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APPROXIMATING THE SOLUTION OF THE DIFFERENTIAL EQUATIONS WITH FRACTIONAL OPERATORS

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Abstract. In this manuscript, we described a brief history of fractional calculus from sixteenth century to twentieth century. Basic functions like gamma function and Mittag-Leffler function are defined which help to understand fractional calculus. Popular fractional integral and derivative operators are also defined. Fractional order ordinary and partial differential equations are also introduced. In this research, we study and proposed two basic methods to solve fractional order ordinary differential equations like semi-analytical method which is followed by Laplace Transform and a numerical method which is followed by Backward (Implicit) Euler's method. To support theses methods we also solved a few examples for better understanding.

1. Introduction

fractional calculus(Research Background) is described in detail. Secondly, a few basic definitions of fractional operators are presented with examples of a few functions. Lastly, some fractional order ordinary differential equations and partial differential equations will be discussed.

1.1. Research Background.

To classify the fractional calculus as a new branch of mathematics is absolutely inaccurate. Actually, the birth of classical and fractional calculus happens just about same. Actually, the fractional calculus started in 17^{th} century. Recently, some books on the topics of fractional calculus are published ([15, 26, 33]) and incorporate bundle of knowledge. A few research articles are also published on the history of fractional calculus ([31, 32]). Accordingly, the introduction this research article involved a brief description of history of fractional calculus which depends on the above books and research articles mentioned.

Newton and Leibniz are considered the pioneers of classical calculus. The startup of fractional calculus is linked with Leibniz. In reality, some questions about the fractional order. Particularly the 1/2 case were asked by L'Hospital who inspired Leibniz to think about fractional calculus [19]. Leibniz also discussed about fractional order derivatives with Johann Bernoulli in 1695 and John Willis in 1697 ([4, 6]). Euler discussed about fractional derivatives in [17]. He linked factorials with gamma functions as a generalization of it. Lagrange [18] worked on integer order power laws that can be moved to fractional order under some conditions. The first disclosed definition of fractional derivatives is given by Laplace in his book [11]. Lactoix, Euler, Fourier, Abel ([20, 30, 35]) were also contributed in the field of fractional calculus in eighteenth century.

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In nineteenth century some work on fractional derivatives is given by Watanabe, Riesz, B.S. Nagy, M. Caputo ([2, 13, 23, 27]).

In twentieth century, an advanced work on fractional calculus started in almost mid of it.

1.1.1. *What is Fractional Operator?* If η_1 and η_2 are two real numbers then the simplest fractional operator is given below

$$\Theta^{\eta_1}\Theta^{\eta_2} = \Theta^{\eta_1+\eta_2}. \tag{1.1}$$

A fractional operator is an operator which is applied to an operand with a non-integral power or a real number time. For example, if in particular $\eta_1 = \eta_2 = 1/2$, then in equation (1.1)

$$\Theta^{1/2}\Theta^{1/2} = \Theta^{1/2+1/2} = \Theta^1 = \Theta. \tag{1.2}$$

1.1.2. *Operator Definition and Properties.* This section includes some useful functions and some popular definitions of fractional derivatives as well as integrals. The literature of fractional calculus includes several fractional derivatives and integrals. Some popular names among them are

i) Riemann-Liouville type
ii) Caputo type
iii) Gr$\ddot{u}$nwald-Letnikof type and Chen type etc.

Their equivalence on some functions can be seen in the standard books ([3, 26, 33]).

About advatages and importance point of view, we can say that these definitions have their own place in several kind of mathematics related problems. First of all we define Gamma functions and then Mittag-Leffler functions.

1.1.3. *The Gamma-Functions.* In the fractional calculus theory, the most important and fundamental function is Gamma function, which is a generalized form of factorial, $n!$ which allows one to take n as a real number as well as a complex number.

This function is defined as

$$\Gamma(k) = \int_0^\infty e^{-\eta}\eta^{k-1}d\eta. \tag{1.3}$$

Its limiting form is defined as

$$\Gamma(k) = \lim_{\theta\to 0}\frac{\theta!\theta^\eta}{\eta(\eta+1)\dots(\eta+\theta)}, Re(\eta) > 0. \tag{1.4}$$

Useful properties of this function are

Firstly

$$\Gamma(k+1) = k\Gamma(k) = \theta(\theta-1)! = \theta!.$$

Secondly, at $\eta = -\theta$, $(\theta = 0, 1, 2, \dots)$ the Gamma function will have sample poles.

1.1.4. *The Mittag-Leffler Fucntion.* This function is a generalization for an exponential function in terms of one-parameter [1].

The generalization is given below

$$E_k(\eta) = \sum_{\theta=0}^{\infty} \frac{\eta^\theta}{\Gamma(k\eta + 1)}. \tag{1.5}$$

For two-parameters form see [36] and is given by

$$E_{k,q}(\eta) = \sum_{\theta=0}^{\infty} \frac{\eta^\theta}{\Gamma(k\eta + q)}, (k > 0, q > 0). \tag{1.6}$$

When $q = 1$, we get Mittag-Leffler Function in one-paramter.

This function can be seen particularly in a relationship with hyperbolic sine and hyperbolic cosine functions by

$$E_{2,1}(\eta^2) = cosh(\eta)$$

and

$$\eta E_{2,2}(\eta^2) = sinh(\eta),$$

and it can also be linked with error function and can also be used as a tool in the solution of systems of fractional order, for further details see ([7, 34]).

1.1.5. *The Riemann-Liouville Type.* Left Riemann-Liouville fractional derivative

$$^\sharp D^k_{x^+}[\phi(\eta)] = \frac{1}{\Gamma(\theta - k)} \frac{d^\theta}{d\eta^\theta} \int_x^\eta (\eta - \nu)^{\theta-k-1} \phi(\nu) d\nu, \eta \geq x. \tag{1.7}$$

Right Riemann-Liouville fractional derivative

$$^\sharp D^k_{y^-}[\phi(\eta)] = \frac{(1)^\theta}{\Gamma(\theta - k)} \frac{d^\theta}{d\eta^\theta} \int_\eta^y (\nu - \eta)^{\theta-k-1} \phi(\nu) d\nu, \eta \leq y. \tag{1.8}$$

For example, consider the function

$$\phi(\eta) = (\eta - x)^v,$$

$$^\sharp D^k[(\eta - x)^v] = \frac{\Gamma(1 + v)}{\Gamma(1 + v + k)} (\eta - x)^{v-k}.$$

Similarly, for $\phi(\eta) = e^{\gamma\eta}$

$$^\sharp D^k[e^{\gamma\eta}] = \eta^{-k} E_{1,1-k}(\gamma).$$

Left Riemann-Liouville fractional integral

$$^\sharp I^k_{x^+}[\phi(\eta)] = \frac{1}{\Gamma(k)} \int_x^\eta (\eta - \nu)^{k-1} \phi(\nu) d\nu, \eta \geq x. \tag{1.9}$$

Right Riemann-Liouville fractional integral

$$^\sharp I^k_{y^-}[\phi(\eta)] = \frac{1}{\Gamma(k)} \int_\eta^y (\nu - \eta)^{k-1} \phi(\nu) d\nu, \eta \leq y. \tag{1.10}$$

1.1.6. *The Caputo Type.* Left Caputo fractional derivative

$$ {}^{\sharp\sharp}D^{k}_{x^{+}}[\phi(\eta)] = \frac{1}{\Gamma(\theta - k)} \int_{x}^{\eta} (\eta - \nu)^{\theta-k-1} \frac{d^{\theta}}{d\nu^{\theta}}[\phi(\nu)]d\nu, \eta \geq x. \tag{1.11} $$

Right Caputo fractional derivative

$$ {}^{\sharp\sharp}D^{k}_{y^{-}}[\phi(\eta)] = \frac{(-1)^{\theta}}{\Gamma(\theta - k)} \int_{\eta}^{y} (\nu - \eta)^{\theta-k-1} \frac{d^{\theta}}{d\nu^{\theta}}[\phi(\nu)]d\nu, \eta \leq y. \tag{1.12} $$

The Caputo fractional integral

$$ {}^{\sharp\sharp}I^{k}[\phi(\eta) = \phi(\eta) - \sum_{i=0}^{\theta-1} \phi^{i}(0)\frac{\eta^{i}}{i!}, (\theta - 1 < k \leq \theta). \tag{1.13} $$

For more details see [16]. The Caputo fractional derivative is useful for both kind of homogeneous and in-homogeneous boundary as well as initial conditions in application of modeling of several real life systems. Relationships between Riemann-Liouville type fractional integrals and derivatives with Caputo type fractional integrals and derivatives is discussed with the help of Laplace transforms, see [25].

1.1.7. *The Grünwald-Letnikof Type.* Left Grünwald-Letnikof fractional derivative

$$ {}^{\sharp\sharp\sharp}D^{k}_{x^{+}}[\phi(\eta)] = \lim_{p\to 0} \frac{1}{p^{k}} \sum_{i=0}^{\theta} (-1)^{i} \frac{\Gamma(k+1)\phi(\eta - ip)}{\Gamma(i+1)\Gamma(k-i+1)}, \theta p = \eta - x. \tag{1.14} $$

Right Grünwald-Letnikof fractional Derivative

$$ {}^{\sharp\sharp\sharp}D^{k}_{y^{-}}[\phi(\eta)] = \lim_{p\to 0} \frac{1}{p^{k}} \sum_{i=0}^{\theta} (-1)^{i} \frac{\Gamma(k+1)\phi(\eta + ip)}{\Gamma(i+1)\Gamma(k-i+1)}, \theta p = y - \eta. \tag{1.15} $$

1.1.8. *The Chen Type.* Left Chen Fractional Derivative

$$ D^{k}_{z}[\phi(\eta)] = \frac{1}{\Gamma(1-k)} \frac{d}{d\eta} \int_{z}^{\eta} (\eta - \nu)^{-k} \phi(\nu) d\nu, \eta > z. \tag{1.16} $$

Right Chen fractional derivative

$$ D^{k}_{z}[\phi(\eta)] = \frac{-1}{\Gamma(1-k)} \frac{d}{d\eta} \int_{\eta}^{z} (\nu - \eta)^{-k} \phi(\nu) d\nu, \eta < z. \tag{1.17} $$

Left Chen fractional integral

$$ I^{k}_{z}[\phi(\eta)] = \frac{1}{\Gamma(k)} \int_{z}^{\eta} (\eta - \nu)^{k-1} \phi(\nu) d\nu, \eta > z. \tag{1.18} $$

Right Chen fractional integral

$$ I^{k}_{z}[\phi(\eta)] = \frac{1}{\Gamma(k)} \int_{\eta}^{z} (\nu - \eta)^{k-1} \phi(\nu) d\nu, \eta < z. \tag{1.19} $$

1.2. **Fractional Order Differential Equations.** In this section, we will define fractional order ordinary and partial differential equations.

1.2.1. *Fractional Order ODE's.* Those ordinary differential equations which involve the derivatives of non-integral order are known as fractional order ordinary differential equations. The researchers in the history of mathematics got an interest in fractional order ordinary differential equations not so far but since the mid of twentieth century. The reason of the development and progress in fractional order ordinary differential equations is that these equations can make complex kind of mathematical models easily. Other than local connections in space-time, this kind of equations solved with the help of kernels present in integral equations. Also these equations are useful in science and engineering almost all over the world of researchers [24].

In this section,firstly, we will define Riemann-Liouville fractional order differential equation. Secondly, we will define Caputo fractional order differential equation. Lastly, we will state a result which will relate Caputo differential equation with Volterra integral equation.

1.2.2. *Riemann-Liouville Fractional Order Differential Equation.*

$$({}^{\sharp}D^{k}_{x^+}\phi)(\eta) = \psi[\eta, \phi(\eta)], k > 0, \eta > x. \tag{1.20}$$

Along with the conditions

$$({}^{\sharp}D^{k-i}_{x^+}\phi)(x^+) = y_i, i = 1, 2, 3, \dots, \theta.$$

1.2.3. *Caputo Fractional Order Differential Equations.*

$$({}^{\sharp\sharp}D^{k}_{x^+}\phi)(\eta) = \psi[\eta, \phi(\eta)], k > 0, \eta > x. \tag{1.21}$$

Along with the initial conditions

$$({}^{\sharp\sharp}D^{i}\phi)(0) = y_i, i = 0, 1, 2, 3, \dots, \theta - 1.$$

Lemma 1.1. *Let $J_c(0) = [0, c]$ be an interval in which the function $\phi(\eta)$ has continuous derivatives with outputs in $[\phi_0 - \xi, \phi_0 + \xi]$, then $\phi(\eta)$ satisfies the equation*

$${}^{\sharp\sharp}D^{k}\phi(\eta) = \psi(\eta, \phi(\eta)), 0 < k \leq 1, \eta > 0, \phi(0) = \phi_0$$

if it satisfies the following Volterra integral equation [5]

$$\phi(\eta) = \phi_0 + \frac{1}{\Gamma(k)}\int_0^{\eta}(\eta - \nu)^{k-1}\psi(\nu, \phi(\nu))d\nu.$$

1.2.4. *Fractional Order PDE's.* Derivation of theorems of existence as well as uniqueness for ordinary differential equations as well as partial differential equations is already a hard problem that is seen by the researchers in the history of mathematics. As a result, each problem is a special case inside itself. Therefore for fractional partial differential equations, the process of finding solutions will be more awful. Hence, in this section we will present only some results for simple fractional partial differential equations.

1.2.5. *Linear and Homogeneous Fractional Order Partial Differential Equation.* Consider

$$\frac{\partial^k \phi(\eta,\xi)}{\partial t^k} = i\frac{\partial^q \phi(\eta,\xi)}{\partial x^q}, \xi \in \mathbb{R}^+, \eta \in \mathbb{R} \tag{1.22}$$

and the initial condition is

$$\phi(\eta, 0) = \phi_0(\eta),$$

where i is a coefficient which is positive, $0 < k < q \leq 1$, $\phi(\eta,\xi)$ is a real valued function, $\frac{\partial^k}{\partial t^k}$ and $\frac{\partial^q}{\partial x^q}$ are Riemann-Liouville fractional partial derivatives and $\phi_0(\eta)$ is continuous over all values of η. The general solution of equation (1.22) is given by the following lemma.

Lemma 1.2. *Assume that both* $\eta, \xi \in \mathbb{R}^+$ *then general solution of equation* (1.22) *is*

$$\phi(\eta,\xi) = \phi_0((\eta^q + \frac{\Gamma(q+1)}{\Gamma(k+1)} i\xi^k)^{\frac{1}{q}}), \tag{1.23}$$

where $\phi_0(\eta)$ *is available in initial condition.*

1.2.6. *In-Homogeneous Fractional Order Partial Differential Equation.* Consider

$$\frac{\partial^k \phi(\eta,\xi)}{\partial t^k} - i\frac{\partial^q \phi(\eta,\xi)}{\partial x^q} = \psi(\eta,\xi), \xi \in \mathbb{R}^+, \eta \in \mathbb{R} \tag{1.24}$$

and the initial condition is

$$\phi(\eta, 0) = \phi_0(\eta).$$

The general solution of equation (1.24) can be found with the help of Fourier transforms.

1.2.7. *Fractional Partial Differential Equation (Wave).* Consider

$$\frac{\partial^{2k} \phi(\eta,\xi)}{\partial t^{2k}} = i^2\frac{\partial^{2q} \phi(\eta,\xi)}{\partial x^{2q}}, \eta \in [x,y], \xi \in \mathbb{R}^2, \eta \in \mathbb{R}. \tag{1.25}$$

Along with the boundary and initial conditions

$$\begin{aligned}
\phi(\eta, 0) &= \phi_0(\eta), \\
\phi(x, \xi) &= 0, \\
\frac{d^k \phi(\eta, 0)}{dt^k} &= \frac{d^k \phi_0(\eta)}{dt^k}, \\
\phi(y, \xi) &= 0.
\end{aligned}$$

Where $0 < 2k < 1$ and $1 < 2q < 2$.

The general solution of equation (1.25) is given below

$$\phi(\eta,\xi) = \psi_1(\eta^q + i\frac{\Gamma(1+q)}{\Gamma(1+k)}\xi^k) + \psi_2(\eta^q - i\frac{\Gamma(1+q)}{\Gamma(1+k)}\xi^k), \tag{1.26}$$

where ψ_1 and ψ_2 are functions given by boundary and initial conditions.

1.2.8. *The Diffusion Fractional Equations.* The diffusion is modeled in a specific kind of porous medium in geometric media is an application of non-integer order derivative. The modeled equation is connected to geometric dimension of the material which must be porous type [14].

Consider

$$D^{1/k-1}\phi(\eta) = M\psi(\eta), \tag{1.27}$$

where $\phi(\eta)$ is across geometric association and a greater or observable flow, M is a constant number, k is geometric dimension and $\psi(\eta)$ is known as driving force.

Another diffusion fractional equation is given in [29].

Consider

$$D^{2/k_1}\phi(\eta,\xi) = \frac{1}{\eta^{k_2-1}}\frac{\partial}{\partial\eta}(\eta^{k_2-1}\frac{\partial\phi(\eta,\xi)}{\partial\eta}), \tag{1.28}$$

where k_1 and k_2 confide in the geometric dimension of media. For more details see [8].

Another diffusion fractional differential equation of one-dimension is given below

$$D^k\phi(\eta,\xi) = \frac{d^2\phi(\eta,\xi)}{d\eta^2}, k \in \mathbb{R}. \tag{1.29}$$

If in particular $k = 1$, then the equation (1.29) becomes diffusion classical wave equation, if in particular $k = 2$, then the equation (1.29) becomes wave classical equation. Lastly for $0 < k < 1$ and for $0 < k < 2$ the equation (1.29) will be called diffusion ultra-slow process and intermediate ultra-slow processes, respectively, see [37].

2. Materials and methods

In this section, we will present two methods to solve fractional order ordinary differential equations by means of two techniques; the first techniques is semi-analytical method which follows Laplace transforms whereas the second technique is a numerical approach in which we present Euler's method for one term and multi-term fractional order ordinary differential equations and lastly we will present some important results of existence and uniqueness of fractional order ordinary differential equations.

2.1. **Methods to Solve Fractional Differential Equations.**

2.1.1. *Method 1: Semi-analytical.* In this section, we will use Laplace transform to solve fractional order ordinary differential equations with the fractional derivative of Riemann-Liouville and Caputo type.

2.1.2. *The Laplace Transform [21].* Let $\phi(\eta)$ be a function with $\eta \in [0,\infty)$ then the Laplace transform of $\phi(\eta)$ is defined as

$$\mathcal{L}[\phi(\eta)] = \hat{\phi}(s) = \int_0^\infty e^{-s\eta}\phi(\eta)d\eta. \tag{2.1}$$

The integral in (2.1) converges only when the function $\phi(\eta)$ is known to be of exponential order $k > 0$ [21], i.e.

$$\lim_{\eta\to\infty} e^{-k\eta}|\phi(\eta)| \leq \eta', \forall \eta > \xi; \eta', \xi \in [0, \infty). \tag{2.2}$$

After transforming our original problem from original variable η to a new variable s, we must need to transform our problem from new variable s to old variable η, so we will need the inverse of Laplace transform in this regard [22].

The inverse of Laplace transform is defined as:

$$\phi(\eta) = \mathcal{L}^{-1}[\hat{\phi}(s)] = \frac{1}{2\pi\iota} \lim_{\xi\to\infty} \int_{\nu-\iota\xi}^{\nu-\iota\xi} e^{s\eta}\hat{\phi}(s)ds, \nu \in \mathbf{R}. \tag{2.3}$$

The formula given in (2.3) used rarely while solving problems but in purposes of theory, the researchers use this formula many times.

The Laplace transform of convolution of two functions:

The convolution of $\phi(\eta)$ and $\psi(\eta)$ is denoted by $\phi(\eta) * \psi(\eta)$ [12] and is defined as

$$\phi(\eta) * \psi(\eta) = \int_0^\eta \phi(\eta - \nu)\psi(\nu)d\nu = \int_0^\eta \phi(\nu)\psi(\eta - \nu)d\nu$$

and when $\eta < 0$, then the convolution $\phi(\eta) * \psi(\eta) = 0$.

The Laplace transform of convolution $\phi(\eta) * \psi(\eta)$ is

$$\mathcal{L}[\phi(\eta) * \psi(\eta)] = \hat{\phi}(s)\hat{\psi}(s) \tag{2.4}$$

with $\hat{\phi}(s)$ and $\hat{\psi}(s)$ exist.

The Laplace transform of a few functions is given below:

Let $b \in \mathbf{R}$

$$\mathcal{L}[e^{b\eta}] = \frac{1}{s-b},$$
$$\mathcal{L}[cos(b\eta)] = \frac{s}{s^2+b^2},$$
$$\mathcal{L}[sin(b\eta)] = \frac{b}{s^2+b^2},$$
$$\mathcal{L}[cosh(b\eta)] = \frac{s}{s^2-b^2},$$
$$\mathcal{L}[sinh(b\eta)] = \frac{b}{s^2-b^2},$$
$$\mathcal{L}[\alpha\eta^k] = \frac{\alpha\Gamma(k)}{s^{k+1}},$$
$$\mathcal{L}[e^{bt}\phi(\eta)] = \hat{\phi}(s-b),$$
$$\mathcal{L}[b\phi(\eta) + c\psi(\eta)] = b\hat{\phi}(\eta) + c\hat{\psi}(\eta).$$

For any positive integer $\sigma > 0$

$$\mathcal{L}[\phi^\sigma(\eta)] = s^\sigma\hat{\phi}(\eta) - s^{\sigma-1}\phi(0) - s^{\sigma-2}\phi'(0) - \ldots - \phi^{(\sigma-1)}(0),$$
$$\mathcal{L}[\eta^{ki-1-1}E_{k,q}^{(i)}(b\eta^k)] = \frac{i!s^{k-1}}{(s^k-b)^{i+1}}.$$

2.1.3. *Laplace Transform of Riemann-Liouville Fractional Order Integral.*

$$\mathcal{L}[{}^{\sharp}\mathbf{I}^k[\phi(\eta)]] = \mathcal{L}[\frac{1}{\Gamma(k)}\int_0^{\eta}(\eta-\nu)^{k-1}\phi(\nu)d\nu].$$

Using the definition of convolution of two functions

$$\mathcal{L}[\frac{1}{\Gamma(k)}\eta^{k-1} * \phi(\eta)] = \mathcal{L}[\frac{1}{\Gamma(k)}\eta^{k-1}]\mathcal{L}[\phi(\eta)],$$

we have

$$\mathcal{L}[{}^{\sharp}\mathbf{I}^k[\phi(\eta)]] = \frac{1}{s^k}\hat{\phi}(s).$$

Note that ${}^{\sharp}\mathbf{I}^k[\phi(\eta)] = D^{-k}[\phi(\eta)]$.

2.1.4. *Laplace Transform of Riemann-Liouville Fractional Order Derivative.* The Riemann-Liouville fractional order derivative is:

$${}^{\sharp}D^k[\phi(\eta)] = \frac{1}{\Gamma(\theta-k)}\frac{d^{\theta}}{d\eta^{\theta}}\int_0^{\eta}(\eta-\nu)^{\theta-k-1}\phi(\nu)d\nu,$$

$${}^{\sharp}D^k[\phi(\eta)] = \psi^{(\theta)}(\eta), \tag{2.5}$$

$${}^{\sharp}D^k[\phi(\eta)] = \frac{d^{\theta}}{d\eta^{\theta}}[\psi(\eta)]. \tag{2.6}$$

Integrating both sides of (2.6) θ times we get

$$\psi(\eta) = D^{-(\theta-k)}[\phi(\eta)], \tag{2.7}$$

where $D^{-(\theta-k)}$ in (2.7) is fold integration of order $(k-\theta)$.

$$\psi(\eta) = \frac{1}{\Gamma(\theta-k)}\int_0^{\eta}(\eta-\nu)^{\theta-k-1}\phi(\nu)d\nu. \tag{2.8}$$

Apply Laplace transform on both sides of (2.8)

$$\hat{\psi}(s) = \mathcal{L}[\frac{1}{\Gamma(\theta-k)}\int_0^{\eta}(\eta-\nu)^{\theta-k-1}\phi(\nu)d\nu],$$

$$\hat{\psi}(s) = s^{-(\theta-k)}\hat{\phi}(s). \tag{2.9}$$

Apply Laplace transform on both sides of (2.5)

$$\begin{aligned}\mathcal{L}[{}^{\sharp}D^k[\phi(\eta)]] &= \mathcal{L}[\psi^{\theta}(\eta)]\\ &= s^{\theta}\hat{\phi}(\eta) - s^{\theta-1}\phi(0) - s^{\theta-2}\phi'(0) - \ldots - \phi^{(\theta-1)}(0)\\ &= s^{\theta}\hat{\phi}(\eta) - \sum_{i=0}^{\theta-1} s^i\psi^{(\theta-i-1)}(0)\end{aligned} \tag{2.10}$$

and we also note that if

$$\psi(\eta) = D^{-(\theta-k)}[\phi(\eta)].$$

Differentiate both sides of above equation with respect to η upto $(\theta-i-1)$ times, we get

$$\psi^{\theta-i-1}(\eta) = {}^{\sharp}D^{k-i-1}[\phi(\eta)] \tag{2.11}$$

at $\eta = 0$ (2.11) becomes

$$\psi^{\theta-i-1}(0) = {}^{\sharp}D^{k-i-1}[\phi(0)] \tag{2.12}$$

put (2.12) in (2.10), we get

$$\mathcal{L}[{}^{\sharp}D^{k}[\phi(\eta)]] = s^{\theta}\hat{\phi}(\eta) - \sum_{i=0}^{\theta-1} s^{i\sharp}D^{(k-i-1)}[\phi(0)]. \tag{2.13}$$

It is important to note that the formula of Laplace transform for integer order derivative of a function is similar to that of the fractional order derivative in Riemann-Liouville type.

2.1.5. *Laplace Transform of Caputo Fractional Order Derivative.* The mathematical definition of Caputo fractional order derivative is:

$$^{\sharp\sharp}D^{k}[\phi(\eta)] = \frac{1}{\Gamma(\theta-k)}\int_{0}^{\eta}(\eta-\nu)^{\theta-k-1}\frac{d^{\theta}}{d\nu^{\theta}}[\phi(\nu)]d\nu. \tag{2.14}$$

The alternate form of definition (2.14) is:

$$^{\sharp\sharp}D^{k}[\phi(\eta)] = D^{-(\theta-k)}D^{\theta}[\phi(\eta)], \theta-1 \leq k < \theta. \tag{2.15}$$

We can also say that the Caputo fractional order derivative is equivalent to $(\theta - k) - th$ order fold integration of $\theta - th$ order differentiation of the function $\phi(\eta)$.

Apply Laplace transform on both sides of (2.15)

$$\mathcal{L}[{}^{\sharp\sharp}D^{k}[\phi(\eta)]] = \mathcal{L}[D^{-(\theta-k)}D^{\theta}[\phi(\eta)]]. \tag{2.16}$$

The right hand side of (2.16) is similar to Riemann-Liouville integral, so

$$\mathcal{L}[{}^{\sharp\sharp}D^{k}[\phi(\eta)]] = s^{-(\theta-k)}\mathcal{L}[D^{\theta}[\phi(\eta)]]. \tag{2.17}$$

Expanding the formula for Laplace transform of integer order differentiation we get

$$\begin{aligned}
&s^{-(\theta-k)}[s^{\theta}\hat{\phi}(s) - \sum_{i=0}^{\theta-1} s^{i}\phi^{(\theta-i-1)}[\phi(0)]] \\
&= s^{-(\theta-k)}[s^{\theta}\hat{\phi}(s) - [s^{0}\phi^{(\theta-1)}(0) + s^{1}\phi^{(\theta-2)}(0) + s^{2}\phi^{(\theta-3})(0) + \ldots + s^{(\theta-1)}\phi(0)]] \\
&= s^{-(\theta-k)}[s^{\theta}\hat{\phi}(s) - \sum_{i=0}^{\theta-1} s^{(\theta-i-1)}\phi^{(i)}(0)]] \\
&= s^{-\theta+k+\theta}\hat{\phi}(s) - \sum_{i=0}^{\theta-1} s^{\theta-i-1-\theta+k}\phi^{(i)}(0).
\end{aligned}$$

Hence, the required formula for Laplace transform of Caputo fractional order derivative is:

$$\mathcal{L}[{}^{\sharp\sharp}D^{k}[\phi(\eta)]] = s^{k}\hat{\phi}(s) - \sum_{i=0}^{\theta-1} s^{k-i-1}\phi^{(i)}(0). \tag{2.18}$$

2.1.6. *Initial Value Problem of Riemann-Liouville Type.* Problem:

$$^{\sharp}D^{k}[\phi(\eta)] - \lambda\phi(\eta) = b\eta, \eta > 0, \theta - 1 < k < \theta. \tag{2.19}$$

Along with the initial conditions

$$^{\sharp}D^{k-i-1}[\phi(\eta)]|_{\eta=0} = c_i, i = 0, 1, 2, \ldots, \theta - 1.$$

Solution:

Take Laplace transform on both sides of (2.19)

$$\mathcal{L}[^{\sharp}D^{k}[\phi(\eta)] - \lambda\phi(\eta)] = \mathcal{L}[b\eta],$$

$$s^{k}\hat{\phi}(s) - \lambda\hat{\phi}(s) = \frac{b}{s^2} + \sum_{i=0}^{\theta-1} c_i s^i,$$

$$(s^{k} - \lambda)\hat{\phi}(s) = \frac{b}{s^2} + \sum_{i=0}^{\theta-1} c_i s^i,$$

$$\hat{\phi}(s) = \frac{bs^{-2}}{(s^k - \lambda)} + \sum_{i=0}^{\theta-1} \frac{c_i s^i}{(s^k - \lambda)}. \tag{2.20}$$

Apply inverse of Laplace transform on both sides of (2.20), we get

$$\mathcal{L}^{-1}[\hat{\phi}(s)] = \mathcal{L}^{-1}[\frac{bs^{-2}}{(s^k - \lambda)} + \sum_{i=0}^{\theta-1} \frac{c_i s^i}{(s^k - \lambda)}].$$

Hence, the required analytical solution is:

$$\phi(\eta) = b\eta^{k+1}E_{k,k+2}(\lambda\eta^{k}) + \sum_{i=0}^{\theta-1} c_i \eta^{k-i-1} E_{k,k-i}(\lambda\eta^{k}). \tag{2.21}$$

2.1.7. *Initial Value Problem of Caputo Type.* Problem:

$$^{\sharp\sharp}D[\phi(\eta)] - \lambda\phi(\eta) = b\eta, \eta > 0, \theta - 1 < k < \theta. \tag{2.22}$$

Along with the initial conditions

$$\phi^{(i)}(0) = c_i, i = 0, 1, 2, \ldots, \theta - 1.$$

Solution:

Take Laplace transform on both sides of (2.22)

$$\mathcal{L}[^{\sharp\sharp}D[\phi(\eta)] - \lambda\phi(\eta)] = \mathcal{L}[b\eta],$$

$$s^{k}\hat{\phi}(s) - \sum_{i=0}^{\theta-1} s^{k-i-1}\phi^{(i)}(0) - \lambda\hat{\phi}(s) = \frac{b}{s^2},$$

$$(s^{k} - \lambda)\hat{\phi}(s) = \frac{b}{s^2} + \sum_{i=0}^{\theta-1} s^{k-i-1} c_i,$$

$$\hat{\phi}(s) = \frac{bs^{-}2}{(s^k - \lambda)} + \sum_{i=0}^{\theta-1} \frac{s^{k-i-1} c_i}{(s^k - \lambda)}. \tag{2.23}$$

Apply inverse of Laplace transform on both sides of (2.23), we get

$$\mathcal{L}^{-1}[\hat{\phi}(s)] = \mathcal{L}^{-1}[\frac{bs^-2}{(s^k-\lambda)} + \sum_{i=0}^{\theta-1} \frac{s^{k-i-1}c_i}{(s^k-\lambda)}].$$

Hence, the required analytical solution is:

$$\phi(\eta) = b\eta^{k+1}E_{k,k+2}(\lambda\eta^k) + \sum_{i=0}^{\theta-1} c_i\eta^i E_{k,i+1}(\lambda\eta^k). \tag{2.24}$$

2.1.8. *Method 2: Numerical.* Numerical methods to solve differential equations are basically of two categories.

(i) One-step methods
(ii) Multi-step methods

One-step methods are suitable only when single iteration is required to obtain a numerical value of the dependent variable using the values from the previous iteration. Whereas in the multi-step methods we use more than one previously obtained iterations to obtain the solution. The better choice of numerical method for fractional order differential equations is to use the multi-step methods. While using multi-step methods in solving fractional order differential equations each iteration involves all the previous iterations and values, so we can say that the solution process may takes more time in computations. Roughly speaking, we can say that the multi-step methods are quadrature type convolution formulas.

Let $p = \lceil k \rceil$ and consider a fractional order differential equation with the derivative of Caputo type:

$$^{\sharp\sharp}D^k[\phi(\eta)] = \psi(\eta, \phi(\eta)) \tag{2.25}$$

with the initial conditions

$$\phi(\eta_0) = \phi_0, \phi'(\eta_0) = \phi_0^{(1)}, \phi''(\eta_0) = \phi_0^{(2)}$$

$$\vdots$$

$$\phi^{(p-1)}(\eta_0) = \phi_0^{(p-1)},$$

where $\psi(\eta, \phi(\eta))$ is continuous and $\phi_0, \phi_0^{(1)}, \phi_0^{(2)}, \ldots, \phi_0^{(p-1)}$ are derivatives at initial value of the independent variable η.

Apply Riemann-Liouville integral on both sides of (2.47):

$$^{\sharp}\mathbf{I}^k[^{\sharp\sharp}D^k[\phi(\eta)]] = {}^{\sharp}\mathbf{I}^k[\psi(\eta, \phi(\eta))]$$

we get,

$$\phi(\eta) - T_{p-1}[\phi; \eta_0](\eta) = \frac{1}{\Gamma(k)} \int_{\eta_0}^{\eta} (\eta - \nu)^{k-1} \psi(\nu, \phi(\nu)) d\nu,$$

$$\phi(\eta) = T_{p-1}[\phi; \eta_0](\eta) + \frac{1}{\Gamma(k)} \int_{\eta_0}^{\eta} (\eta - \nu)^{k-1} \psi(\nu, \phi(\nu)) d\nu, \tag{2.26}$$

where $T_{p-1}[\phi;\eta_0](\eta)$ is a polynomial called Taylor's polynomial whose degree is $p-1$ which is for the function $\phi(\eta)$ is given below centered at η_0 [10].

$$T_{p-1}[\phi;\eta_0](\eta) = \sum_{i=0}^{p-1} \frac{(\eta-\eta_0)^i}{i!}\phi^{(i)}(\eta_0). \tag{2.27}$$

The multi-step method for (2.26) is written as a convolution type formula:

$$\phi_n = \xi_n + \sum_{i=0}^{n} b_{n-i}\psi_i, \psi_i = \psi(\eta_i, \phi_i), \tag{2.28}$$

where ξ_n and b_n are the coefficients and $\eta_n = \eta_0 + nh$ is a grid which is assigned with a fixed step-size. The coefficients ξ_n and b_n will be derived using two categories of multi-step methods.

(i) Product-integration method
(ii) Fractional multi-step method of linear type

The Product-integration method and fractional multi-step method of linear type are completely based on the integral in (2.47).

2.1.9. *Product-Integration Methods.* The Product-integration methods were originally introduced in 1954 [28]. The Product-integration methods are used to solve weakly-singular Volterra integral equations of second kind and as in (2.26) we derived a weakly-singular Volterra integral equations of second kind, so we can relate Product-integration method to fractional order ordinary differential (2.47).

The Product-integration method for (2.26) is given by a grid $\eta_n = \eta_0 + nh, h > 0$ is a fixed step size.

$$\phi(\eta_n) = T_{p-1}[\phi;\eta_0](\eta) + \frac{1}{\Gamma(k)} \sum_{i=0}^{n-1} \int_{\eta_i}^{\eta_{i+1}} (\eta_n - \nu)^{k-1}\psi(\nu, \phi(\nu))d\nu, \tag{2.29}$$

where we approximate $\psi(\nu, \phi(\nu))$ in each $[\eta_i, \eta_{i+1}]$, with the help of some polynomials through interpolation.

The Forward Euler's method for fractional order differential equations:

$$\phi_n = T_{p-1}[\phi;\eta_0](\eta_n) + h^k \sum_{i=0}^{n-1} c_{n-i-1}^{(k)}\psi(\eta_i, \phi_i). \tag{2.30}$$

The Backward Euler's method for fractional order differential equations:

$$\phi_n = T_{p-1}[\phi;\eta_0](\eta_n) + h^k \sum_{i=0}^{n} c_{n-i}^{(k)}\psi(\eta_i, \phi_i), \tag{2.31}$$

where

$$c_n^{(k)} = \frac{(n+1)^k - n^k}{\Gamma(k+1)}$$

and

$$\psi(\nu, \phi(\nu)) = \psi(\eta_{i+1}, \phi_{i+1}) + \frac{\nu - \eta_{i+1}}{h}[\psi(\eta_{i+1}, \phi_{i+1}) - \psi(\eta_i, \phi_i)], \nu \in [\eta_i, \eta_{i+1}].$$

The order of convergence is $h > 0$,i.e.

$$|\phi(\eta_n) - \phi_n| = O(h)$$

with $h \to 0$.

Here note that $\phi(\eta_n)$ is analytical solution of fractional differential equation(2.47).

2.1.10. *Fractional Multi-step Methods of Linear Type.* As we know that the Riemann-Liouville fractional integral is:

$$^{\sharp}\mathbf{I}^{k}_{\eta_0}[\phi(\eta)] = \frac{1}{\Gamma(k)}\int_{\eta_0}^{\eta}(\eta-\nu)^{k-1}\phi(\nu)d\nu. \tag{2.32}$$

The fractional multi-step form of (2.32) is given by Lubich [28].

$$^{\sharp}_{h}\mathbf{I}^{k}_{\eta_0} = h^k\sum_{i=0}^{n}\Theta^{(k)}_{n-i}\phi(\eta_i), \tag{2.33}$$

where $\Theta^{(k)}_n$ are obtained below

$$\sum_{i=0}^{\infty}\Theta^{(k)}_n\mu^n = \Theta^{(k)}(\mu), \Theta^{(k)}(\mu) = (\delta(\mu))^{-k}, \tag{2.34}$$

where

$$\delta(\mu) = \frac{\rho(1/\mu)}{\sigma(1/\mu)}, \tag{2.35}$$

$$\rho(x) = \rho_0 x^i + \rho_0 x^{i-1} + \rho_1 x^{i-2} + \ldots + \rho_i,$$

$$\sigma(x) = \sigma_0 x^i + \sigma_0 x^{i-1} + \sigma_1 x^{i-2} + \ldots + \sigma_i,$$

and $\delta(\mu)$ is a generating function for multi-step method of linear type which has not any zeros in the dist $|\mu| \leq 1$ that is also closed.

The approximation in (2.33) is of order $q > 0$ and its convergence is followed by (Theorem 2.1) in [28].

$$|^{\sharp}\mathbf{I}^{k}_{\eta_0}[\phi(\eta_n)] - {}^{\sharp}_{h}\mathbf{I}^{k}_{\eta_0}[\phi(\eta_n)]| \leq A(\eta_n - \eta_0)^{k-1-qh^q}, \tag{2.36}$$

where A is a constant and is not depending on $h > 0$. Note that $\phi(\eta)$ is smooth sufficiently. When $\phi(\eta)$ is non-smooth then the approximation form of (2.33) is:

$$^{\sharp}_{h}\mathbf{I}^{k}_{\eta_0}[\phi(\eta_n)] = h^k\sum_{i=0}^{v}\Theta_{n,i}\phi(\eta_i) + h^k\sum_{i=0}^{n}\Theta^{(k)}_{n-i}\phi(\eta_i). \tag{2.37}$$

As a result for fractional differential equations, the fractional multi-step method of linear type of (2.26) is:

$$\phi_n = T_{p-1}[\phi;\eta_0](\eta_n) + h^k\sum_{i=0}^{s}\Theta_{n,i}\psi(\eta_i,\phi_i) + h^k\sum_{i=0}^{n}\Theta^{k}_{n-i}\psi(\eta_i,\phi_i) \tag{2.38}$$

for more details see [9].

2.1.11. *Numerical Approach to Fractional Differential Equations With Many Terms.* The equation involving fractional derivatives in many terms is:

$$\lambda_\theta D^{k_\theta}_{\eta_0}[\phi(\eta)] + \lambda_{\theta-1} D^{k_{\theta-1}}_{\eta_0}[\phi(\eta)] + \cdots + \lambda_2 D^{k_2}_{\eta_0}[\phi(\eta)] + \lambda_1 D^{k_1}_{\eta_0}[\phi(\eta)] = \psi(\eta, \phi(\eta)) \tag{2.39}$$

in (2.39). $\lambda_1, \lambda_2, \dots, \lambda_{\theta-1}, \lambda_\theta$ are real coefficients and $k_\theta, k_{\theta-1}, \dots, k_2, k_1$ are fractional orders of derivatives either Riemann-Liouville or Caputo and are in descending order such that $k_\theta > k_{\theta-1} > \dots > k_2 > k_1$ with the leading term $\lambda_\theta \neq 0$. Here $p_j = \lceil k_j \rceil, j = 1, 2, \dots, \theta$.

The initial conditions to (2.39) are:

$$\begin{aligned} \phi(\eta_0) &= \phi_0, \\ \frac{d}{d\eta}\phi(\eta_0) &= \phi_0^{(1)}, \\ &\vdots \\ \frac{d^{p_\theta - 1}}{d\eta^{p_\theta - 1}}\phi(\eta_0) &= \phi_0^{p_\theta - 1}. \end{aligned}$$

(2.39) is known as multi-terms fractional order differential equation ([9, 28]).

The discrete form of (2.39) as a numerical approach is obtained by applying Riemann-Liouville integral operator on both sides of (2.39).

We set

$$\begin{aligned} \phi(\eta) = T_{p_{\theta-1}}[\phi; \eta_0](\eta) - \sum_{j=1}^{\theta-1} \frac{\lambda_j}{\lambda_\theta} \sharp\mathbf{I}^{k_\theta - k_j}_{\eta_0}[\phi(\eta) - T_{p_{j-1}}[\phi; \eta_0](\eta)] \\ + \frac{1}{\lambda_\theta} \sharp\mathbf{I}^{k_\theta}_{\eta_0}[\psi(\eta, \phi(\eta))]. \end{aligned} \tag{2.40}$$

Forward Euler's method for multi-term fractional order differential equation:

$$\begin{aligned} \phi_n = T_{p_{\theta-1}}[\phi; \eta_0](\eta) + \sum_{j=1}^{\theta-1} \frac{\lambda_j}{\lambda_\theta} \sum_{i=0}^{p_{j-1}} \frac{(\eta - \eta_0)^{i - k_\theta - k_j}}{\Gamma(i + k_\theta - k_j + 1} \phi^{(i)}(\eta_0) \\ - \sum_{j=1}^{\theta-1} \frac{\lambda_j}{\lambda_\theta} h^{k_\theta - k_j} \sum_{s=0}^{n-1} c^{(k_\theta - k_j)}_{n-s-1} \phi_s + \frac{1}{\lambda^\theta} h^{k_\theta} \sum_{s=0}^{n-1} c^{(k_\theta)}_{n-s-1} \psi(\eta_s, \phi_s). \end{aligned} \tag{2.41}$$

Backward Euler's method for multi-term fractional order differential equation:

$$\begin{aligned} \phi_n = T_{p_{\theta-1}}[\phi; \eta_0](\eta) + \sum_{j=1}^{\theta-1} \frac{\lambda_j}{\lambda_\theta} \sum_{i=0}^{p_{j-1}} \frac{(\eta - \eta_0)^{i - k_\theta - k_j}}{\Gamma(i + k_\theta - k_j + 1} \phi^{(i)}(\eta_0) \\ - \sum_{j=1}^{\theta-1} \frac{\lambda_j}{\lambda_\theta} h^{k_\theta - k_j} \sum_{s=1}^{n} c^{(k_\theta - k_j)}_{n-s} \phi_s + \frac{1}{\lambda^\theta} h^{k_\theta} \sum_{s=1}^{n-1} c^{(k_\theta)}_{n-s-1}. \end{aligned} \tag{2.42}$$

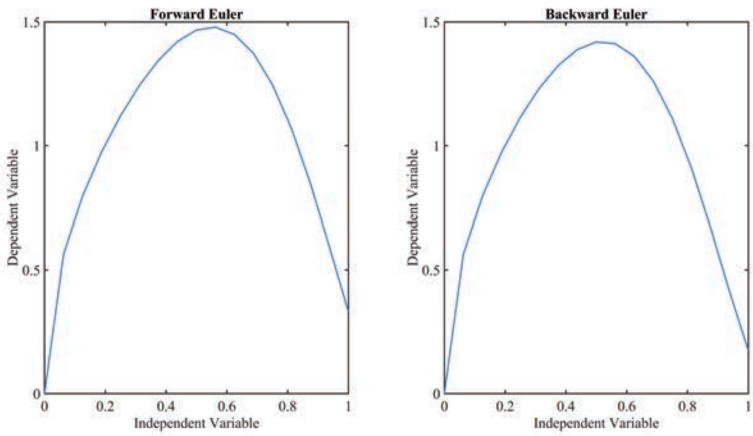
Example 2.1. Consider a fractional order ordinary differential equation:

$$D^k_{\eta_0}[\phi(\eta)] = -(\phi(\eta))^{1.5} + (1.5\eta^{0.5k} - \eta^4)^3 + 2.25\Gamma(k+1) - \frac{3\Gamma(5+0.5k)}{\Gamma(5-0.5k)} + \frac{40320\eta^{8-k}}{\Gamma(9-k)}. \tag{2.43}$$

Along with the initial condition $\phi(0) = 0$. The analytical solution [9] to (2.43) is

$$\phi(\eta) = \eta^8 - 3\eta^{4+0.5k} + \frac{9}{4}\eta^k. \tag{2.44}$$

We apply forward and backward Euler's method on (2.43) with a step size of $h = \frac{1}{2^4}$ and $k = \frac{1}{2}$ in the interval $[0, 1]$.



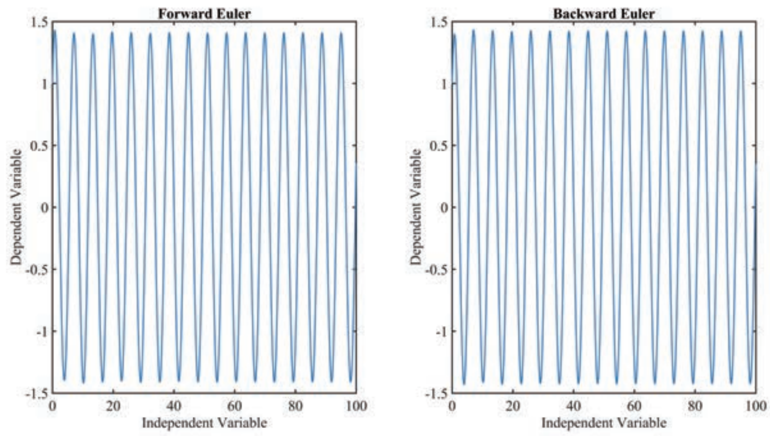
Example 2.2. Consider a Multi-term Fractional Order Ordinary Differential Equation:

$$\phi'''(\eta) + D^{2.5}_0\phi(\eta) + \phi''(\eta) + 4\phi'(\eta) + D^{0.5}_0\phi(\eta) + 4\phi(\eta) = 6cos(\eta). \tag{2.45}$$

Along with the initial conditions $\phi(0) = 1, \phi'(0) = 1, \phi''(0) = -1$.
The analytical solution to (2.45) is:

$$\phi(\eta) = \sqrt{2}sin(\frac{\pi}{4} + \eta). \tag{2.46}$$

We apply forward and backward Euler's method on (2.43) with a step size of $h = \frac{1}{2^4}$ in the interval $[0, 100]$.



2.2. **Important Results.** In this section we will present existence and uniqueness theorems for fractional order ordinary differential equations of Caputo Type [9].

Let $p = \lceil k \rceil$ and consider a fractional order differential equation with the derivative of Caputo type:

$$ {}^{\sharp\sharp}D^k[\phi(\eta)] = \psi(\eta, \phi(\eta)) \tag{2.47} $$

with the initial conditions

$$ \phi(\eta_0) = \phi_0, \phi'(\eta_0) = \phi_0^{(1)}, \phi''(\eta_0) = \phi_0^{(2)}, $$
$$ \vdots $$
$$ \phi^{(p-1)}(\eta_0) = \phi_0^{(p-1)}, $$

where $\psi(\eta, \phi(\eta))$ is continuous and $\phi_0, \phi_0^{(1)}, \phi_0^{(2)}, \ldots, \phi_0^{(p-1)}$ are derivatives at initial value of the independent variable η.

Theorem 2.3 (Existence [9])**.** *Let $\mathcal{D} = [0, M] \times [\phi_0^0 - q, \phi_0^0 + q]; M, q > 0$ also let the function $\psi : \mathcal{D} \to \mathbf{R}$ is continuous. Define $M^* = min\{M, (\frac{q\Gamma(k+1)}{||\phi||_\infty})^{\frac{1}{k}}\}$. There must exist the function $\phi : [0, M^*] \to \mathbf{R}$.*

Theorem 2.4 (Uniquenes [9])**.** *Let $\mathcal{D} = [0, M] \times [\phi_0^0 - q, \phi_0^0 + q]; M, q > 0$ also let the function $\psi : \mathcal{D} \to \mathbf{R}$ is bounded on the set $\mathcal{D}$ and ψ satisfies the condition below:*

For some $\xi \in \mathcal{D}$

$$ |\psi(\eta, \phi) - \psi(\eta, \xi)| \leq W|\phi - \xi|, $$

where $W > 0$ is a real constant.

3. Conclusions

In this present research article, we have examined Laplace Transform Method as a Semi-Analytical Method and Numerical Method as Backward Euler's Method and solved fractional order ordinary differential equations. One can easily apply these techniques to solve fractional order ordinary differential equations of Riemann-Liouville and Caputo Type operators. For initial level of understanding of the subject of fractional calculus, this research article is specially designed.

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