Applied Analysis and Optimization
Volume 6, Number 1, 2022, 123–147

Yokohama Publishers
ISSN 2189-1664 Online Journal

FRACTIONAL STUDY OF MHD CNTS MAXWELL NANOFLUID WITH HEAT GENERATION, RADIATION AND THERMODIFFUSION

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Abstract. Nanofluids are well known for their thermal conductivity as well as heat transfer phenomena. Carbon nanotubes have a great impact in nanotechnology. They have marvellous range of applications in electronics, energy storage, materials science, reinforcement of turbine blades and chemical processing. This paper is an analysis of flow of MHD carbon nanotubes (CNTs) of Maxwell nanofluid with heat generation, radiation and thermodiffusion. The equations for heat, mass and momentum are established in the terms of Caputo (C), Caputo-Fabrizio (CF) and the Atangana-Baleanu in Caputo sense (ABC) fractional derivatives. The solutions are evaluated by employing Laplace transform and inversion algorithm. The flow in momentum profile due to variability in the values of parameters are graphically illustrated among C, CF and ABC models. It is concluded that Atangana Baleanu fractional operator presents larger memory effects than Caputo and Caputo-Fabrizio fractional operators.

1. Introduction

Nowadays, carbon nanotubes (CNTs) have fascinated several researchers worldwide. Their stunning physical properties and very small dimensions makes them impressively useful in applied sciences, normal and artificial phenomena such as cooling instruments, crystal glowing, thermal exchange and various organic sciences. Pandey et al. [40] studied the effects of viscous dissipation and suction/injection on MHD flow of a nanofluid. At room temperature, thermal conductivity of CNT's nanofluid is six times greater than other material's nanofluid. This result has been diagnosed by Murshed et al. [36]. Ahmad et al. [5] discussed heat and mass transfer in Maxwell nanofluid with CNTs.

Through the past thirty years, fractional derivatives have fascinated multiple investigators as compared to classical derivatives. Also, fractional derivatives are more credible in mathematical modeling of real world problems [18,24,25,33]. In classical calculus, derivatives and integrals are uniquely computed. The similar situation exists in the case of fractional integrals. For example, Samko et al. [48], Podlubny [41], Oldham and Spanier [38], Miller and Ross [35] used the similar definition to compute the fractional integrals. But the circumstances are complicated in fractional order derivative (FOD) because several different competing definitions exist in the literature. For instance, a few of those approaches includes, the Riemann-Liouville, the Caputo, the Hadamard, the Marchaud, the Granwald-Letinkov, the Erdelyi-Kober, the Riesz-Feller, the Caputo-Fabrizio and the Atangana-Baleanu approach. These

2020 *Mathematics Subject Classification.* 26A33, 44A10.
Key words and phrases. Power Law, exponential law, nanofluid, carbon nanotube, non-local and non-singular kernel, thermodiffusion.

Table 1. Nomenclature

Symbol	Quantity
$\widetilde{W}$	Velocity of the fluid
$\widetilde{\Upsilon}$	Temperature of the fluid
$\widetilde{\Lambda}$	Concentration of the fluid
g	Acceleration due to gravity
Pr_{eff}	Effective Prandtl number
Sc	Schmidt number
Gr	Thermal Grashof number
Gm	Mass Grashof number
$\widetilde{\Upsilon}_w$	Fluid temperature at the plate
$\widetilde{\Upsilon}_\infty$	Fluid temperature far away from the plate
$\widetilde{\Lambda}_w$	Fluid concentration on the plate
$\widetilde{\Lambda}_\infty$	Fluid concentration far away from the plate
$(C_p)_{nf}$	Specific heat at constant pressure
u	Laplace transforms parameter
ρ_{nf}	Density of nano fluid
k_{nf}	Thermal Conductivity
χ	Fractional parameter
μ_{nf}	Dynamic viscosity of nano fluid
ν_{nf}	Kinematic viscosity of nano fluid
$\beta_{\widetilde{\Upsilon}}$	Volumetric coefficient of thermal expansion
$\beta_{\widetilde{\Lambda}}$	Volumetric coefficient of expansion for mass concentration
Q	Thermal sink/source
Sr	Thermodiffusion parameter
M	Magnetic field
θ	Volume fraction parameter
β_{nf}	Expansion Coefficient
N	Fractional contribution of Gm with respect to Gr

definitions coincide only for some particular cases. Among all these approaches to define fractional differentiation and fractional integration, the approach of Riemann-Liouville is significant. But the approach of Riemann-Liouville does not properly address the physics of some fractional derivative initial-boundary value problems. Also this definition can exhibit the derivative of a constant function other than zero. To overcome this problem Caputo proposed an alternate definition of FOD in 1967 [20] and it was used in fluid dynamics to explain the theory of viscoelasticity [21]. Recently, Caputo and Fabrizio [22] provides the modern definition of non integer order derivative including exponential function and the Atangana and Baleanu [14] based on Mittag-Leffler function.

FOD inherit nonlocal nature, so it is an excellent tool to get better understanding of hereditary properties of different processes and materials. Possibly, the first utilization of non integer calculus in physical problems was noticed due to the work of

Abel [2] while finding the solution of integer order equation, known as tautochrone problem. In this problem, the curve of an object (frictionless wire), lying in a vertical plane, was determined by using the operator $D_0^{\frac{1}{2}}$, and assuming the dependence of the time position not on the starting point. Bagley [17] presented the first Ph.D thesis on the applications of FC in viscoelasticity models. Recently, the applications of FC have been observed in psychology to determine the time variation of emotions of mankind [6, 51]. The applications of FOD problems can be seen in dynamics and control systems [8], marine sciences and wave dynamics [10, 30, 31, 37], diffusion processes [34, 39, 58], solid mechanics [27, 46, 56], medical sciences [13, 15, 16, 54] and many more [3, 9, 11, 12, 23, 43, 45, 47, 55, 59–61]. Cao et al. [19] analyzed fractional model of nanofluid over a moving plate. Abro et al. [4] applied a fractional derivative with non-singular and non-local kernal on a nano-fluid under magnetism. Saqib et al. [49] highlighted the strong memory effect of Atangana Baleanu fractional model of CNTs nanofluid.

The abeyance of nano-particles in fluid airing notable enhancement of their properties at reticent nano-particle concentrations are known as nanofluids [57]. Nanofluids parade enhanced thermal conductivity that escalates with growing volumetric fraction of nano-particles. In computers and other electronics, nanofluids are used to cool microchips. Lin et al. [32] inspected silver nanoparticles in vibrant thermal pipes. Saqib et al. [50] applied ABC derivative to MHD flow of CMC based CNTs nanofluid. Ikram et al. [29] diagnosed that Caputo fractional model exhibits greater memory effect as compared to Caputo-Fabrizio fractional model of nanofluid.

The nanofluid with injection/suction, concentration and temperature are used in dissolution of garbage from nuclear reactors, filtration and absorption of chemicals by a soft medium. The motion of nano-fluid under the impact of suction/injection in conical domain was discussed by Sreedevi et al. [52]. Du et al. [23] investigated the memory effect with derivative's order. Hayat et al. [26] inspected 3-D flow of nano-fluid under the influence of heat generation and magnetic field. Ahmad et al. [7] studied MHD CNTs of second grade nanofluid under the influence of first order chemical reaction and thermal generation.

This paper reveals study of three different fractional models to show the memory effect of CNTs Maxwell nanofluid model along with magnetic effect, heat generation, radiation and thermodiffusion. The impact of emerging parameters for momentum, mass and energy solutions are plotted by different graphs with real justifications. Finally, a comparison has been made among C, CF and ABC fractional models.

2. Mathematical model

Suppose a Maxwell nanofluid with carbon nano tube upon an upright and unbounded plate with the impact of magnetic field having strength B_0, heat generation, radiation and thermodiffusion. Initially, nanofluid and plate are at rest with ambient temperature $\widetilde{\Upsilon}_\infty$ and ambient concentration $\widetilde{\bigwedge}_\infty$, respectively. As time increases from 0, plate starts vibrating with velocity $U_o f(t)$. As $\jmath$-axis is normal to plane, therefore flow field depends only on time τ and $\jmath$. Physical interpretation of the problem can be given as follows by [5]:

$$(1+\lambda_1\partial_\tau)\,\partial_\jmath\widetilde{W}(\jmath,\tau)=\nu_{nf}\partial^2_{\jmath\jmath}\widetilde{W}(\jmath,\tau)+g\left(\beta_{\widetilde{\Upsilon}}\right)_{nf}(1+\lambda_1\partial_\tau)\left(\widetilde{\Upsilon}(\jmath,\tau)-\widetilde{\Upsilon}_\infty\right)$$
$$+g\left(\beta_{\widetilde{\Lambda}}\right)_{nf}(1+\lambda_1\partial_\tau)\left(\widetilde{\Lambda}(\jmath,\tau)-\widetilde{\Lambda}_\infty\right)$$
$$-\frac{\sigma B_o^2}{\rho_{nf}}(1+\lambda_1\partial_\tau)\,\widetilde{W}(\jmath,\tau), \tag{2.1}$$

$$K_{nf}(1+Nr)\,\partial^2_{\jmath\jmath}\widetilde{\Upsilon}(\jmath,\tau)+(\rho C_p)\,\partial_\tau\widetilde{\Upsilon}(\jmath,\tau)=Q_o\left(\widetilde{\Upsilon}(\jmath,\tau)-\widetilde{\Upsilon}_\infty\right), \tag{2.2}$$

$$D_m\partial^2_{\jmath\jmath}\widetilde{\Lambda}(\jmath,\tau)+\frac{D_mK_T}{Tm}\partial^2_{\jmath\jmath}\widetilde{\Upsilon}(\jmath,\tau)=\partial_\tau\widetilde{\Lambda}(\jmath,\tau). \tag{2.3}$$

The suitable initial and boundary conditions are

$$\widetilde{W}(\jmath,0)=0,\ \widetilde{\Upsilon}(\jmath,0)=\Upsilon_\infty,\ \widetilde{\Lambda}(\jmath,0)=\Lambda_\infty,$$
$$\widetilde{W}(0,\tau)=U_of(\widetilde{\tau}),\ \widetilde{\Upsilon}(0,\tau)=\Upsilon_w,\ \widetilde{\Lambda}(0,\tau)=\Lambda_w,$$
$$\widetilde{W}(\jmath,\tau)=0,\ \widetilde{\Upsilon}(\jmath,\tau)=\widetilde{\Upsilon}_\infty,\ \widetilde{\Lambda}(\jmath,\tau)=\widetilde{\Lambda}_\infty,\ as\ \jmath\to\infty. \tag{2.4}$$

Dimensionless variables are given below

$$\jmath=\frac{U_o\widetilde{\jmath}}{\nu_{nf}},\ \tau=\frac{U_o^2\widetilde{\tau}}{\nu_{nf}},\ W=\frac{\widetilde{W}}{U_o},\ \Upsilon=\frac{\widetilde{\Upsilon}-\Upsilon_\infty}{\Upsilon_w-\Upsilon_\infty},\ \Lambda=\frac{\widetilde{\Lambda}-\Lambda_\infty}{\Lambda_w-\Lambda_\infty},\ Pr_{eff}=\frac{Pr}{1+Nr},$$
$$Sc=\frac{\nu_{nf}}{D_m},\ N=\frac{(\rho\beta_\Lambda)_{nf}(\Lambda_w-\Lambda_\infty)}{(\rho\beta_\Upsilon)_{nf}(\Upsilon_w-\Upsilon_\infty)},\ Sr=\frac{D_mK_T(\Upsilon_w-\Upsilon_\infty)}{\nu_{nf}T_m(\Lambda_w-\Lambda_\infty)},\ M=\frac{\sigma\nu_{nf}B_0^2}{\rho_{nf}U_o^2},$$
(2.5)
$$Q=\frac{\nu_{nf}Q_o}{(\rho C_p)_{nf}U_o^2},\ \lambda=\frac{U_o^2\lambda_1}{\nu_{nf}},\ f(\widetilde{\tau})=f(\frac{\nu_{nf}\tau}{U_O^2}),$$

and dimensionless set of governing equations are:

(2.6)
$$(1+\lambda\partial_\tau)\,\partial_\tau W(\jmath,\tau)=\partial^2_{\jmath\jmath}W(\jmath,\tau)+(1+\lambda\partial_\tau)\left[\Upsilon(\jmath,\tau)+N\Lambda(\jmath,\tau)-MW(\jmath,\tau)\right],$$

$$\partial_\tau\Upsilon(\jmath,\tau)=\frac{1}{Pr_{eff}}\partial^2_{\jmath\jmath}\Upsilon(\jmath,\tau)-Q\Upsilon(\jmath,\tau), \tag{2.7}$$

$$\partial_\tau\Lambda(\jmath,\tau)=\frac{1}{Sc}\partial^2_{\jmath\jmath}\Lambda(\jmath,\tau)+Sr\partial^2_{\jmath\jmath}\Upsilon(\jmath,\tau). \tag{2.8}$$

Also dimensionless initial and boundary conditions are:

$$W(\jmath,0)=0,\ \Upsilon(\jmath,0)=0,\ \Lambda(\jmath,0)=0,$$
$$W(0,\tau)=f(\tau),\ \Upsilon(0,\tau)=1,\ \Lambda(0,\tau)=1,$$
$$W(\jmath,\tau)=0,\ \Upsilon(\jmath,\tau)=0,\ \Lambda(\jmath,\tau)=0,\ as\ \jmath\to\infty. \tag{2.9}$$

2.1. **Preliminaries.** The Caputo time derivative and its Laplace transform is given by [1]:

$$^{C}D_{\tau}^{\chi}m(\jmath,\tau)=\frac{1}{\Xi(i-\chi)}\int_{b}^{\tau}\left(\frac{g^{(i)}(\varrho)}{(\tau-\varrho)^{\chi+1-i}}\right)d\varrho, \tag{2.10}$$

$$\mathcal{L}\left(^{C}D_{\tau}^{\chi}m(\jmath,\tau)\right)=u^{\chi}\mathcal{L}(m(\jmath,\tau))-u^{\chi-1}g(\jmath,0). \tag{2.11}$$

where $\Xi(\cdot)$ is gamma function.
The Caputo-Fabrizio time derivative its Laplace transform is given by [28]:

$$^{CF}D_{\tau}^{\chi}m(\jmath,\tau)=\frac{G(\chi)}{1-\chi}\int_{b}^{\tau}\exp\left(-\frac{\chi(\tau-\chi)}{1-\chi}\right)\frac{\partial g(\jmath,\varrho)}{\partial\varrho}d\varrho, \tag{2.12}$$

$$\mathcal{L}\left(^{CF}D_{\tau}^{\chi}m(\jmath,\tau)\right)=\frac{u\mathcal{L}(m(\jmath,\tau))-g(\jmath,0)}{(1-\chi)u+\chi}. \tag{2.13}$$

where $G(\chi)$ is a normalization function.
The Atangana-Baleanu time derivative in Caputo sense its Laplace transform is given by [42, 44]:

$$^{ABC}D_{\tau}^{\chi}m(\jmath,\tau)=\frac{AB(\chi)}{1-\chi}\int_{b}^{\tau}E_{\chi}\left(-\frac{\chi(\tau-\chi)}{1-\chi}\right)\frac{\partial g(\jmath,\varrho)}{\partial\varrho}d\varrho, \tag{2.14}$$

$$\mathcal{L}\left(^{ABC}D_{\tau}^{\chi}m(\jmath,\tau)\right)=\frac{u^{\chi}\mathcal{L}(m(\jmath,\tau))-u^{\chi-1}g(\jmath,0)}{(1-\chi)u^{\chi}+\chi}. \tag{2.15}$$

where $AB(\chi)$ is a normalization function.

3. Solutions via Caputo

Following are the fractional solutions of heat, mass and flow with Caputo fractional derivative.

3.1. **Temperature profile.** By using definition of Caputo derivative eq. (2.10), we have

$$^{C}D_{\tau}\Upsilon_{c}(\jmath,\tau)=\frac{1}{Pr_{eff}}\partial_{\jmath\jmath}^{2}\Upsilon_{c}(\jmath,\tau)-Q\Upsilon_{c}(\jmath,\tau), \tag{3.1}$$

Taking Laplace of eq. (3.1) by using eq. (2.11)

$$\partial_{\jmath\jmath}^{2}\overline{\Upsilon}_{c}(\jmath,u)-Pr_{eff}(u^{\chi}+Q)\overline{\Upsilon}_{c}(\jmath,u)=0. \tag{3.2}$$

Solution of eq. (3.2) is

$$\overline{\Upsilon}_{c}(\jmath,u)=\frac{1}{u}e^{-\jmath\sqrt{Pr_{eff}(u^{\chi}+Q)}}. \tag{3.3}$$

3.2. **Concentration profile.** The following Caputo derivative form of concentration eq. (2.10) is

$$ {}^{C}D_{\tau}\Lambda_c(\jmath,\tau) = \frac{1}{Sc}\partial^2_{\jmath\jmath}\Lambda_c(\jmath,\tau) + Sr\partial^2_{\jmath\jmath}\Upsilon_c(\jmath,\tau). \tag{3.4} $$

Taking Laplace of eq. (3.4) by using eq. (2.11), we get

$$ \partial^2_{\jmath\jmath}\overline{\Lambda}_c(\jmath,\tau) - Scu^{\chi}\overline{\Lambda}_c(\jmath,\tau) == -ScSr\partial^2_{\jmath\jmath}\overline{\Upsilon}_c(\jmath,\tau). \tag{3.5} $$

Solution of eq. (3.5) is

$$ \overline{\Lambda}_c(\jmath,u) = \left(\frac{1}{u} + Z\right)e^{-\jmath\sqrt{Scu^{\chi}}} - (Z)\,e^{-\jmath\sqrt{Pr_{eff}(u^{\chi}+Q)}}. \tag{3.6} $$

where

$$ Z = \frac{ScSrPr_{eff}(u^{\chi}+Q)}{\left(\frac{Pr_{eff}-Sc}{Pr_{eff}Q}\right)\left(\frac{1}{u} - \frac{1}{u^{\chi}+\frac{Pr_{eff}-Sc}{Pr_{eff}Q}}\right)}. \tag{3.7} $$

3.3. **Velocity Profile.** The following Caputo derivative form of velocity eq. (2.10) is

$$ \begin{aligned} &\left(1+\lambda^{C}D_{\tau}\right)\partial_{\tau}W_c(\jmath,\tau) \\ &\quad = \partial^2_{\jmath\jmath}W_c(\jmath,\tau) + \left(1+\lambda^{C}D_{\tau}\right)\left[\Upsilon_c(\jmath,\tau) + N\Lambda_c(\jmath,\tau) - MW_c(\jmath,\tau)\right]. \end{aligned} \tag{3.8} $$

Taking Laplace of eq. (3.8) by using eq. (2.11), we get solution

(3.9)

$$ \begin{aligned} \overline{W}_c(\jmath,u) &= -F(u)\,e^{-\jmath A_5} \\ &+ \left(\frac{1+\lambda u^{\chi}}{\lambda}\right)\left(\frac{1}{Z_2} - \frac{1}{\sqrt{4\left(\frac{Pr_{eff}Q-M}{\lambda}\right) - \left(\frac{Pr_{eff}-(1+\lambda M)}{\lambda}\right)^2}}\left(\frac{1}{A_1} - \frac{1}{A_2}\right)\right) \\ &\times \left(e^{-\jmath\sqrt{Pr_{eff}(u^{\chi}+Q)}} - e^{-\jmath A_5}\right) - \left(\frac{(1+\lambda u^{\chi})N}{\lambda\sqrt{\left(\frac{Sc-(1+\lambda M)}{\lambda}\right)^2 - \frac{4M}{\lambda}}}\right)\left(\frac{1}{A_3} - \frac{1}{A_4}\right) \\ &\times \left(\frac{1}{u}e^{-\jmath\sqrt{Scu^{\chi}}} + \frac{ScSrPr_{eff}(u^{\chi}+Q)}{\frac{1}{u} - \frac{1}{\frac{Pr_{eff}-Sc}{Pr_{eff}Q}}}\left(e^{-\jmath\sqrt{Scu^{\chi}}} - e^{-\jmath\sqrt{Pr_{eff}(u^{\chi}+Q)}}\right) - \frac{1}{u}e^{-\jmath A_5}\right), \end{aligned} $$

where

(3.10)

$$ Z_1 = \frac{\sqrt{4\left(\frac{Pr_{eff}Q-M}{\lambda}\right) - \left(\frac{Pr_{eff}-(1+\lambda M)}{\lambda}\right)^2}}{2}, $$

(3.11)
$$Z_2 = \left(\left(\frac{Pr_{eff} - (1+\lambda M)}{2\lambda} \right)^2 - \frac{\left(4\left(Pr_{eff}Q - M\right) - \left(\frac{Pr_{eff} - (1+M\lambda)}{\lambda} \right)^2 \right)}{4} \right) u^{\chi},$$

(3.12)
$$A_1 = \left(\left(\frac{Pr_{eff} - (1+\lambda M)}{2\lambda} \right) - Z_1 \right) \left(u^{\chi} - Z_1 \right),$$

(3.13)
$$A_2 = \left(\left(\frac{Pr_{eff} - (1+\lambda M)}{2\lambda} \right) + Z_1 \right) \left(u^{\chi} - Z_1 \right),$$

(3.14)
$$A_3 = u^{\chi} - \left(\left(\frac{Sc - (1+\lambda M)}{2\lambda} \right) - \frac{\sqrt{\left(\frac{Sc-(1+\lambda M)}{\lambda} \right)^2 - \frac{4M}{\lambda}}}{2} \right),$$

(3.15)
$$A_4 = u^{\chi} - \left(\left(\frac{Sc - (1+\lambda M)}{2\lambda} \right) + \frac{\sqrt{\left(\frac{Sc-(1+\lambda M)}{\lambda} \right)^2 - \frac{4M}{\lambda}}}{2} \right),$$

(3.16)
$$A_5 = \sqrt{\lambda} \sqrt{\left[u^{\chi} + \left(\frac{1+\lambda M}{2\lambda} \right) \right]^2 - \left[\frac{(1+\lambda M)^2 - 4\lambda M}{4\lambda^2} \right]^2}.$$

4. Solutions via Caputo-Fabrizio

The non-integer order solutions via singular kernel Caputo-Fabrizio fractional derivative for temperature, concentration and velocity are as follow:

4.1. **Temperature profile.** By using definition of Caputo-Fabrizio derivative eq. (2.12), we have

$$ {}^{CF}D_{\tau}\Upsilon_{cf}(\jmath,\tau) = \frac{1}{Pr_{eff}} \partial^2_{\jmath\jmath}\Upsilon_{cf}(\jmath,\tau) - Q\Upsilon_{cf}(\jmath,\tau), \tag{4.1}$$

Taking Laplace of eq. (4.1) by using eq. (2.13)

$$\partial^2_{\jmath\jmath}\overline{\Upsilon}_{cf}(\jmath,u) - \left(\frac{\frac{Pr_{eff}u}{1-\chi}}{u + \frac{\chi}{1-\chi}} + Pr_{eff}Q \right) \overline{\Upsilon}_{cf}(\jmath,u) = 0. \tag{4.2}$$

Solution of eq. (4.2) is

$$\overline{\Upsilon}_{cf}(\jmath,u) = \frac{1}{u} e^{-\jmath\sqrt{\frac{\frac{Pr_{eff}u}{1-\chi}}{u+\frac{\chi}{1-\chi}} + Pr_{eff}Q}}. \tag{4.3}$$

4.2. **Concentration profile.** The following CF derivative form of concentration eq. (2.12) is

$$ {}^{CF}D_\tau \Lambda_{cf}(\jmath,\tau) = \frac{1}{Sc}\partial^2_{\jmath\jmath}\Lambda_{cf}(\jmath,\tau) + Sr\partial^2_{\jmath\jmath}\Upsilon_{cf}(\jmath,\tau). \tag{4.4} $$

Taking Laplace of eq. (4.4) by using eq. (2.13), we get

$$ \partial^2_{\jmath\jmath}\overline{\Lambda}_{cf}(\jmath,\tau) - \frac{Scu}{(1-\chi)\left(u+\frac{\chi}{1-\chi}\right)}\overline{\Lambda}_{cf}(\jmath,\tau) = -ScSr\partial^2_{\jmath\jmath}\overline{\Upsilon}_{cf}(\jmath,\tau). \tag{4.5} $$

Solution of eq. (4.5) is

$$ \overline{\Lambda}_{cf}(\jmath,u) = \left(\frac{1}{u}+Z_3\right)e^{-\jmath\sqrt{\frac{Scu}{(1-\chi)\left(u+\frac{\chi}{1-\chi}\right)}}} - (Z_3)\,e^{-\jmath\sqrt{\frac{\left(\frac{Pr_{eff}u}{1-\chi}\right)}{u+\frac{\chi}{1-\chi}}+Pr_{eff}Q}}. \tag{4.6} $$

where

$$ Z_3 = \left(\frac{ScSr\left(\frac{Pr_{eff}u}{1-\chi}+\left(u+\frac{\chi}{1-\chi}\right)Pr_{eff}Q\right)}{\left(\frac{Pr_{eff}u}{1-\chi}-\frac{Sc}{1-\chi}+Pr_{eff}Q\right)\left(\frac{\frac{Pr_{eff}Q\chi}{1-\chi}}{\left(\frac{Pr_{eff}}{1-\chi}-\frac{Sc}{1-\chi}+Pr_{eff}Q\right)}\right)}\right) \left(\frac{1}{u}-\frac{1}{u+\frac{\frac{Pr_{eff}Q\chi}{1-\chi}}{\left(\frac{Pr_{eff}}{1-\chi}-\frac{Sc}{1-\chi}+Pr_{eff}Q\right)}}\right). \tag{4.7} $$

4.3. **Velocity Profile.** Applying definition of Caputo-Fabrizio (2.12) to non-dimensional flow eq. (2.6)

$$ \begin{aligned} &\left(1+\lambda^{CF}D_\tau\right)\partial_\tau W_{cf}(\jmath,\tau) \\ &\quad = \partial^2_{\jmath\jmath}W_{cf}(\jmath,\tau) + \left(1+\lambda^{CF}D_\tau\right)\left[\Upsilon_{cf}(\jmath,\tau)+N\Lambda_{cf}(\jmath,\tau)-MW_{cf}(\jmath,\tau)\right], \end{aligned} \tag{4.8} $$

Taking Laplace of equation (4.8), we get solution

$$ \begin{aligned} \overline{W}_{cf}(\jmath,u) = &-F(u)e^{-\jmath M_3} \\ &+\left(\frac{\left(\left(1+\frac{\lambda}{1-\chi}\right)u+\left(\frac{\chi}{1-\chi}\right)\right)N}{\left(\frac{Pr_{eff}}{1-\chi}+Pr_{eff}Q-\left(1+\frac{\lambda}{1-\chi}\right)\right)(2B_1)}\right)\left(\frac{1}{B_2+B_1}-\frac{1}{B_2-B_1}\right) \\ &\times\left(e^{-\jmath M_3}-e^{-\jmath M_2}\right) \\ &-\left(\frac{\left(1+\frac{\lambda}{1-\chi}\right)Mu+\left(\frac{\chi}{1-\chi}\right)}{\left(1-\frac{\lambda}{1-\chi}\right)\sqrt{\frac{Sc-\chi+(1-\chi+\lambda)}{\lambda(1-\chi+\lambda)}-\left(\frac{\chi}{1-\chi+\lambda}\right)}}\right)\left(\frac{1}{B_3-B_4}\right) \\ &\times\left(\frac{1}{u}e^{-\jmath M_1}+B_5\left(e^{-\jmath M_1}-e^{-\jmath M_2}\right)-\frac{1}{u}e^{-\jmath M_3}\right), \end{aligned} \tag{4.9} $$

where

(4.10)

$$B_1=\sqrt{\left[\frac{Pr_{eff}Q\chi}{1-\chi}-\frac{\chi}{1-\chi}-\left(1+\frac{\lambda}{1-\chi}\right)M\right]^2-\frac{\frac{4M\chi}{1-\chi}}{\frac{Pr_{eff}}{1-\chi}+Pr_{eff}Q-\left(1+\frac{\lambda}{1-\chi}\right)M}},$$

(4.11)

$$B_2=u+\left(\frac{Pr_{eff}Q\chi-\chi-(1-\chi+\lambda)\,M}{2\,(Pr_{eff}+(1-\chi)\,Pr_{eff}Q-(1-\chi+\lambda)\,M)}\right),$$

(4.12)

$$B_3=u-\left(\frac{Sc-\chi+(1-\chi+\lambda)}{2\,(1-\chi+\lambda)}-\sqrt{\frac{Sc-\chi+(1-\chi+\lambda)}{2\,(1-\chi+\lambda)}-\left(\frac{\chi}{1-\chi+\lambda}\right)}\right),$$

(4.13)

$$B_4=u-\left(\frac{Sc-\chi+(1-\chi+\lambda)}{2\,(1-\chi+\lambda)}+\sqrt{\frac{Sc-\chi+(1-\chi+\lambda)}{2\,(1-\chi+\lambda)}-\left(\frac{\chi}{1-\chi+\lambda}\right)}\right),$$

(4.14)

$$B_5=\left(\frac{ScSr\left(\frac{Pr_{eff}u}{1-\chi}+\left(u+\frac{\chi}{1-\chi}\right)Pr_{eff}Q\right)}{\left(\frac{Pr_{eff}u}{1-\chi}-\frac{Sc}{1-\chi}+Pr_{eff}Q\right)\left(\frac{\frac{Pr_{eff}Q\chi}{1-\chi}}{\frac{Pr_{eff}u}{1-\chi}-\frac{Sc}{1-\chi}+Pr_{eff}Q}\right)}\right)\left(\frac{1}{q}-\frac{1}{\frac{u+\frac{Pr_{eff}Q\chi}{1-\chi}}{\frac{Pr_{eff}}{1-\chi}-\frac{Sc}{1-\chi}+Pr_{eff}Q}}\right),$$

(4.15)

$$M_1=\sqrt{\frac{Scu}{(1-\chi)\left(u+\frac{\chi}{1-\chi}\right)}},\ M_2=\sqrt{\frac{Pr_{eff}u}{(1-\chi)\left(u+\frac{\chi}{1-\chi}\right)}},$$

$$M_3=\sqrt{(u+M)\left(\frac{(1-\chi+\lambda)\,u+\chi}{(1-\chi)\,u+\chi}\right)}.$$

5. Solutions via Atangana Baleanu

The non-integer order solutions via Atangana Baleanu derivative of temperature, concentration and velocity are as follow:

5.1. **Temperature profile.** By using definition of Atangana Baleanu derivative eq. (2.14), we have

$$^{ABC}D_\tau\Upsilon_{abc}\,(\jmath,\tau)=\frac{1}{Pr_{eff}}\partial^2_{\jmath\jmath}\Upsilon_{abc}\,(\jmath,\tau)-Q\Upsilon_{abc}\,(\jmath,\tau)\,, \tag{5.1}$$

Taking Laplace of eq. (5.1) by using eq. (2.15)

$$\partial^2_{\jmath\jmath}\overline{\Upsilon}_{abc}(\jmath,u)-\left(\frac{\frac{Pr_{eff}u^{\chi}}{1-\chi}}{u^{\chi}+\frac{\chi}{1-\chi}}+Pr_{eff}Q\right)\overline{\Upsilon}_{abc}(\jmath,u)=0 \tag{5.2}$$

Solution of eq. (5.2) is

$$\overline{\Upsilon}_{abc}(\jmath,u)=\frac{1}{u}e^{-\jmath\sqrt{\frac{\frac{Pr_{eff}u^{\chi}}{1-\chi}}{u^{\chi}+\frac{\chi}{1-\chi}}+Pr_{eff}Q}}. \tag{5.3}$$

5.2. **Concentration profile.** The following ABC derivative form of concentration eq. (2.14) is

$$^{CF}D_{\tau}\Lambda_{abc}(\jmath,\tau)=\frac{1}{Sc}\partial^2_{\jmath\jmath}\Lambda_{abc}(\jmath,\tau)+Sr\partial^2_{\jmath\jmath}\Upsilon_{abc}(\jmath,\tau). \tag{5.4}$$

Taking Laplace of eq. (5.4) by using eq. (2.15), we get

$$\partial^2_{\jmath\jmath}\overline{\Lambda}_{abc}(\jmath,\tau)-\frac{Scu^{\chi}}{(1-\chi)\left(u^{\chi}+\frac{\chi}{1-\chi}\right)}\overline{\Lambda}_{abc}(\jmath,\tau)==-ScSr\partial^2_{\jmath\jmath}\overline{\Upsilon}_{abc}(\jmath,\tau). \tag{5.5}$$

Solution of eq. (5.5) is

$$\overline{\Lambda}_{abc}(\jmath,u)=\left(\frac{1}{u}+Z_4\right)e^{-\jmath\sqrt{\frac{Scu^{\chi}}{(1-\chi)\left(u^{\chi}+\frac{\chi}{1-\chi}\right)}}}-(Z_4)\,e^{-\jmath\sqrt{\frac{\left(\frac{Pr_{eff}u^{\chi}}{1-\chi}\right)}{u^{\chi}+\frac{\chi}{1-\chi}}+Pr_{eff}Q}}. \tag{5.6}$$

where

$$Z_4=\left(\frac{ScSr\left(\frac{Pr_{eff}u^{\chi}}{1-\chi}+\left(u^{\chi}+\frac{\chi}{1-\chi}\right)Pr_{eff}Q\right)}{\left(\frac{Pr_{eff}u^{\chi}}{1-\chi}-\frac{Sc}{1-\chi}+Pr_{eff}Q\right)\left(\frac{\frac{Pr_{eff}Q\chi}{1-\chi}}{\left(\frac{Pr_{eff}}{1-\chi}-\frac{Sc}{1-\chi}+Pr_{eff}Q\right)}\right)}\right)\left(\frac{1}{u}-\frac{1}{u^{\chi}+\frac{\frac{Pr_{eff}Q\chi}{1-\chi}}{\left(\frac{Pr_{eff}}{1-\chi}-\frac{Sc}{1-\chi}+Pr_{eff}Q\right)}}\right). \tag{5.7}$$

5.3. **Velocity Profile.** Applying definition of Atangana Baleanu (2.14) to non-dimensional flow eq. (2.6)

$$\begin{aligned}&\left(1+\lambda^{ABC}D_{\tau}\right)\partial_{\tau}W_{abc}(\jmath,\tau)\\&=\partial^2_{\jmath\jmath}W_{abc}(\jmath,\tau)+\left(1+\lambda^{ABC}D_{\tau}\right)\left[\Upsilon_{abc}(\jmath,\tau)+N\Lambda_{abc}(\jmath,\tau)-MW_{abc}(\jmath,\tau)\right],\end{aligned} \tag{5.8}$$

Taking Laplace of eq. (5.8), we get solution

$$\overline{W}_{abc}(\jmath,u)=-F(u)e^{-\jmath P_3}$$

$$+\left(\frac{\left(\left(1+\frac{\lambda}{1-\chi}\right)u^{\chi}+\left(\frac{\chi}{1-\chi}\right)\right)N}{\left(\frac{Pr_{eff}}{1-\chi}+Pr_{eff}Q-\left(1+\frac{\lambda}{1-\chi}\right)\right)(2B_1)}\right)\left(\frac{1}{C_2+B_1}-\frac{1}{C_2-B_1}\right)$$
$$\times\left(e^{-\jmath P_3}-e^{-\jmath P_2}\right)$$
$$-\left(\frac{\left(1+\frac{\lambda}{1-\chi}\right)Mu^{\chi}+\left(\frac{\chi}{1-\chi}\right)}{\left(1-\frac{\lambda}{1-\chi}\right)\sqrt{\frac{Sc-\chi+(1-\chi+\lambda)}{\lambda(1-\chi+\lambda)}-\left(\frac{\chi}{1-\chi+\lambda}\right)}}\right)\left(\frac{1}{C_3-C_4}\right)$$
$$\times\left(\frac{1}{u}e^{-\jmath P_1}+C_5\left(e^{-\jmath P_1}-e^{-\jmath P_2}\right)-\frac{1}{u}e^{-\jmath P_3}\right), \tag{5.9}$$

where

$$B_1=\sqrt{\left[\frac{Pr_{eff}Q\chi}{1-\chi}-\frac{\chi}{1-\chi}-\left(1+\frac{\lambda}{1-\chi}\right)M\right]^2-\frac{\frac{4M\chi}{1-\chi}}{\frac{Pr_{eff}}{1-\chi}+Pr_{eff}Q-\left(1+\frac{\lambda}{1-\chi}\right)M}}, \tag{5.10}$$

$$C_2=u^{\chi}+\left(\frac{Pr_{eff}Q\chi-\chi-(1-\chi+\lambda)M}{2\left(Pr_{eff}+(1-\chi)Pr_{eff}Q-(1-\chi+\lambda)M\right)}\right), \tag{5.11}$$

$$C_3=u^{\chi}-\left(\frac{Sc-\chi+(1-\chi+\lambda)}{2(1-\chi+\lambda)}-\sqrt{\frac{Sc-\chi+(1-\chi+\lambda)}{2(1-\chi+\lambda)}-\left(\frac{\chi}{1-\chi+\lambda}\right)}\right), \tag{5.12}$$

$$C_4=u^{\chi}-\left(\frac{Sc-\chi+(1-\chi+\lambda)}{2(1-\chi+\lambda)}+\sqrt{\frac{Sc-\chi+(1-\chi+\lambda)}{2(1-\chi+\lambda)}-\left(\frac{\chi}{1-\chi+\lambda}\right)}\right), \tag{5.13}$$

$$C_5=\left(\frac{ScSr\left(\frac{Pr_{eff}u^{\chi}}{1-\chi}+\left(u^{\chi}+\frac{\chi}{1-\chi}\right)Pr_{eff}Q\right)}{\left(\frac{Pr_{eff}u^{\chi}}{1-\chi}-\frac{Sc}{1-\chi}+Pr_{eff}Q\right)\left(\frac{\frac{Pr_{eff}Q\chi}{1-\chi}}{\frac{Pr_{eff}u^{\chi}}{1-\chi}-\frac{Sc}{1-\chi}+Pr_{eff}Q}\right)}\right)\left(\frac{1}{q}-\frac{1}{\frac{u^{\chi}+\frac{Pr_{eff}Q\chi}{1-\chi}}{\frac{Pr_{eff}}{1-\chi}-\frac{Sc}{1-\chi}+Pr_{eff}Q}}\right), \tag{5.14}$$

$$P_1=\sqrt{\frac{Scu^{\chi}}{(1-\chi)\left(u^{\chi}+\frac{\chi}{1-\chi}\right)}},\ P_2=\sqrt{\frac{Pr_{eff}u^{\chi}}{(1-\chi)\left(u^{\chi}+\frac{\chi}{1-\chi}\right)}}, \tag{5.15}$$

$$P_3 = \sqrt{(u^\chi + M)\left(\frac{(1-\chi+\lambda)\,u^\chi+\chi}{(1-\chi)\,u^\chi+\chi}\right)}.$$

The results obtained for velocity, temperature and Concentration after solving the fractionalized models are complex and generalized in nature. As the models were fractionalized with power Law, exponential and non-local kernel to see the significance of memory effects. These are generalized results and as we use fractional parameter goes to 1 and ignore slip condition, we recovered the result calculated for integer order case. The detail is as by taking $\chi \to 1$ with no slip condition in eq. (24), we will get similar results as for integer order obtained by Ahmad et al. [5] [eq. (20)]. As $\chi \to 1$ and ignoring slip condition in eq. (40), identical results are exist in Ahmad et al. [5] [eq. (20)]. And as $\chi \to 1$ in eq. (55) and omit slip condition results for fractional order reduce to integer order presented by Ahmad et al. [5] [eq. (20)]. This shows worth of our results.
The roots are complex in nature in order to find their inverses we use inversion algorithm namely Stehfest's formula [53] one of the simplest algorithm we use to sort out the inverse Laplace transform.
$v(r,t) = \frac{e^{4.7}}{t}\left[\frac{1}{2}\bar{v}\left(r, \frac{4.7}{t}\right) + Re\left\{\sum_{k=1}^{N_1}(-1)^k \bar{v}\left(r, \frac{4.7+k\pi i}{t}\right)\right\}\right],$
where Re(.) is the real part, i is the imaginary unit and N_1 is a natural number.

6. Results and Discussion

MHD Maxwell CNT's flow of nano-fluid has been studied, in this paper for three particular time derivative (C, CF and ABC). Fractional models of flow, heat and mass are introduced. Graphs has been drawn to elaborate the effects of χ, M, λ, Pr_{eff}, Sc, Sr and N for fractional models.

6.1. **Effect of χ:** From figure 1, we can observe that velocity of Maxwell nano-fluid decreases as the order of fractional derivative increases. As there is non-local and non-singular kernel present in Atangana Baleanu time derivative, velocity is highest for ABC.

6.2. **Effect of M:** Figure 2 illustrates that as magnetic field amplifies, fluid flow de-escalates. Lorentz force is a frictional force caused by magnetic field. When magnetic field maximizes, there is an increase in Lorentz force at outermost layer of fluid. this leads into decrease in velocity of nano-fluid. ABC fractional model bears highest flow rate as compared to Caputo and Caputo-Fabrizio fractional MHD models.

6.3. **Effect of λ:** Figure 3 shows that as λ increases, velocity profile decreases. The velocity function has also seen for variable time. As relaxation time enhances the backward flow and reduces the velocity. Also, when λ ascends, thickness of momentum boundary layer minimizes that leads to downshift the flow. Since increment in relaxation time suggests that fluid will acquire additional time to relax, it affirms decrease in velocity.

6.4. **Effect of Pr_{eff}:** The flow profile declines providing the values of Pr_{eff} rise as displayed in figure 4. Pr_{eff} is directly and inversely proportional to momentum and thermal diffusivity respectively. Fluid's viscosity increases granting decrease in thermal diffusivity. In response, flow field decreases and it is minimal for Caputo.

6.5. **Effect of Sc:** From figure 5, as the values of Sc increases, it increases kinematic viscosity but decreases mass diffusivity. This successively descends flow profile of nano-fluid as presented in figure 6. Velocity field for ABC, CF and C is supreme, adequate and least respectively.

6.6. **Effect of Sr:** The fluid flow increases the way Sr escalates as shown in figure 6. As velocity profile increases, the momentum boundary layer becomes thicker. It is greatest for ABC fractional MHD Maxwell model.

6.7. **Effect of N:** Figure 7 and 8 elaborates the effect of N on velocity. Flow profile increases when $N > 0$ and opposite for $N < 0$. Velocity is maximal, average and minimal for ABC, CF and C respectively.

7. Conclusion

This article inquires time-dependent, MHD Maxwell CNTs flow of nanofluid. The C, CF and ABC operators are used to construct equation for heat, mass and velocity. Solutions of model equations are presented by Laplace transform. Several graphs are given to illustrate impact of incipient parameters for solutions. Some remarkable finding are:
1. Fluid flow descends for χ, M, Pr_{eff} and Sc.
2. Increment in flow field has been generated when λ and Sr increases.
3. For $N > 0$, velocity of nano-fluid escalates. Whereas for $N < 0$, flow shows opposite behavior.
4. Among C, CF and ABC, flow profile is maximum for ABC.

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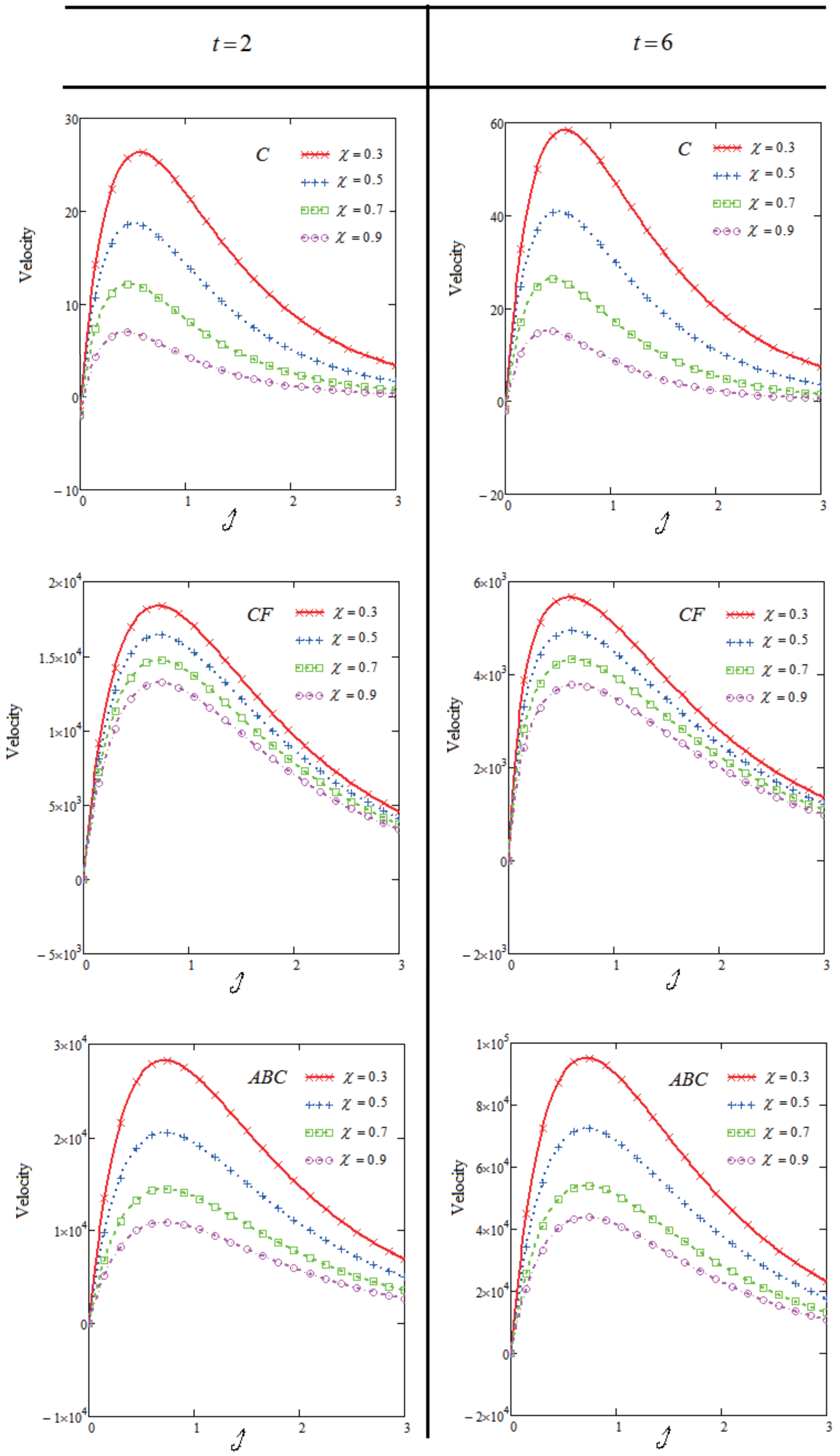

FIGURE 1. Graphs for velocity with variable fractional parameter and $M = 2$, $Sc = 0.66$, $Q = 0.1$, $Pr_{eff} = N = Sr = \lambda = 5$ and $f(\tau) = 1$.

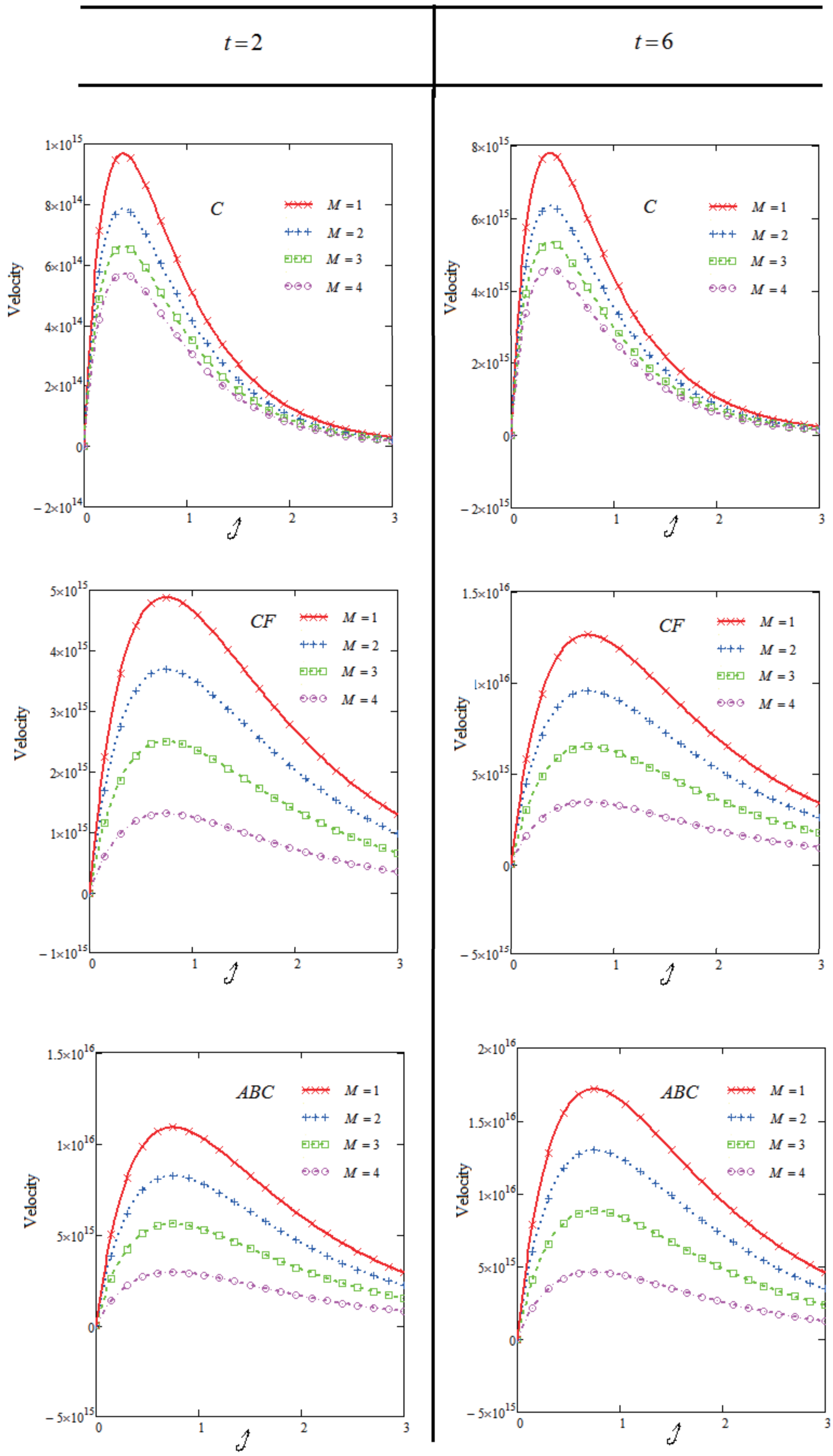

FIGURE 2. Graphs for velocity with variable M and $\chi = 2$, $Sc = 0.66$, $Q = 0.1$ $Pr_{eff} = N = Sr = \lambda = 5$ and $f(\tau) = 1$.

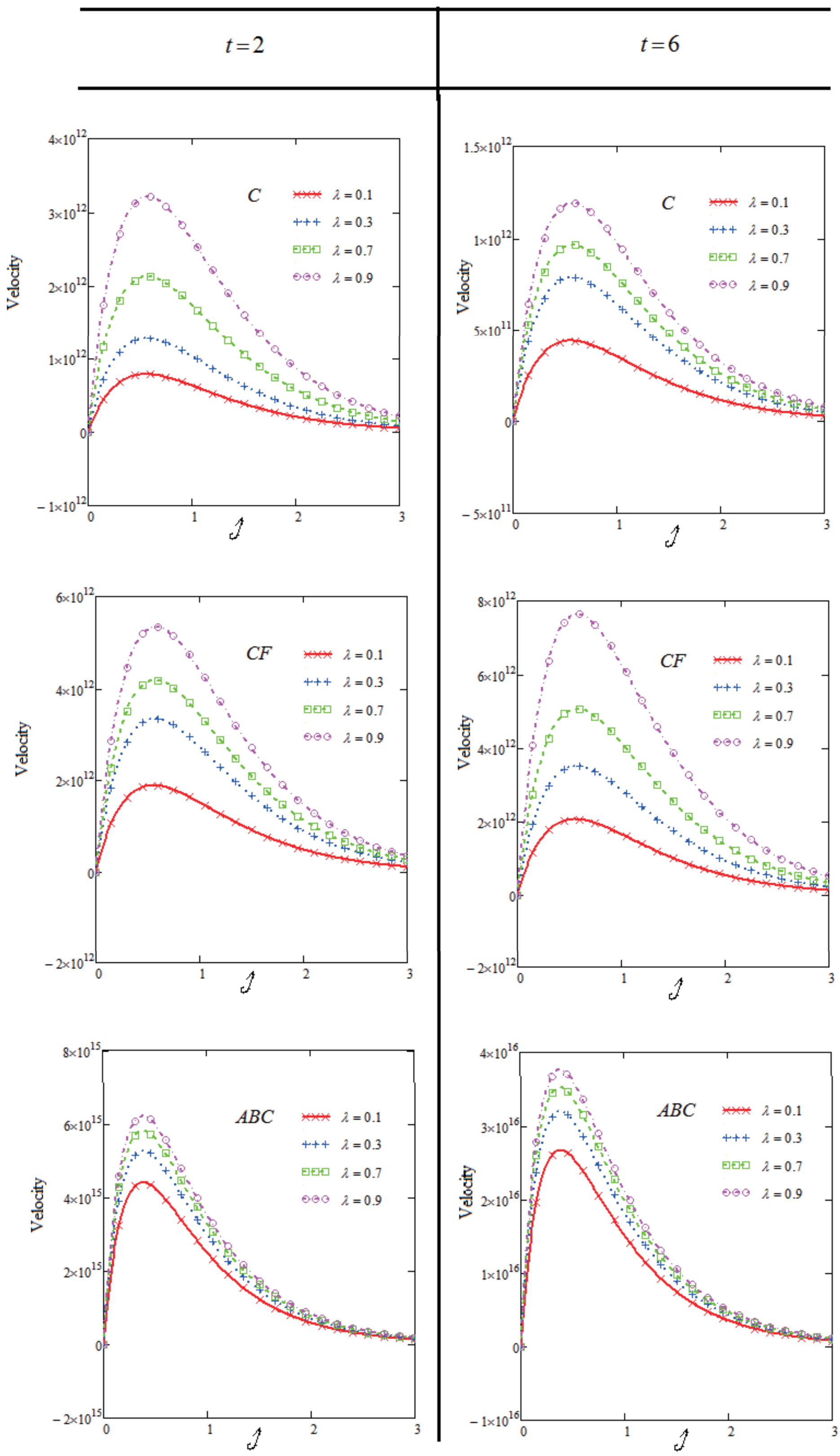

FIGURE 3. Graphs for velocity with variable λ and $\chi = 2$, $Sc = 0.66$, $Q = 0.1$, $Pr_{eff} = N = Sr = 5$ and $f(\tau) = 1$.

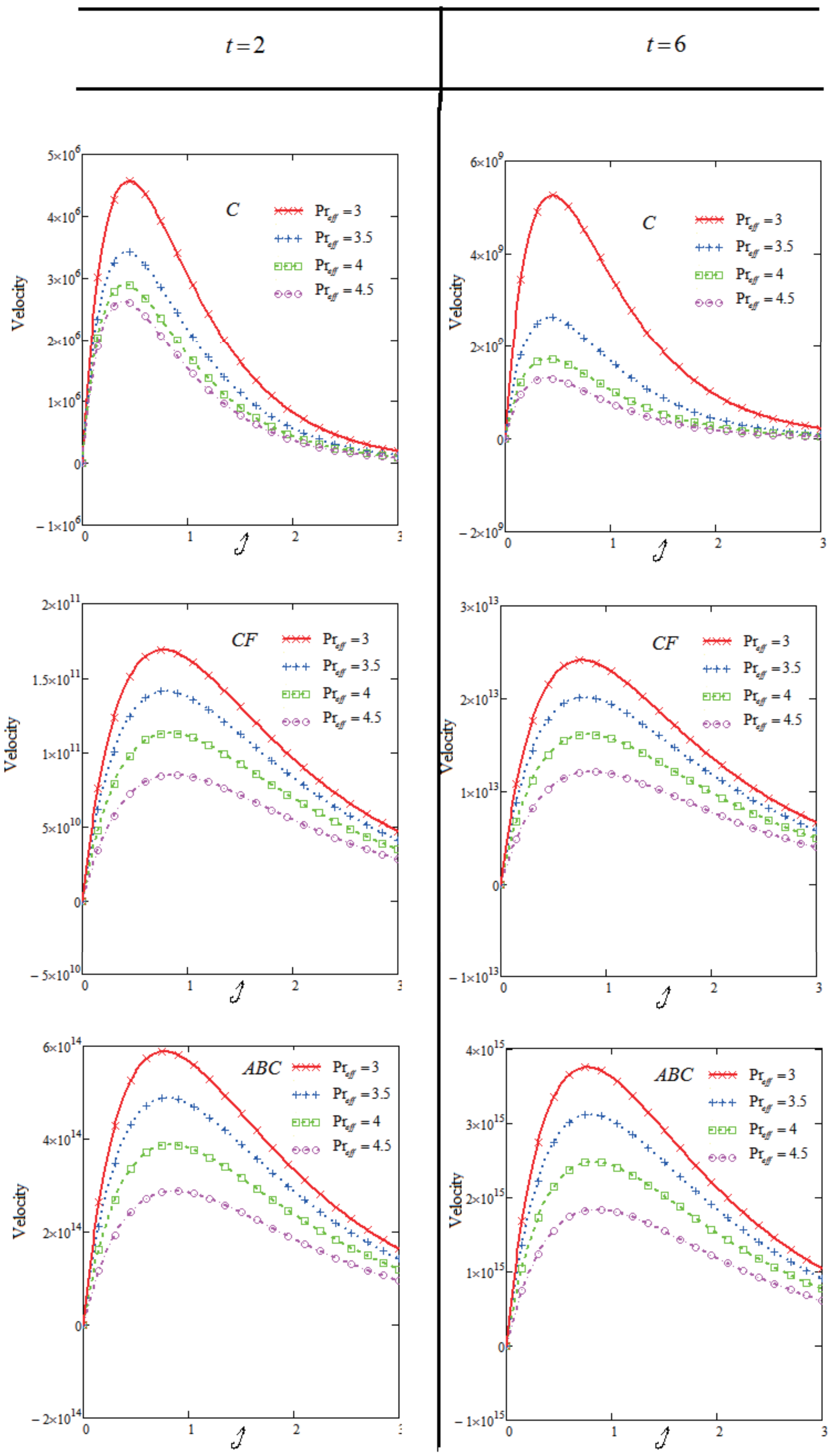

FIGURE 4. Graphs for velocity with variable Pr_{eff} and $\chi = 2$, $Sc = 0.66$, $Q = 0.1$, $N = Sr = \lambda = 5$ and $f(\tau) = 1$.

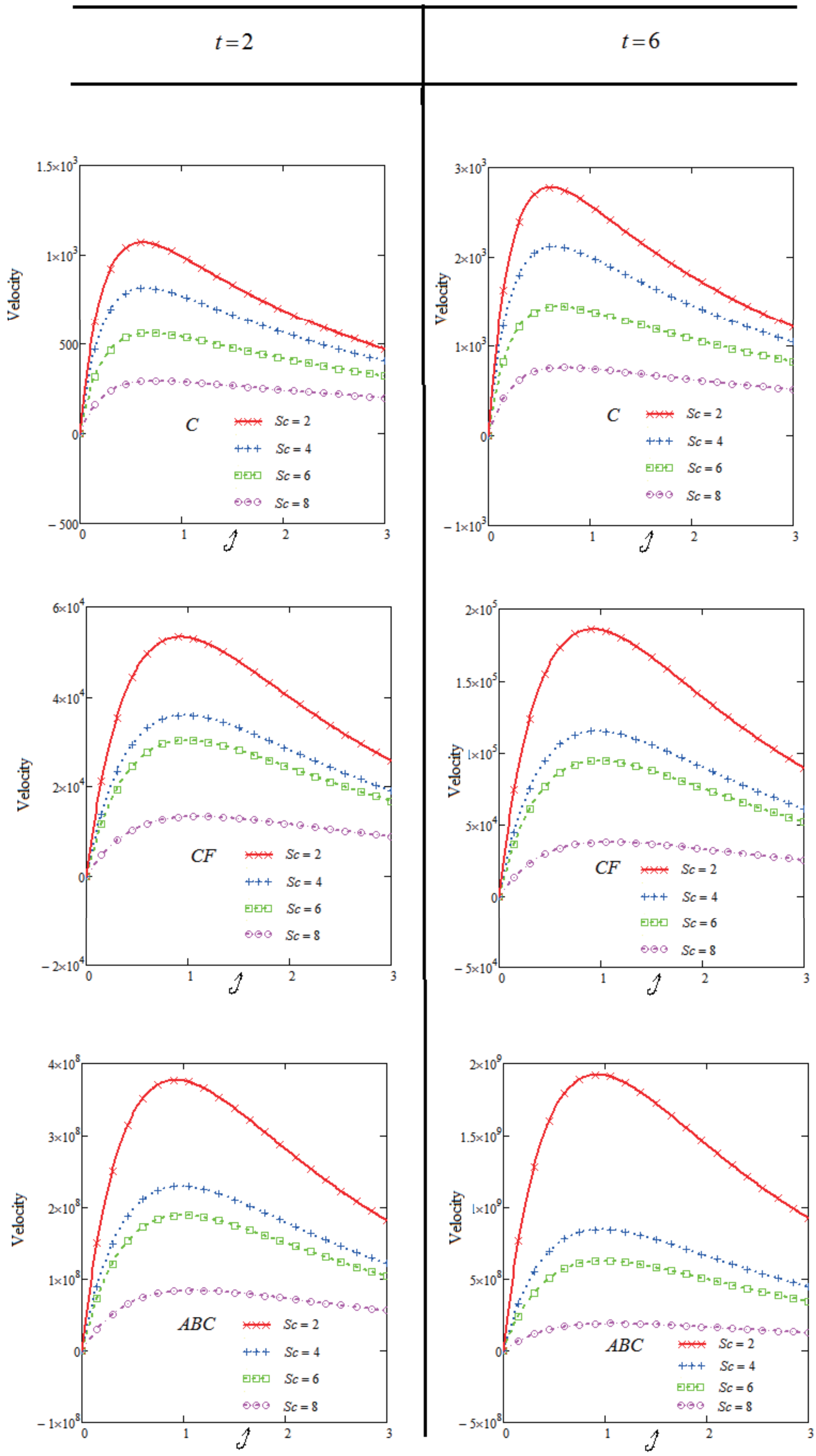

FIGURE 5. Graphs for velocity with variable Sc and $\chi = 2$, $M = 2$, $Q = 0.1$, $Pr_{eff} = N = Sr = \lambda = 5$ and $f(\tau) = 1$.

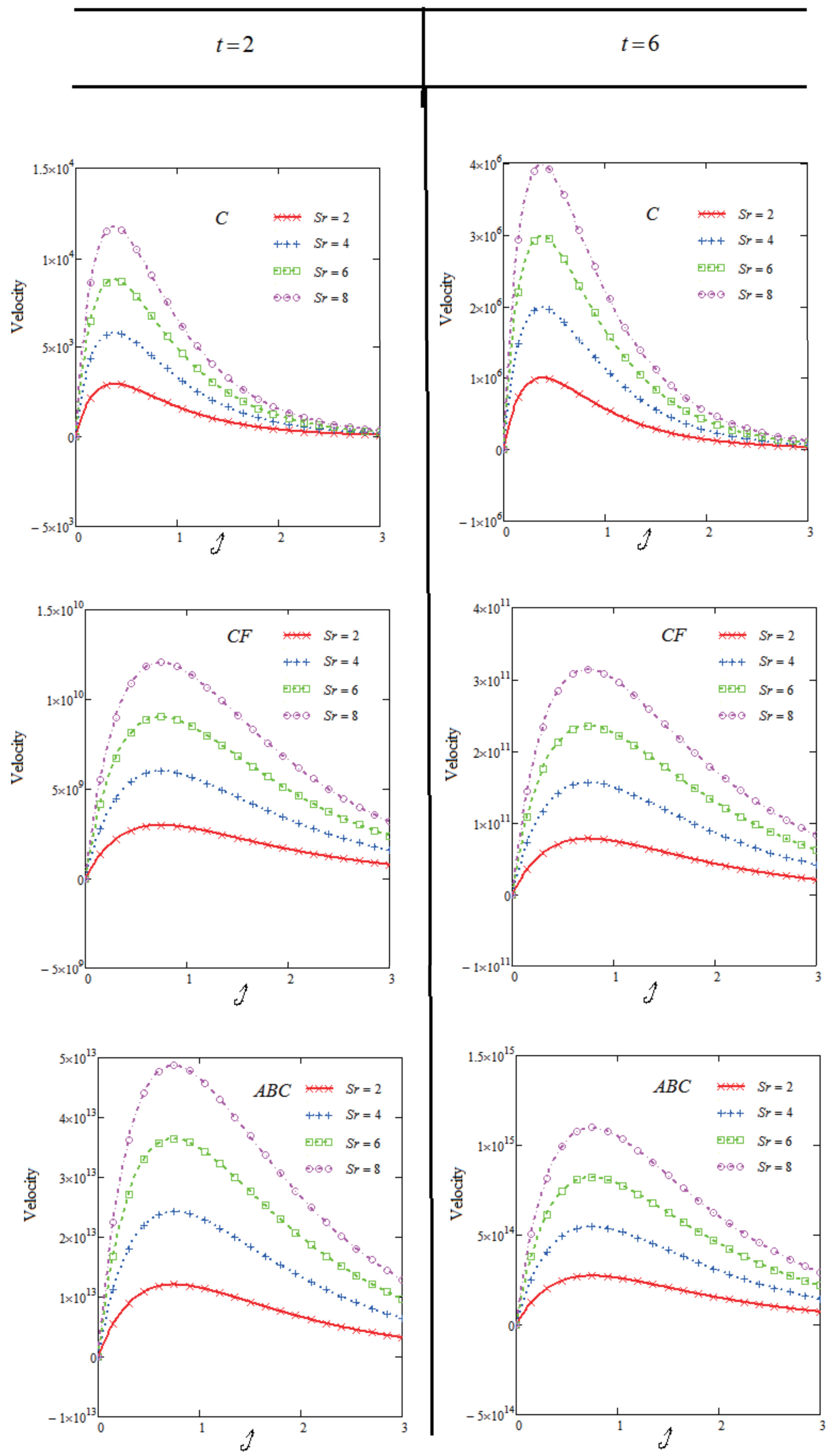

FIGURE 6. Graphs for velocity with variable Sr and $\chi = 2$, $M = 2$, $Sc = 0.66$, $Q = 0.1$, $Pr_{eff} = N = \lambda = 5$ and $f(\tau) = 1$.

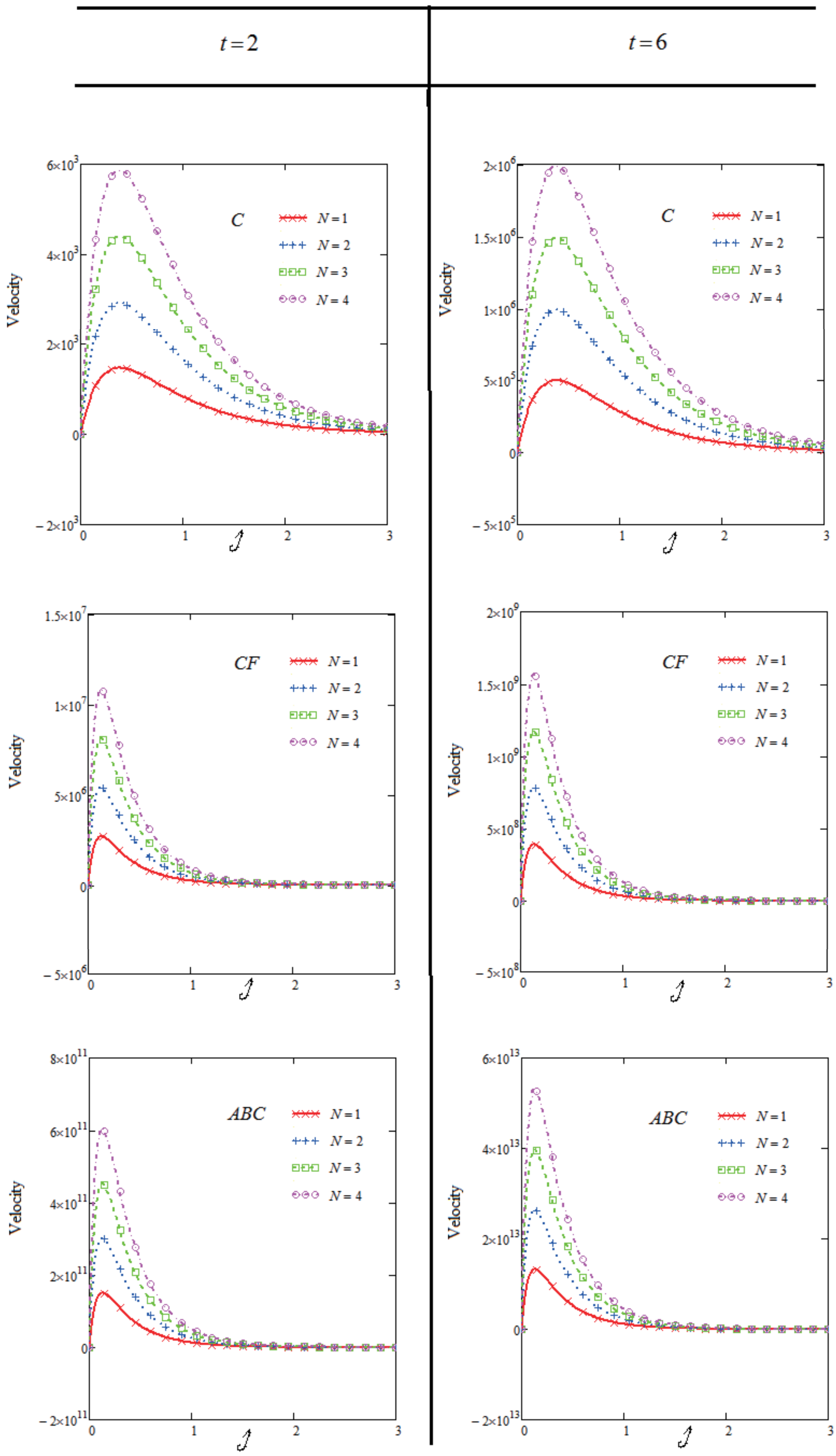

FIGURE 7. Graphs for velocity with variable positive values of N and $\chi = 2$, $M = 2$, $Q = 0.1$, $Sc = 0.66$, $Pr_{eff} = Sr = \lambda = 5$ and $f(\tau) = 1$.

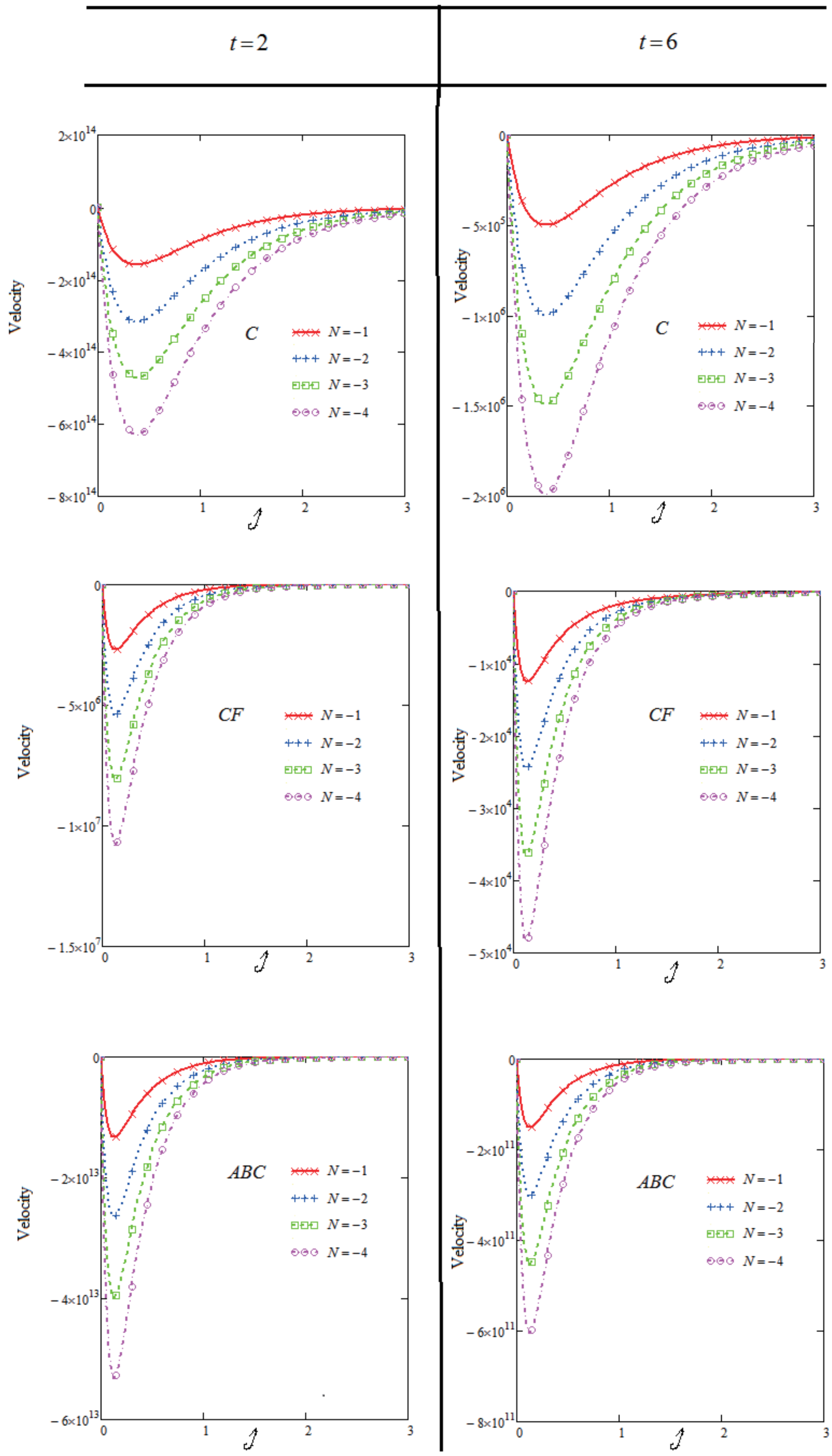

FIGURE 8. Graphs for velocity with variable negative values of N and $\chi = 2$, $M = 2$, $Q = 0.1$, $Sc = 0.66$, $Pr_{eff} = Sr = \lambda = 5$ and $f(\tau) = 1$.

Manuscript received August 5 2021
revised October 10 2021

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