution* $x \in C[0,T]$.

Proof. Define the set S_L by

$$S_L = \left\{x \in C[0,T] : |x(t_2) - x(t_1)| \leq L|t_2 - t_1|\right\} \subset C[0,T].$$

where $L = \frac{M+b|x_0|}{1-bT}$.

It clear that S_L is nonempty, closed, bounded and convex subset of $C[0,T]$. Define the operator F associated with equation (1.1) by

$$Fx(t) = x_0 + \int_0^t f(s, x(\phi(x(s))))ds \ t \in [0,T].$$

First, we prove that F is uniformly bounded. Let $x \in C[0,T]$, then for $t \in [0,T]$ we get

$$\begin{aligned} |Fx(t)| &\leq |x_0| + \int_0^t |f(s, x(\phi(x(s))))|ds \ t \in [0,T] \\ &\leq |x_0| + \int_0^t \{m(s) + b|x(\phi(x(s))))|\}ds \\ &\leq |x_0| + \int_0^t \{M + b|x(\phi(x(s))))|\}ds. \end{aligned}$$

But

$$|x(\phi(x(t))))| - |x_0| \leq L|x(\phi(x(t)))) - x(0)| \leq L|\phi(x(s))|,$$

thus

$$|x(\phi(x(t))))| \leq L|\phi(x(s))| + |x_0|. \tag{2.1}$$

Using (2.1), we get

$$\begin{aligned} |Fx(t)| &\leq |x_0| + \int_0^t \{M + b(L|\phi(x(s))| + |x_0|)\}ds \\ &\leq |x_0| + \int_0^t \{M + b\ (Ls + |x_0|)\}ds \\ &\leq |x_0| + (M + b(LT + |x_0|))t \\ &\leq LT + |x_0|. \end{aligned}$$

This proves that the class functions $\{Fx\}$ is uniformly bounded. Second, we will show that $F : S_L \to S_L$ and the class of functions $\{Fx\}$ is equi-continuous. Let

$x \in S_L$ and $t_1,\ t_2 \in [0,T]$ with $t_1 < t_2$ such that $|t_2, -t_1| < \delta$, then

$$\begin{aligned}
|Fx(t_2) - Fx(t_1)| &= |\int_{t_1}^{t_2} f(s, x(\phi(x(s))))ds| \\
&\leq \int_{t_1}^{t_2} |f(s, x(\phi(x(s))))|ds \\
&\leq \int_{t_1}^{t_2} \{M + b|x(\phi(x(s))))|\}ds \\
&\leq \int_{t_1}^{t_2} \{M + b(L|\phi(x(s))| + |x_0|)\}ds \\
&\leq \int_{t_1}^{t_2} (M + b(Ls + |x_0|))ds \\
&\leq L|t_2 - t_1|.
\end{aligned}$$

This proves that $F : S_L \to S_L$ and the class of functions $\{Fx\}$ is equi-continuous. Now by Arzela-Ascoli Theorem [17], F is compact. Finally, we will show that F is continuous. Let $\{x_n\} \subset S_L$ such that $x_n \to x$ uniformly on $[0,T]$, (i.e, $|x_n(t) - x(t)| \leq \epsilon_1$) this implies that $|x_n(\phi(x(t))) - x(\phi(x(t)))| \leq \epsilon_2$ thus for arbitrary $\epsilon_1, \epsilon_2 \geq 0$, we have

$$\begin{aligned}
&|x_n(\phi(x_n(t))) - x(\phi(x(t)))| \\
&\quad = |x_n(\phi(x_n(t))) - x_n(\phi(x(t))) + x_n(\phi(x(t))) - x(\phi(x(t)))| \\
&\quad \leq |x_n(\phi(x_n(t))) - x_n(\phi(x(t)))| + |x_n(\phi(x(t))) - x(\phi(x(t)))| \\
&\quad \leq L|\phi(x_n(t)) - \phi(x(t))| + |x_n(\phi(x(t))) - x(\phi(x(t)))| \\
&\quad \leq \epsilon
\end{aligned}$$

which implies that

$$x_n(\phi(x_n(t))) \to x(\phi(x(t)))\ in\ S_L.$$

Now the function f is continuous in the second argument, then

$$f\big(t, x_n(\phi(x_n(t)))\big) \to f\big(t, x(\phi(x(t)))\big).$$

Using assumption (2) and Lebesgues dominated convergence Theorem [5], we have

$$\begin{aligned}
\lim_{n\to\infty} x_n(t) &= x_0 + \lim_{n\to\infty} \int_0^t f\big(s, x_n(\phi(x_n(s)))\big)ds \\
&= x_0 + \int_0^t f\big(s, x(\phi(x(s)))\big)ds.
\end{aligned}$$

Then F is continuous. Now all conditions of Schauder fixed point Theorem [16] , are satisfied, then the operator F has at least one fixed point $x \in S_L$. Consequently there exist at leat one solution $x \in C[0,T]$ of equation (1.3) which completes the proof. □

Corollary 2.2. *Let the assumptions of Theorem 1.1 be satisfied, if $\phi(x(t)) = t$, then the integral equation*

$$x(t) = x_0 + \int_0^t f(s, x(s))ds,\ t \in [0,T]$$

has at least one solution $x \in C[0,T]$. *Consequently the initial value problem*

$$\begin{aligned} \frac{dx(t)}{dt} &= f(t,x(t)) \ a.e. \ t \in (0,T] \\ x(0) &= x_0 \end{aligned}$$

has at least one solution $x \in AC[0,T]$.

2.2. **Uniqueness of the solution.** In this section we prove the uniqueness of the solution for the integral equation (1.3). For this aim we assume that:

(1′) $|f(t,x) - f(t,y)| \leq b\ |x-y|$.
(2′) $|f(t,0)| \leq M$.
(3′) $|\phi(t) - \phi(s)| \leq k|t-s|$.

Theorem 2.3. *Let the assumptions (1),(3),(4) of Theorem 3.1 and* (1′), (2′) *and* (3′) *be satisfied, if* $bT(Lk+1) < 1$, *then the solution of equation (1.3) is unique.*

Proof. Assumption (2) of Theorem 3.1 can be deduced from assumptions (1′) *and* (2′). By putting $y = 0$ in (1′), we get

$$|f(t,x)| \leq b\ |x| + |f(t,0)|,$$

hence we deduce that all assumptions of Theorem 3.1 are satisfied. Then the solution of equation (1.3) exists. Now let $x,\ y$ be two solutions of (1.3), then

$$\begin{aligned} |x(t) - y(t)| &= \Big| \int_0^t f(s,x(\phi(x(s))))ds - \int_0^t f(s,y(\phi(y(s))))ds \Big| \\ &\leq \int_0^t |f(s,x(\phi(x(s)))) - f(s,y(\phi(y(s))))|ds \\ &\leq b \int_0^t |x(\phi(x(s))) - y(\phi(y(s)))|ds \\ &\leq b \int_0^t |x(\phi(x(s))) - x(\phi(y(s)))|ds \\ &+ b \int_0^t |x(\phi(y(s))) - y(\phi(y(s)))|ds \\ &\leq bL \int_0^t |\phi(x(s)) - \phi(y(s))|ds \\ &+ b \int_0^t |x(\phi(y(s))) - y(\phi(y(s)))|ds \\ &\leq b\ L\ k \int_0^t |x(s) - y(s)|ds \\ &+ b \int_0^t |x(\phi(y(s))) - y(\phi(y(s)))|ds \\ &\leq b\ L\ k\ T\|x-y\| + b\ T\|x-y\| \\ &= b\ T\ (L\ k+1)\|x-y\|, \end{aligned}$$

thus we have

$$\|x-y\| \leq b\ T\ (L\ k+1)\ \|x-y\|.$$

Since $b\ T\ (L\ k+1) < 1$, then we get $x = y$ and hence the solution of (1.3) is unique. □

2.3. **Continuous dependence.** Here we prove that the solution of equation (1.3) depends continuously of the initial data x_0.

Definition 2.4. The solution of the integral equation (1.3) depends continuously on the initial data x_0 *if* $\forall\ \epsilon > 0\ \exists\ \delta(\epsilon) > 0$ *such that*,

$$|x_0 - x_0^*| \leq \delta \Rightarrow \|x - x^*\| \leq \epsilon$$

where x^* is the unique solution of the equation

$$x^*(t) = x_0^* + \int_0^t f(s, x^*(\phi(x^*(s))))ds,\ t \in [0, T]. \tag{2.2}$$

Theorem 2.5. *Let the assumptions of Theorem 3.3 be satisfied, assume that* $|x_0 - x_0^*| \leq \delta$, *then the solution of (1.3) depends continuously on the initial data* x_0.

Proof.

$$\begin{aligned}
|x(t) - x^*(t)| &= |x_0 + \int_0^t f(s, x(\phi(x(s))))ds - x_0^* + \int_0^t f(s, x^*(\phi(x^*(s))))ds| \\
&\leq |x_0 - x_0^*| + \int_0^t |f(s, x(\phi(x(s)))) - f(s, x^*(\phi(x^*(s))))|ds \\
&\leq |x_0 - x_0^*| + b \int_0^t |x(\phi(x(s))) - x^*(\phi(x^*(s)))|ds \\
&\leq |x_0 - x_0^*| + b\ L \int_0^t |\phi(x(s)) - \phi(x^*(s))|ds \\
&+ b \int_0^t |x(\phi(x^*(s))) - x^*(\phi(x^*(s)))|ds \\
&\leq |x_0 - x_0^*| + b\ L\ k \int_0^t |x(s) - x^*(s)|ds \\
&+ b \int_0^t |x(\phi(x^*(s))) - x^*(\phi(x^*(s)))|ds \\
&\leq \delta + b\ L\ k\ \|x - x^*\|\ T + b\|x - x^*\|\ T,
\end{aligned}$$

thus

$$\|x - x^*\| \leq \delta + b\ T(L\ k\ + 1)\|x - x^*\|,$$

hence

$$\|x - x^*\| \leq \frac{\delta}{(1 - b\ T(L\ k\ + 1))}.$$

Since $b\ T(L\ k\ + 1) < 1$, then the solution of (1.3) depends continuously on the initial data x_0. □

3. Group (2) of conditions

(*i*) $f : [0,T] \times [0,T] \to R$ satisfies Carathéodory condition i.e. f is measurable in t for all $x \in C[0,T]$ and continuous in x for almost all $t \in [0,T]$.
(*ii*) There exist a measurable bounded function $m(t)$ and a constant $b \geq 0$ such that $|f(t,x)| \leq m(t) + b|x|$.
(*iii*) $\phi : [0,T] \to [0,T]$ is continuous such that $\phi(x(t)) \leq x(t)$.
(*iv*) $L_2 = M + b\,T < 1$.
(*v*) $L_2 T + |x(0)| \leq T$.

Some examples for the function ϕ

(1) $\phi(x(t)) = \frac{x(t)}{1+x^2(t)}$.
(2) $\phi(x(t)) = \frac{x(t)\ e^{-x^2(t)}}{1+\sin^2 x(t)}$.
(3) $\phi(x(t)) = \frac{x(t)}{1+e^{-x^2(t)}}$.

3.1. **Existence of solutions.**

Theorem 3.1. *Let the assumptions* $(i) - (v)$ *be satisfied, then the initial value problem (1.1), (1.2) has at least one solution* $x \in C[0,T]$.

Proof. Define the set S_{L_2} by

$$S_{L_2} = \left\{x \in C[0,T] : |x(t_2) - x(t_1)| \leq L_2|t_2 - t_1|\right\} \subset C[0,T].$$

where $L_2 = M + b\,T$.

It clear that S_{L_2} is nonempty, closed, bounded and convex subset of $C[0,T]$.
Define the operator F associated with equation (1.1) by

$$Fx(t) = x_0 + \int_0^t f(s, x(\phi(x(s))))ds \ t \in [0,T].$$

First, we prove that F is uniformly bounded. Let $x \in C[0,T]$, then for $t \in [0,T]$, we get

$$\begin{aligned} |Fx(t)| &\leq |x_0| + \int_0^t |f(s, x(\phi(x(s))))|ds \\ &\leq |x_0| + \int_0^t \{m(s) + b|x(\phi(x(s))))|\}ds \\ &\leq |x_0| + \int_0^t \{M + b|x(\phi(x(s))))|\}ds. \end{aligned}$$

But

$$|x(\phi(x(t))))| - |x_0| \leq L_2|x(\phi(x(t)))) - x(0)| \leq L_2|\phi(x(s))|,$$

thus

$$|x(\phi(x(t))))| \leq L_2|\phi(x(s))| + |x_0|. \tag{3.1}$$

Using (3.1), we get

$$\begin{aligned}
|Fx(t)| &\leq |x_0| + \int_0^t \{M + b(L_2|x(s)| + |x_0|)\}ds \\
&\leq |x_0| + \int_0^t \{M + b\ (LT + |x_0|)\}ds \\
&\leq |x_0| + (M + b(LT + |x_0|))t \\
&\leq |x_0| + (M + bT)T \\
&\leq L_2T + |x_0| \\
&\leq T.
\end{aligned}$$

This proves that the class functions $\{Fx\}$ is uniformly bounded. Second, we will show that $F : S_L \to S_L$ and the class of functions $\{Fx\}$ is equi-continuous. Let $x \in S_L$ and $t_1,\ t_2 \in [0, T]$ with $t_1 < t_2$ such that $|t_2, -t_1| < \delta$, then

$$\begin{aligned}
|Fx(t_2) - Fx(t_1)| &= |\int_{t_1}^{t_2} f(s, x(\phi(x(s))))ds| \\
&\leq \int_{t_1}^{t_2} |f(s, x(\phi(x(s))))|ds \\
&\leq \int_{t_1}^{t_2} \{M + b|x(\phi(x(s))))|\}ds \\
&\leq \int_{t_1}^{t_2} \{M + b(L_2|\phi(x(s))| + |x_0|)\}ds \\
&\leq \int_{t_1}^{t_2} (M + b(L_2|x(s)| + |x_0|))ds \\
&\leq \int_{t_1}^{t_2} (M + bT)ds \\
&\leq L_2|t_2 - t_1|.
\end{aligned}$$

This proves that $F : S_{L_2} \to S_{L_2}$ and the class of functions $\{Fx\}$ is equi-continuous. Now by Arzela-Ascoli Theorem F is compact. Finally, we will show that F is continuous. Let $\{x_n\} \subset S_{L_2}$ such that $x_n \to x$ uniformly on $[0, T]$, (i.e. $|x_n(t) - x(t)| \leq \epsilon_1$) this implies that $|x_n(\phi(x(t))) - x(\phi(x(t)))| \leq \epsilon_2$ thus for arbitrary $\epsilon_1, \epsilon_2 \geq 0$, we have

$$\begin{aligned}
&|x_n(\phi(x_n(t))) - x(\phi(x(t)))| \\
&= |x_n(\phi(x_n(t))) - x_n(\phi(x(t))) + x_n(\phi(x(t))) - x(\phi(x(t)))| \\
&\leq |x_n(\phi(x_n(t))) - x_n(\phi(x(t)))| + |x_n(\phi(x(t))) - x(\phi(x(t)))| \\
&\leq L_2|\phi(x_n(t)) - \phi(x(t))| + |x_n(\phi(x(t))) - x(\phi(x(t)))| \\
&\leq \epsilon
\end{aligned}$$

which implies that

$$x_n(\phi(x_n(t))) \to x(\phi(x(t)))\ in\ S_{L_2}.$$

Now the function f is continuous in the second argument, then

$$f\big(t, x_n(\phi(x_n(t)))\big) \to f\big(t, x(\phi(x(t)))\big).$$

Using assumption (2) and Lebesgues dominated convergence theorem, we have

$$\begin{aligned}\lim_{n\to\infty} x_n(t) &= x_0 + \lim_{n\to\infty}\int_0^t f\big(s, x_n(\phi(x_n(s)))\big)ds \\ &= x_0 + \int_0^t f\big(s, x(\phi(x(s)))\big)ds.\end{aligned}$$

Then F is continuous. Now all conditions of Schauder fixed point Theorem are satisfied, then the operator F has at least one fixed point $x \in S_{L_2}$. Consequently there exist at least one solution $x \in C[0,T]$ of equation (1.3) which completes the proof. □

Corollary 3.2. *Let the assumptions of Theorem 1.1 be satisfied, if $\phi(x(t)) = x(t)$, then the integral equation*

$$x(t) = x_0 + \int_0^t f(s, x(x(s)))ds, \ t \in [0,T],$$

has at least one solution $x \in C[0,T]$. Consequently the initial value problem

$$\begin{aligned}\frac{dx(t)}{dt} &= f(t, x(x(t))) \ a.e. \ t \in (0,T] \\ x(0) &= x_0\end{aligned}$$

has at least one solution $x \in AC[0,T]$.

3.2. **Uniqueness of the solution.** In this section we prove the uniqueness of the solution for the integral equation (1.3). For this aim we assume that:

(1′) $|f(t,x) - f(t,y)| \le b\,|x-y|$
(2′) $|f(t,0)| \le M$,
(3′) $|\phi(t) - \phi(s)| \le k|t-s|$

Theorem 3.3. *Let the assumptions (1),(3),(4) of Theorem 3.1 and* (1′), (2′) *and* (3′) *be satisfied, if $bT(L_2\ k+1) < 1$, then the solution of equation (1.3) is unique.*

Proof. Assumption (2) of Theorem 3.1 can be deduced from assumptions (1′) *and* (2′). By putting $y = 0$ in (1′), we get

$$|f(t,x)| \ \le \ b\,|x| + |f(t,0)|,$$

hence we deduce that all assumptions of Theorem 3.1 are satisfied. Then the solution of equation (1.3) exists. Now let $x,\ y$ be two solutions of (1.3), then

$$\begin{aligned}
|x(t)-y(t)| &= \Big|\int_0^t f(s,x(\phi(x(s))))ds - \int_0^t f(s,y(\phi(y(s))))ds\Big| \\
&\leq \int_0^t |f(s,x(\phi(x(s)))) - f(s,y(\phi(y(s))))|ds \\
&\leq b\int_0^t |x(\phi(x(s))) - y(\phi(y(s)))|ds \\
&\leq b\int_0^t |x(\phi(x(s))) - x(\phi(y(s)))|ds \\
&+ b\int_0^t |x(\phi(y(s))) - y(\phi(y(s)))|ds \\
&\leq bL_2\int_0^t |\phi(x(s)) - \phi(y(s))|ds \\
&+ b\int_0^t |x(\phi(y(s))) - y(\phi(y(s)))|ds \\
&\leq b\ L_2\ k\ \int_0^t |x(s) - y(s)|ds \\
&+ b\int_0^t |x(\phi(y(s))) - y(\phi(y(s)))|ds \\
&\leq b\ L_2\ k\ T\|x-y\| + b\ T\|x-y\| \\
&= b\ T\ (L_2\ k+1)\|x-y\|,
\end{aligned}$$

thus we have

$$\|x-y\| \quad \leq \quad b\ T\ (L_2\ k+1)\ \|x-y\|.$$

Since $b\ T\ (L_2\ k+1) < 1$, then we get $x = y$ and hence the solution of (1.3) is unique. □

3.3. **Continuous dependence.** Here we prove that the solution of equation (1.3) depends continuously of the initial data x_0.

Definition 3.4. The solution of the integral equation (1.3) depends continuously on the initial data x_0 *if* $\forall\ \epsilon > 0\ \exists\ \delta(\epsilon) > 0$ *such that*,

$$|x_0 - x_0^*| \leq \delta \Rightarrow \|x - x^*\| \leq \epsilon$$

where x^* is the unique solution of the equation

$$x^*(t) = x_0^* + \int_0^t f(s,x^*(\phi(x^*(s))))ds,\ t \in [0,T]. \tag{3.2}$$

Theorem 3.5. *Let the assumptions of Theorem 3.3 be satisfied, assume that* $|x_0 - x_0^*| \leq \delta$, *then the solution of (1.3) depends continuously on the initial data* x_0.

Proof.

$$\begin{aligned}
|x(t)-x^*(t)| &= |x_0+\int_0^t f(s,x(\phi(x(s))))ds - x_0^* + \int_0^t f(s,x^*(\phi(x^*(s))))ds| \\
&\leq |x_0-x_0^*| + \int_0^t |f(s,x(\phi(x(s)))) - f(s,x^*(\phi(x^*(s))))|ds \\
&\leq |x_0-x_0^*| + b\int_0^t |x(\phi(x(s))) - x^*(\phi(x^*(s)))|ds \\
&\leq |x_0-x_0^*| + b\ L\int_0^t |\phi(x(s)) - \phi(x^*(s))|ds \\
&+ b\int_0^t |x(\phi(x^*(s))) - x^*(\phi(x^*(s)))|ds \\
&\leq |x_0-x_0^*| + b\ L_2\ k\int_0^t |x(s)-x^*(s)|ds \\
&+ b\int_0^t |x(\phi(x^*(s))) - x^*(\phi(x^*(s)))|ds \\
&\leq \delta + b\ L_2\ k\ \|x-x^*\|\ T + b\|x-x^*\|\ T,
\end{aligned}$$

thus

$$\|x-x^*\| \leq \delta + b\ T(L_2\ k\ +1)\|x-x^*\|,$$

hence

$$\|x-x^*\| \leq \frac{\delta}{(1-b\ T(L_2\ k\ +1))}.$$

Since $b\,T(L\ k\ +1) < 1$ then the solution of (1.3) depends continuously on the initial data x_0. □

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