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NEW RESULTS ON THE PROPERTIES OF SOLUTIONS OF A CAPUTO FRACTIONAL DIFFERENTIAL SYSTEM

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Abstract. This paper considers a perturbed nonlinear system of fractional ordinary differential equations (FrODEs) with Caputo fractional derivative. The aim of paper is to study uniform stability, asymptotic stability, Mittag-Leffer stability of zero solution as well as boundedness of non-zero solutions of this system of FrODEs with Caputo fractional derivative. We obtain four new theorems on these mathematical concepts by applying the second Lyapunov method. In order to explain the obtained results and show their applications, an example is given, which satisfies the constructed conditions. The results of this paper extend some results that can be found in the literature and they have contributions to theory of FrODEs.

1. Introduction

As we all know from fractional calculus, traditional definitions of integral and derivative of a function are generalized from integer orders to real orders. Hence, together with order differential equations, numerous forms of fractional order differential equations raised in the relevant literature. During the past decades, qualitative behaviors of fractional order ordinary differentials have been paid more and more attentions due to their significant applications in sciences and engineering. From the relevant literature, it follows that, in recent years, different and the same kind of qualitative problems for various scalar FrODEs and systems of FrODEs with Caputo fractional derivative of the from

$$ {}_{t_o}^{C}D_t^q x(t) = A(t)x(t), $$

$$ {}_{t_o}^{C}D_t^q x(t) = f(t, x(t)),\ x(t_0) = x_0,\ f(t, 0) = 0, $$

and numerous others forms of these FrODEs have been investigated and are still being investigated by many authors (see, for example, [1], [2], [4-7], [9-11], [13-17], [27-33] and the references of these sources). Today, continuing investigations on various problems in qualitative theory of FrODEs is a very attractive topic in the literature due to effective roles of FrODEs in sciences and engineering. Therefore, it deserves to investigate behaviors of solutions of FrODEs with Caputo fractional derivative.

In this paper, motivated by the recent literature on qualitative theory of FrODEs, we study a nonlinear perturbed system of FrODEs of Caputo type

$$ {}_{t_o}^{C}D_t^q x(t) = -A(t)x(t) - f(t, x(t)) + g(t, x(t)), \tag{1.1} $$

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where $x = (x_1, ..., x_n)^T \in \mathbb{R}^n$, $t \in \mathbb{R}^+$, $\mathbb{R}^+ = [0, \infty)$, $q \in (0, 1)$, $A(t) = (a_{ij}(t)) \in C(\mathbb{R}^+, \mathbb{R}^{n\times n})$, $A(t)$ is a symmetric matrix, $(i, j = 1, ..., n)$, $f = (f_1, ..., f_n)^T \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $f_i(t, x(t)) = f_i(t, x_1(t), ..., x_n(t))$, $f(t, 0) = 0$, $g = (g_1, ..., g_n)^T \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $g_i(t, x(t)) = g_i(t, x_1(t), ..., x_n(t))$ and $g(t, 0) = 0$. Then, the system of FrODEs (1.1) with Caputo derivative includes the zero solution.

The motivation to consider the perturbed system of FrODEs (1.1) and to study the mentioned qualitative concepts comes from the books ([3], [6], [11], [15], [16], [33]), the papers ([1], [2],[4],[5],[7-10], [12-14], [17-32]) and the references of these sources. Here, our aim is to obtain some new results on the uniform stability, asymptotic stability, Mittag-Leffler stability of zero solution of system of FrODEs (1.1) as well as boundedness of solutions of non-zero solutions of system of FrODEs (1.1) using the second Lyapunov method and Caputo derivative. The main results of this paper are new, original and they have contributions to the qualitative theory of FrODEs with Caputo derivatives.

2. Basic Results

We now consider a system of FrODEs with a Caputo derivative:

$$ {}_{t_o}^{C}D_t^q x(t) = F(t, x(t)), \tag{2.1} $$

where $q \in (0, 1)$, $F(t, 0) = 0$, $F \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $\mathbb{R}^+ = [0, \infty)$. The following basis results and concepts are needed in the remainder of this paper.

Definition 2.1. *([13]). A continuous function $\gamma_1 : [0, t) \to [0, +\infty)$ is said to belong to class-K if it is strictly increasing and $\gamma_1(0) = 0$.*

Theorem 2.1. *([7]). Let $x = 0$ be an equilibrium point for the non-autonomous system of FrODEs (2.1). Let us assume that there exist a continuous Lyapunov function $V(t, x(t))$ and a scalar class$-K$ function $\gamma_1(.)$ such that, $\forall x \neq 0$,*

$$ \gamma_1(\|x(t)\|) \le V(t, x(t)), $$

and

$$ {}_{t_0}^{C}D_t^q V(t, x(t)) \le 0 \text{ with } q \in (0, 1). $$

Then the origin of the system of FrODEs (2.1) is Lyapunov stable.

If, furthermore, there is a scalar class-K function $\gamma_2(.)$ satisfying

$$ V(t, x(t)) \le \gamma_2(\|x(t)\|) $$

then the origin of the system of FrODEs (2.1) is Lyapunov uniformly stable.

Theorem 2.2. *([13]). Let $x = 0$ be an equilibrium point for the non-autonomous system of FrODEs (2.1). Assume that there exist a Lyapunov function $V(t, x(t))$ and class-K functions γ_i $(i = 1, 2, 3)$ satisfying*

$$ \gamma_1(\|x\|) \le V(t, x) \le \gamma_2(\|x\|), $$

$$ {}_{t_0}^{C}D_t^q V(t, x(t)) \le -\gamma_3(\|x\|), $$

where $q \in (0, 1)$. Then the system of FrODEs (2.1) is asymptotically stable.

Definition 2.2. *([14]). The trivial solution of the system of FrODEs (2.1) is said to be Mittag-Leffler stable if*

$$\|x(t)\| \le [m(x(t_0))E_v(-\sigma(t-t_0)^v)]^\mu,$$

where $v \in (0,1]$, $\sigma \ge 0$, $\mu > 0$, $m(0) = 0$, $m(x) \ge 0$, $m(x)$ is locally Lipschitz with Lipschitz constant m_0, and

$$E_v(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(vk+1)}$$

is the one-parameter Mittag-Leffler function, and Γ denotes the Gamma function.

Lemma 2.1. *([14]). Let $x \in \mathbb{R}^n$ be a vector of differentiable functions. If a continuous function $V : [t_0,\infty) \times \mathbb{R}^n \to \mathbb{R}^+$ satisfies*

$${}_{t_o}^{C}D_t^q V(t,x(t)) \le -\alpha V(t,x(t)),$$

then

$$V(t,x(t)) \le V(t_0,x(t_0))E_q(-\alpha(t-t_0)^q),$$

where $\alpha > 0$ and $0 < q < 1$.

Lemma 2.2. *([16]).*

$${}_{t_o}^{C}D_t^q(ax(t)+by(t)) = a\,{}_{t_o}^{C}D_t^q x(t) + b\,{}_{t_o}^{C}D_t^q y(t),$$

where $q \in (0,1]$.

Let $x \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n\times n}$. In this paper, the vector and matrix norms are defined by $\|x\| = \left(\sum_{i=1}^{n} |x_i|\right)$ and $\|M\| = \max_{1\le j\le n} \left(\sum_{i=1}^{n} |m_{ij}|\right)$, respectively. We will use the definitions of both of these norms, when it is needed.

3. Analyses of Behaviors of Solutions

We will use the following conditions in our main results.

(C1) There is a positive constant ρ_0 such that

$$a_{ii}(t) - \sum_{j=1,j\ne i}^{n} |a_{ji}(t)| > \rho_0, t \in \mathbb{R}^+.$$

(C2)

$$f(t,0) = g(t,0) = 0,$$
$$x_i(t)f_i(t,x(t)) > 0, (x_i(t) \ne 0), \text{ for all } t \in \mathbb{R}^+ \text{ and all } x \in \mathbb{R}^n.$$

(C3) There is a positive constant ρ_0 from (C1) such that

$$\rho_0 \|x(t)\| + \|f(t,x(t))\| - \|g(t,x(t))\| \ge 0 \text{ for all } t \in \mathbb{R}^+ \text{ and all } x \in \mathbb{R}^n.$$

(C4) There are positive constants ρ_0 from (C1), f_0, g_0 and ρ_2 such that

$$\|f(t,x(t))\| \ge f_0 \|x(t))\|,$$
$$\|g(x(t))\| \le g_0 \|x(t))\| \text{ for all } t \in \mathbb{R}^+ \textit{ and } x \in \mathbb{R}^n,$$

and

$$\rho_0 + f_0 - g_0 \ge \rho_2.$$

Our first result, Theorem 3.1, studies the uniformly stability of the zero solution of the system of FrODEs (1.1).

Theorem 3.1. *The zero solution of the system of FrODEs (1.1) is uniformly stable if conditions (C1)-(C3) are satisfied.*

Proof. We define a LF $\Xi := \Xi(t, x(t))$ by

$$\Xi(t, x(t)) := \|x(t)\| = \sum_{i=1}^{n} |x_i(t)| = |x_1(t)| + ... + |x_n(t)|. \tag{3.1}$$

From this point, it is clear that the LF $\Xi(t, x(t))$in (3.1) satisfies the following relations:

$$\Xi(t, 0) = 0, k_1 |x_1(t)| + ... + k_1 |x_n(t)| = k_1 \|x\| \leq \Xi(t, x(t)),$$

and

$$\Xi(t, x(t)) \leq k_2 |x_1(t)| + ... + k_2 |x_n(t)| = k_2 \|x(t)\|,$$

i.e., we have

$$k_1 \|x\| \leq \Xi(t, x(t)) \leq k_2 \|x\|,$$

where $k_1 \in (0, 1)$ and $k_2 \geq 1$.

Calculating the Caputo fractional derivative of the LF $\Xi(t, x(t))$ in (3.1) along the system of FrODEs (1.1), using conditions (C1), (C2) and elementary calculations, we obtain

$$\begin{aligned}
{}_{t_o}^{C}D_t^q \Xi(t, x(t)) &= {}_{t_o}^{C}D_t^q \Xi \|x(t)\| \\
&= {}_{t_o}^{C}D_t^q (|x_1(t)| + ... + |x_n(t)|) \\
&= {}_{t_o}^{C}D_t^q |x_1(t)| + ... + {}_{t_o}^{C}D_t^q |x_n(t)| \\
&= \sum_{i=1}^{n} x'_i(t) sgn x_i(t + 0) \\
&= {}_{t_o}^{C}D_t^q x_1(t)[sgn x_1(t + 0)] + ... + {}_{t_o}^{C}D_t^q x_n(t)[sgn x_n(t + 0)] \\
&= -\sum_{i=1}^{n} [(a_{ii}(t) x_i(t)) \times sgn x_i(t + 0)
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{i=1}^{n} \sum_{j=1, i \neq j}^{n} a_{ji}(t) x_i \times sgn x_i(t+0) \\
&- \sum_{i=1}^{n} \left[f_i(t, x(t)) - g_i(t, x(t)) \right] \times sgn x_i(t+0) \\
\leq &- \sum_{i=1}^{n} \left[a_{ii}(t) - \sum_{j=1, j \neq i}^{n} |a_{ji}(t)| \right] |x_i(t)| \\
&- \|f(t, x(t))\| + \|g(t, x(t))\| \\
= &- \left[\sum_{i=1}^{n} a_{ii}(t) - \sum_{j=1, j \neq i}^{n} |a_{ji}(t)| \right] |x_i| \\
&- \|f(t, x(t))\| + \|g(t, x(t))\| \\
\leq &- \rho_0 \|x\| - \|f(t, x(t))\| + \|g(t, x(t))\| .
\end{aligned}
$$

Then, it follows that

$$
{}_{t_o}^{C}D_t^q \Xi(t, x(t)) \leq - \left[\rho_0 \|x(t)\| + \|f(t, x(t))\| - \|g(t, x(t)\| \right] .
$$

Next, using condition (C3), we have

$$
{}_{t_o}^{C}D_t^q \Xi(t, x(t)) \leq 0. \tag{3.2}
$$

Thus, from Theorem 2.1, the zero solution of the system of FrODEs (1.1) is uniformly stable. This result completes the proof of Theorem 3.1

Our next result studies the boundedness of solutions of the system of FrODEs (1.1).

Theorem 3.2. *The solutions of the system of FrODEs (1.1) are bounded at the infinity if conditions (C1)-(C3) are satisfied.*

Proof. Using conditions (C1)-(C3) from Theorem 3.1, we have (3.2), i.e.,

$$
{}_{t_o}^{C}D_t^q \Xi(t, x(t)) \leq 0.
$$

Next, it is clear that

$$
\Xi(t, x(t)) \leq \Xi(t_0, x(t_0)), \ t \geq t_0.
$$

As for the next step, we derive

$$
\begin{aligned}
\|x(t)\| &= |x_1(t)| + ... + |x_n(t)| \\
&= \Xi(t, x(t)) \\
&\leq \Xi(t_0, x(t_0)) = \|x(t_0)\| = |x_1(t_0)| + ... + |x_n(t_0)| ,
\end{aligned}
$$

i.e., we have

$$
\|x(t)\| \leq \|x(t_0)\| .
$$

Let

$$
\rho_1 = \|x(t_0)\| = |x_1(t_0)| + ... + |x_n(t_0)| > 0.
$$

Hence, we obtain

$$
\|x(t)\| = |x_1(t)| + ... + |x_n(t)| \leq \rho_1.
$$

Thus, it is clear that if $t \to \infty$, then $\|x(t)\| \leq \rho_1$. This result completes the proof of Theorem 3.2.

Our next result deals with the asymptotic stability of the zero solution of the system of FrODEs (1.1).

Theorem 3.3. *The zero solution of the system of FrODEs (1.1) is asymptotically stable if conditions (C1), (C2) and (C4) are satisfied.*

Proof. Applying conditions (C1), (C2) as in Theorem 3.1, we have

$$ {}_{t_o}^{C}D_t^q \Xi(t, x(t)) \leq -\rho_0 \|x(t)\| - \|f(t, x(t))\| + \|g(t, x(t)\| . $$

As for the next step, using condition (C4) we derive

$$ \begin{aligned} {}_{t_o}^{C}D_t^q \Xi(t, x(t)) &\leq -\rho_0 \|x(t)\| - f_0 \|x(t)\| + g_0 \|x(t)\| \\ &= -(\rho_0 + f_0 - g_0) \|x(t)\| \\ &\leq -(\rho_2) \|x(t)\| < 0, \|x(t)\| \neq 0. \end{aligned} $$

Hence, the zero solution of the system of FrODEs (1.1) is asymptotically stable by Theorem 2.2. This is the end of the proof of Theorem 3.3.

The following theorem shows the Mittag-Leffler stability of the zero solution of the system FrODEs (1.1).

Theorem 3.4. *The zero solution of the system of FrODEs (1.1) is Mittag-Leffler stable if conditions (C1), (C2) and (C4) are satisfied.*

Proof. Using conditions C1), (C2) and (C4) as in Theorem 3.3, we have

$$ {}_{t_o}^{C}D_t^q \Xi(t, x(t)) \leq -\rho_2 \|x(t)\| = -\rho_2 \Xi(t, x(t)), $$

i.e.,

$$ {}_{t_o}^{C}D_t^q \Xi(t, x(t)) \leq -\rho_2 \Xi(t, x(t)). $$

Using Lemma 2.1, we derive

$$ \begin{aligned} \|x(t)\| &= \Xi(t, x(t)) \leq \Xi(t_0, x(t_0)) E_q(-\rho_2 (t - t_0)^q) \\ &= \|x(t_0)\| E_q(-\rho_2 (t - t_0)^q) \\ &= [m(x(t_0)) E_q(-\rho_2 (t - t_0)^q)] , \end{aligned} $$

where $m(x) = \|x(t)\|$,$m(0) = 0$, $m(x) \geq 0$, $m(x)$ is locally Lipschitz with Lipschitz constant $m_0 = 1$. Thus, the proof of Theorem 3.4 is completed by using Definition 2.2.

4. Numerical Applications

Example 4.1. Consider the following system of FrODEs with Caputo derivative, $q \in (0,1)$:

$$\begin{pmatrix} {}_{t_o}^{C}D_t^q x_1(t) \\ {}_{t_o}^{C}D_t^q x_2(t) \end{pmatrix} = -\begin{bmatrix} 9+\exp(-t) & 1 \\ 1 & 9+\exp(-t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$
$$- \begin{bmatrix} 2x_1(t) + \frac{x_1(t)}{1+\exp(t)+|x_1(t)|} \\ 2x_2(t) + \frac{x_2(t)}{1+\exp(t)+|x_2(t)|} \end{bmatrix} \begin{bmatrix} \frac{x_1(t)}{1+\exp(2t)+|x_1(t)|} \\ \frac{x_2(t)}{1+\exp(2t)+|x_2(t)|} \end{bmatrix}. \tag{4.1}$$

Comparing the systems of FrODEs (4.1) and (1.1), we have the following relations:

$$A(t) = \begin{bmatrix} 9+\exp(-t) & 1 \\ 1 & 9+\exp(-t) \end{bmatrix},$$

$$a_{ii}(t) - \sum_{j=1, j\neq i}^{n} |a_{ji}(t)| = 9+\exp(-t) - 1 \geq 8 > 7 = \rho_0;$$

$$f(t,x(t)) = \begin{bmatrix} f_1(t,x_1(t),x_2(t)) \\ f_2(t,x_1(t),x_2(t)) \end{bmatrix} = \begin{bmatrix} 2x_1(t) + \frac{x_1(t)}{1+\exp(t)+|x_1(t)|} \\ 2x_2(t) + \frac{x_2(t)}{1+\exp(t)+|x_2(t)|} \end{bmatrix},$$

$$f(t,0) = 0,$$

$$\begin{aligned}\|f(t,x(t))\| &= \left\| \begin{bmatrix} f_1(t,x_1(t),x_2(t)) \\ f_2(t,x_1(t),x_2(t)) \end{bmatrix} \right\| \\ &= \left\| \begin{bmatrix} 2x_1(t) + \frac{x_1(t)}{1+\exp(t)+|x_1(t)|} \\ 2x_2(t) + \frac{x_2(t)}{1+\exp(t)+|x_2(t)|} \end{bmatrix} \right\| \\ &\geq 2|x_1(t)| - \frac{|x_1(t)|}{1+\exp(t)+|x_1(t)|} + 2|x_2(t)| - \frac{|x_2(t)|}{1+\exp(t)+|x_2(t)|} \\ &\geq |x_1(t)| + |x_2(t)| = \|x(t)\|,\end{aligned}$$

$$f_0 = 1;$$

$$g(t,x(t)) = \begin{bmatrix} g_1(t,x_1(t),x_2(t)) \\ g_2(t,x_1(t),x_2(t)) \end{bmatrix} = \begin{bmatrix} \frac{x_1(t)}{1+\exp(2t)+|x_1(t)|} \\ \frac{x_2(t)}{1+\exp(2t)+|x_2(t)|} \end{bmatrix},$$

$$g(t,0) = 0,$$

$$\begin{aligned}\|g(t,x(t))\| &= \left\| \begin{bmatrix} g_1(t,x_1(t),x_2(t)) \\ g_2(t,x_1(t),x_2(t)) \end{bmatrix} \right\| = \left\| \begin{bmatrix} \frac{x_1(t)}{1+\exp(2t)+|x_1(t)|} \\ \frac{x_2(t)}{1+\exp(2t)+|x_2(t)|} \end{bmatrix} \right\| \\ &\leq \frac{|x_1(t)|}{1+\exp(t)+|x_1(t)|} + \frac{|x_2(t)|}{1+\exp(t)+|x_2(t)|} \\ &\leq |x_1| + |x_2| = \|x\|,\end{aligned}$$

$$g_0 = 1;$$

As for the next step, from the above relations, we derive

$$\begin{aligned}&\rho_0 \|x(t)\| + \|f(t,x(t))\| - \|g(t,x(t))\| \\ &\quad \geq 7\|x(t)\| + \|x(t)\| - \|x(t)\| = 7\|x(t)\| \geq 0\end{aligned}$$

and

$$\rho_0 + f_0 - g_0 \geq 7 + 1 - 1 = 7 = \rho_2.$$

Considering the above discussion, we can verify that conditions (C1)-(C3) of Theorem 3.1 and Theorem 3.2 and conditions (C1), (C2) and (C4) of Theorem 3.3 and Theorem 3.4 are satisfied. For these reasons, the zero solution of the system of FrODEs (4.1) is uniformly stable, asymptotically stable and Mittag-Leffler stable and nonzero solutions of FrODEs (4.1) are bounded at infinity.

5. Conclusion

This paper investigates qualitative properties of solutions of a nonlinear system of FrODEs with Caputo fractional derivative. Indeed, here, a new perturbed system of FrODEs (1.1) with Caputo fractional derivative is taken into account. Four new results related to uniform stability, asymptotic stability, Mittag-Leffer stability of zero solution as well as boundedness of non-zero solutions of the system of FrODEs (1.1) are obtained. The results of this paper are proved by the second Lyapunov method. When we compare the main results of this paper with those obtained in the literature, our results are more general, suitable and convenient for applications. An example is given as numerical application to verify the applications of the conditions of the main results. The results of this paper have new contributions to the theory of FrODEs and the relevant literature

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