Applied Analysis and Optimization
Volume 5, Number 3, 2021, 343–356

Yokohama Publishers
ISSN 2189-1664 Online Journal

GENERIC WELL-POSEDNESS OF SYMMETRIC MINIMIZATION PROBLEMS

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Abstract. In this paper we study four classes of symmetric optimization problems which are identified with the corresponding spaces of objective functions, equipped with appropriate complete metrics. Using the Baire category approach, for any of these classes we show the existence of subset of the space of functions, which is a countable intersection of open and everywhere dense sets, such that for every objective function from this intersection the corresponding symmetric optimization problem possesses a solution. These results are obtained as realizations of a general variational principle which is established in this paper.

1. Introduction

Assume that (X, ρ) is a complete metric space. Denote by $\mathcal{M}_l$ the set of all lower semicontinuous and bounded from below functions $f : X \to R^1$. We equip the set $\mathcal{M}_l$ with the uniformity determined by the following base

$$\mathcal{E}(\epsilon) = \{(f, g) \in \mathcal{M}_l \times \mathcal{M}_l : \ |f(x) - g(x)| \le \epsilon \text{ for all } x \in X\},$$

where $\epsilon > 0$. It is known that this uniformity is metrizable (by a metric d) and complete [14].

Consider a minimization problem

$$f(x) \to \min, \ x \in X,$$

where $f \in \mathcal{M}_l$.

This problem has a solution when X is compact or when f satisfies a growth conditions and bounded subsets of X are compact. When X does not satisfy a compactness assumption the existence problem becomes more difficult and less understood. It is possible to overcome this difficulty by using the so-called generic approach which is applied fruitfully in many areas of Analysis (see, for example, [1, 2, 5–7, 9, 14] and the references mentioned there). According to the generic approach we say that a property holds for a generic (typical) element of a complete metric space (or the property holds generically) if the set of all elements of the metric space possessing this property contains a G_δ everywhere dense subset of the metric space which is a countable intersection of open everywhere dense sets. In particular, it is known that the optimization problem stated above (see [2, 9, 10, 14] and the references mentioned therein) can be solved generically (for a generic objective function). Namely, there exists a set $\mathcal{F} \subset \mathcal{M}_l$ which is a countable intersection of open and everywhere dense sets such that for each $f \in \mathcal{F}$ the minimization problem has a unique solution which is a limit of any minimizing sequence. This result and its numerous

2020 *Mathematics Subject Classification.* 49J27, 90C31.
Key words and phrases. Complete metric space, generic element, lower semicontinuos function, uniformity.

extensions are collected in [14]. It should be mentioned that generic existence results in optimal control and the calculus of variations are discussed in [15] while generic results in nonlinear analysis are presented in [3, 8, 11–13]. In particular, [15] contains generic results on the existence of solutions for large classes of optimal optimal control problems without convexity assumptions, generic existence results for best approximation problems are presented in [1, 3, 11], generic existence of fixed points for nonlinear operators is shown in [7, 8, 11] and the generic existence of a unique zero of maximally monotone operators is shown in [13]. In the present paper our goal is to obtain a generic existence of minimization problems with symmetry. These results are important because has applications in crystallography [4]. The first such result was obtained in [16].

In this paper we study four classes of symmetric optimization problems which are identified with the corresponding spaces of objective functions, equipped with appropriate complete metrics. Using the Baire category approach, for any of these classes we show the existence of subset of the space of functions, which is a countable intersection of open and everywhere dense sets, such that for every objective function from this intersection the corresponding symmetric optimization problem possesses a solution and is well-posed. These results are obtained as realizations of a general variational principle which is established in this paper. This variational principle extends the variational principle of [2].

2. A generic variational principle

We will obtain our well-posedness results as a realization of a variational principle which is considered in this section. This variational principle is an extension of the variational principle of [2].

We consider a complete metric space (X, ρ) which is called the domain space and a complete metric space $(\mathcal{A}, d)$ which is called the data space. We always consider the set X with the topology generated by the metric ρ. For the space $\mathcal{A}$ we consider the topology generated by the metric d. This topology will be called the strong topology. In addition to the strong topology we also consider a weaker topology on $\mathcal{A}$ which is not necessarily Hausdorff. This topology will be called the weak topology. (Note that these topologies can coincide.)

For each function $h : Y \to [-\infty, \infty]$, where Y is nonempty, set

$$\inf(h) = \inf\{h(y) : \ y \in Y\}$$

and

$$\mathrm{dom}(h) = \{y \in Y : \ h(y) < \infty\}.$$

For each $x \in X$ and each nonempty set $D \subset X$ put

$$\rho(x, D) = \inf\{\rho(x, y) : \ y \in D\}.$$

For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \ \rho(x, y) \le r\}.$$

If Z in a topological space and $Y \subset Z$, then Y is equipped with a relative topology.

We assume that with every $a \in \mathcal{A}$ a lower semicontinuous function f_a on X is associated with values in $\bar{R} = [-\infty, \infty]$.

Assume that a mapping $T : X \to X$ is continuous and the mapping $T^2 = T \circ T$ is an identity mapping in X:

$$T^2(x) = x \text{ for all } x \in X. \tag{2.1}$$

This implies that $T(X) = X$, if $x_1, x_2 \in X$ and $T(x_1) = T(x_2)$, then $x_1 = x_2$ and that there exists $T^{-1} = T$.

Assume that $\mathcal{A}_T \subset \mathcal{A}$ is a closed set in the strong topology such that

$$f_a \circ T = f_a \text{ for all } a \in \mathcal{A}_T. \tag{2.2}$$

The space $\mathcal{A}_T \subset \mathcal{A}$ is equipped with the relative weak and strong topologies. (Note that these spaces can coincide.)

In our study we use the following basic hypothesis about the functions.

(H) For any $a \in \mathcal{A}_T$, any $\epsilon > 0$ and any $\gamma > 0$ there exist a nonempty open set $\mathcal{W}$ in $\mathcal{A}$ with the weak topology, $x \in X$, $\alpha \in R^1$ and $\eta > 0$ such that

$$\mathcal{W} \cap \{b \in \mathcal{A}_T : \ d(a, b) < \epsilon\} \neq \emptyset$$

and for any $b \in \mathcal{W}$,

(i) $\inf(f_b)$ is finite;

(ii) if $z \in X$ is such that $f_b(z) \leq \inf(f_b) + \eta$, then $\rho(x, \{z, T(z)\}) \leq \gamma$ and $|f_b(z) - \alpha| \leq \gamma$.

We show (see Theorem 2.1 below) that if (H) holds, then for a generic $a \in \mathcal{A}$ the problem minimize $f_a(x)$ subject to $x \in X$, has a solution.

Given $a \in \mathcal{A}_T$ we say that the problem of minimization of f_a on X is well-posed with respect to data in $\mathcal{A}$ (or just with respect to $\mathcal{A}$) if the following assertions hold:

(1) $\inf(f_a)$ is finite and there exists $x_a \in X$ such that

$$\{x \in X : \ f_a(x) = \inf(f_a)\} = \{x_a, \ T(x_a)\}.$$

(2) For each $\epsilon > 0$ there are a neighborhood $\mathcal{V}$ of a in $\mathcal{A}$ with the weak topology and $\delta > 0$ such that for each $b \in \mathcal{V}$, $\inf(f_b)$ is finite and if $z \in X$ satisfies $f_b(z) \leq \inf(f_b) + \delta$, then

$$|f_b(z) - f_a(x_a)| \leq \epsilon$$

and

$$\min\{\rho(z, \{x_a, T(x_a)\}), \ \rho(T(z), \{x_a, T(x_a)\})\} \leq \epsilon.$$

Theorem 2.1. *Assume that (H) holds. Then there exists an everywhere dense (in the strong topology) set $\mathcal{B} \subset \mathcal{A}_T$ which is a countable intersection of open (in the weak topology) subsets of $\mathcal{A}_T$ such that for any $a \in \mathcal{B}$ the minimization problem of f_a on X is well-posed with respect to $\mathcal{A}$.*

Following the tradition, we can summarize the theorem by saying that under the assumption (H) the minimization problem for f_a on (X, ρ) is generically well-posed with respect to $\mathcal{A}$ or that the minimization problem for f_a is well-posed with respect to $\mathcal{A}$ for a generic $a \in \mathcal{A}$.

Proof. Let $a \in \mathcal{A}_T$. By (H) for any natural $n = 1, 2, \dots$ there are a nonempty open set $\mathcal{U}(a,n)$ in $\mathcal{A}$ with the weak topology, $x(a,n) \in X$, $\alpha(a,n) \in R^1$ and $\eta(a,n) > 0$ such that

$$\mathcal{U}(a,n) \cap \{b \in \mathcal{A}_T : \ d(a,b) < 1/n\} \neq \emptyset \tag{2.3}$$

and for any $b \in \mathcal{U}(a,n)$, $\inf(f_b)$ is finite and if $z \in X$ satisfies

$$f_b(z) \le \inf(f_b) + \eta(a,n),$$

then

$$|f_b(z) - \alpha(a,n)| \le 1/n$$

and

$$\rho(x(a,n), \{z, T(z)\}) \le 1/n.$$

Define

$$\mathcal{B}_n = (\cup\{\mathcal{U}(a,m) : \ a \in \mathcal{A}_T, \ m \ge n\}) \cap \mathcal{A}_T \tag{2.4}$$

for $n = 1, 2, \dots$. Clearly for each integer $n \ge 1$, the set $\mathcal{B}_n \subset \mathcal{A}_T$ is open in the relative weak topology and in view of (2.3), it is everywhere dense in the relative strong topology. Set

$$\mathcal{B} = \cap_{n=1}^{\infty} \mathcal{B}_n. \tag{2.5}$$

Since for each integer $n \ge 1$ the set $\mathcal{B}_n$ is also open in the relative strong topology generated by the complete metric d we conclude that $\mathcal{B}$ is everywhere dense in the relative strong topology.

Let

$$b \in \mathcal{B}. \tag{2.6}$$

Evidently $\inf(f_b)$ is finite. By (2.4)-(2.6), there are a sequence $\{a_n\}_{n=1}^{\infty} \subset \mathcal{A}_T$ and a strictly increasing sequence of natural numbers $\{k_n\}_{n=1}^{\infty}$ such that

$$b \in \mathcal{U}(a_n, k_n), \ n = 1, 2, \dots. \tag{2.7}$$

Assume that $\{z_n\}_{n=1}^{\infty} \subset X$ and

$$\lim_{n \to \infty} f_b(z_n) = \inf(f_b). \tag{2.8}$$

Let $m \ge 1$ be an integer. By (2.8), for all large enough n the inequality

$$f_b(z_n) < \inf(f_b) + \eta(a_m, k_m) \tag{2.9}$$

is true and it follows from the definition of $\mathcal{U}(a_m, k_m)$, (2.3), (2.7) and (2.9) that

$$|f_b(z_n) - \alpha(a_m, k_m)| \le k_m^{-1} \tag{2.10}$$

and

$$\rho(x(a_m, k_m), \{z_n, T(z_n)\}) \le k_m^{-1} \tag{2.11}$$

for all large enough n. Since m is an arbitrary natural number we conclude that there exists a subsequence $\{z_{i_p}\}_{p=1}^{\infty}$ such that at least one of the sequences $\{z_{i_p}\}_{p=1}^{\infty}$ and $\{T(z_{i_p})\}_{p=1}^{\infty}$ converge. Since the mapping T is continuous and T^2 is the identity mapping we conclude that these both sequences converges. Denote

$$x_b = \lim_{p \to \infty} z_{i_p}. \tag{2.12}$$

Then

$$T(x_b) = \lim_{p\to\infty} T(z_{i_p}).$$

As f_b is lower semicontinuous, by (2.4)-(2.6) and (2.8),

$$f_b(x_b) = f_b(T(x_b)) = \inf(f_b). \tag{2.13}$$

By (2.10) and (2.11) with $z_n = x_b$, $n = 1, 2, \ldots$,

$$|f_b(x_b) - \alpha(a_m, k_m)| \le k_m^{-1}, \tag{2.14}$$

$$\rho(x(a_m, k_m), \{x_b, T(x_b)\}) \le k_m^{-1}. \tag{2.15}$$

Assume that $\xi \in X$ satisfies

$$f_b(\xi) = \inf(f_b). \tag{2.16}$$

By (2.10), (2.11) and (2.16) with $z_n = \xi$, $n = 1, 2, \ldots$,

$$\rho(x(a_m, k_m), \{\xi, T(\xi)\}) \le k_m^{-1}. \tag{2.17}$$

Together with (2.15) and (2.17) these relations imply that

$$\min\{\rho(\xi, x_b),\ \rho(\xi, T(x_b)),\ \rho(T(\xi), x_b),\ \rho(T(\xi), T(x_b)\} \le 2k_m^{-1}.$$

Since m is any natural number we obtain that

$$\min\{\rho(\xi, x_b),\ \rho(\xi, T(x_b)),\ \rho(T(\xi), x_b),\ \rho(T(\xi), T(x_b))\} = 0,$$

$$\xi \in \{x_b, T(x_b)\}$$

and

$$\{x \in X :\ f_b(x) = \inf(f_b)\} = \{x_b, T(x_b)\}.$$

Let $\epsilon > 0$. Choose a natural number m for which

$$4k_m^{-1} < \epsilon. \tag{2.18}$$

Let

$$a \in \mathcal{U}(a_m, k_m). \tag{2.19}$$

Clearly $\inf(f_a)$ is finite. Let $z \in X$ and

$$f_a(z) \le \inf(f_a) + \eta(a_m, k_m). \tag{2.20}$$

By the definition of $\mathcal{U}(a_m, k_m)$, (2.3), (2.19) and (2.20),

$$|f_a(z) - \alpha(a_m, k_m)| \le k_m^{-1}, \tag{2.21}$$

$$\rho(x(a_m, k_m), \{z, T(z)\}) \le k_m^{-1}. \tag{2.22}$$

By (2.13)-(2.15), (2.18), (2.21) and (2.22),

$$|\inf(f_b) - f_a(z)| \le 2k_m^{-1} < \epsilon,$$

$$\min\{\rho(z, x_b),\ \rho(z, T(x_b)),\ \rho(T(z), x_b),\ \rho(T(z), T(x_b))\} \le \epsilon.$$

Theorem 2.1 is proved. $\square$

3. The first generic result

Assume that (X, ρ) is a complete metric space. Denote by $\mathcal{M}_l$ the set of all lower semicontinuous and bounded from below functions $f : X \to R^1$. We equip the set $\mathcal{M}_l$ with the uniformity determined by the following base

$$\mathcal{E}(\epsilon) = \{(f, g) \in \mathcal{M}_l \times \mathcal{M}_l : \ |f(x) - g(x)| \le \epsilon \text{ for all } x \in X\},$$

where $\epsilon > 0$. It is known that this uniformity is metrizable (by a metric d) and complete [14].

Denote by $\mathcal{M}_c$ the set of all continuous functions $f \in \mathcal{M}_l$. It is not difficult to see that $\mathcal{M}_c$ is a closed subset of $\mathcal{M}_l$.

Consider a minimization problem

$$f(x) \to \min, \ x \in X,$$

where $f \in \mathcal{M}_l$. Set

$$\mathcal{A} = \mathcal{M}_l$$

and $f_a = a$ for all $a \in \mathcal{A}$. For the space $\mathcal{A}$ the strong and weak topologies coincide.

Assume that a mapping $T : X \to X$ is continuous and the mapping $T^2 = T \circ T$ is an identity mapping in X:

$$T^2(x) = x \text{ for all } x \in X.$$

In this section $\mathcal{A}_T$ is either the set of all $f \in \mathcal{M}_l$ such that

$$f(T(x)) = f(x) \text{ for all } x \in X$$

or the set of all $f \in \mathcal{M}_c$ satisfying the equation above.

Clearly, $\mathcal{A}_T$ is a closed subset of $\mathcal{A}$. It is equipped with the relative topology induced by the metric d.

In [16] it was shown that there exists a set $\mathcal{F} \subset \mathcal{A}_T$ which is a countable intersection of open and everywhere dense sets in $\mathcal{A}_T$ such that for each $f \in \mathcal{F}$, the minimization problem of f on X is well-posed with respect to $\mathcal{A}$.

Here we deduce this result from Theorem 2.1. The following lemma was obtained in [16].

Lemma 3.1. *Assume that $f \in \mathcal{A}_T$, $\epsilon \in (0, 1)$, $\gamma > 0$,*

$$\delta \in (0, 8^{-1}\epsilon\gamma),$$

$\bar{x} \in X$ satisfies

$$f(\bar{x}) \le \inf(f) + \delta,$$

$$\bar{f}(x) = f(x) + \gamma \min\{1, \ \rho(x, \bar{x}), \ \rho(T(x), \bar{x})\}, \ x \in X$$

and that

$$U = \{g \in \mathcal{M}_l : \ (\bar{f}, g) \in \mathcal{E}(\delta)\}.$$

Then $\bar{f} \in \mathcal{A}_T$ and for each $g \in U$ and each $z \in X$ satisfying

$$g(z) \le \inf(g) + \delta$$

the following inequality holds:

$$\min\{\rho(z, \bar{x}), \ \rho(T(z), \bar{x})\} < \epsilon.$$

We can easily proof the following auxiliary result.

Lemma 3.2. *Let $f \in \mathcal{M}_l$ and let ϵ be a positive number. Then there exists a neighborhood U of f in $\mathcal{M}_l$ with the weak topology such that for each $g \in U$,*

$$|\inf(g) - \inf(f)| \le \epsilon$$

and if $x \in X$ satisfies $g(x) \le \inf(g) + \epsilon$, then

$$|g(x) - \inf(f)| \le 2\epsilon.$$

The following result holds.

Theorem 3.3. *There exists an everywhere dense set $\mathcal{B} \subset \mathcal{A}_T$ which is a countable intersection of open subsets of $\mathcal{A}_T$ such that for any $f \in \mathcal{B}$ the minimization problem of f on X is well-posed with respect to $\mathcal{A}$.*

By Theorem 2.1, in order to prove this result it is sufficient to show that (H) holds. The hypothesis (H) follows from Lemmas 3.1 and 3.2.

4. The second and third generic result

Let (X, ρ) be a complete metric space. Fix $\theta \in X$. Denote by $\mathcal{M}$ the set of all bounded from below lower semicontinuous functions $f : X \to R^1 \cup \{\infty\}$ such that

$$\mathrm{dom}(f) \ne \emptyset \text{ and } f(x) \to \infty \text{ as } \rho(x, \theta) \to \infty. \tag{4.1}$$

We equip the set $\mathcal{M}$ with strong and weak topologies.

For each function $h : Y \to R^1 \cup \{\infty\}$, where Y is nonempty, set

$$\mathrm{epi}(h) = \{(y, \alpha) \in Y \times R^1 : \ \alpha \ge h(y)\}.$$

Assume that a mapping $T : X \to X$ is continuous and the mapping $T^2 = T \circ T$ is an identity mapping in X:

$$T^2(x) = x \text{ for all } x \in X. \tag{4.2}$$

Set

$$\mathcal{M}_T = \{f \in \mathcal{M} : \ f \circ T = f\}. \tag{4.3}$$

For the set $\mathcal{M}$ we consider the uniformity determined by the following base:

$$E_s(n) = \{(f, g) \in \mathcal{M} \times \mathcal{M} :$$

$$f(x) \le g(x) + n^{-1} \text{ and } \ g(x) \le f(x) + n^{-1} \text{ for all } x \in X\}, \tag{4.4}$$

where n is a natural number. Clearly this uniform space $\mathcal{M}$ is metrizable and complete. Denote by τ_s the topology in $\mathcal{M}$ induced by this uniformity. The topology τ_s is called the strong topology.

Now we equip the set $\mathcal{M}$ with a weak topology. We consider the complete metric space $X \times R^1$ with the metric $\Delta(\cdot, \cdot)$ defined by

$$\Delta((x_1, \alpha_1), (x_2, \alpha_2)) = \rho(x_1, x_2) + |\alpha_1 - \alpha_2|, \ x_1, x_2 \in X \ , \alpha_1, \alpha_2 \in R^1. \tag{4.5}$$

For each lower semicontinuous bounded from below function $f : X \to R^1 \cup \{\infty\}$ with a nonempty epigraph define a function $\Delta_f : X \times R^1 \to R^1$ by

$$\Delta_f(x, \alpha) = \inf\{\Delta((x, \alpha), (y, \beta)) : \ (y, \beta) \in \ \mathrm{epi}(f)\}, \ (x, \alpha) \in X \times R^1. \tag{4.6}$$

For each natural number n denote by $E_w(n)$ the set of all pairs $(f, g) \in \mathcal{M} \times \mathcal{M}$ which have the following property:

C(i) For each $(x,\alpha) \in X \times R^1$ satisfying $\rho(x,\theta) + |\alpha| \le n$,

$$|\Delta_f(x,\alpha) - \Delta_g(x,\alpha)| \le n^{-1}; \tag{4.7}$$

C(ii) For each $(x,\alpha) \in X \times R^1$ satisfying

$$\alpha \le n, \ \min\{\Delta_f(x,\alpha), \Delta_g(x,\alpha)\} \le n \tag{4.8}$$

the inequality (4.7) is valid.

It was shown in Section 4.4 of [14] that for the set $\mathcal{M}$ there exists the uniformity generated by the base $E_w(n)$, $n = 1, 2, \dots$. This uniformity is metrizable (by a metric $\widehat{\Delta}_w$) and it induces in $\mathcal{M}$ a topology τ_w which is weaker than τ_s. The topology τ_w is called the weak topology.

For the set $\mathcal{M}$ we consider the metrizable uniformity determined by the following base:

$$\mathcal{E}(n) = \{(f,g) \in \mathcal{M} \times \mathcal{M} : \text{ (4.7) is valid for all } (x,\alpha) \in X \times R^1$$

$$\text{satisfying } \rho(x,\theta) + |\alpha| \le n\} \tag{4.9}$$

where $n = 1, 2, \dots$. Denote by τ_* the topology induced by this uniformity. The topology τ_* is called the epi-distance topology [14]. Clearly the topology τ_* is weaker than τ_w.

Let $\phi \in \mathcal{M}$. Denote by $\mathcal{M}(\phi)$ the set of all $f \in \mathcal{M}$ satisfying $f(x) \ge \phi(x)$ for all $x \in X$. It is easy to verify that $\mathcal{M}(\phi)$ is a closed subset of $\mathcal{M}$ with the topology τ_w. We consider the topological subspace $\mathcal{M}(\phi) \subset \mathcal{M}$ with the relative weak and strong topologies. Since the function ϕ satisfies (4.1) with $f = \phi$ for any natural number n there exists a natural number $m > n$, depending on n and ϕ, such that (4.8) implies

$$\rho(x,\theta) + |\alpha| \le m.$$

This implies that the topologies τ_w and τ_* induce the same relative topology on $\mathcal{M}(\phi)$. Set

$$\mathcal{M}_T(\phi) = \mathcal{M}_T \cap \mathcal{M}(\phi). \tag{4.10}$$

Clearly, $\mathcal{M}_T$ is a close set in $\mathcal{M}$ with the relative strong topology and $\mathcal{M}_T(\phi)$ is a closed set in $\mathcal{M}(\phi)$ with the relative strong topology. We prove the following two results.

Theorem 4.1. *Let $\mathcal{A} = \mathcal{M}$, $f_a = a$, $a \in \mathcal{A}$ and $\mathcal{A}_T = \mathcal{M}_T$. Then there exists an everywhere dense (in the strong topology) set $\mathcal{B} \subset \mathcal{A}_T$ which is a countable intersection of open (in the weak topology) subsets of $\mathcal{A}_T$ such that for any $f \in \mathcal{B}$ the minimization problem of f on X is well-posed with respect to $\mathcal{A}$.*

Theorem 4.2. *Let $\mathcal{A} = \mathcal{M}(\phi)$, $\mathcal{A}_T = \mathcal{M}_T(\phi)$ and $f_a = a$, $a \in \mathcal{M}(\phi)$. Then there exists an everywhere dense (in the strong topology) set $\mathcal{B} \subset \mathcal{M}_T(\phi)$ which is a countable intersection of open (in the weak topology) subsets of $\mathcal{M}_T(\phi)$ such that for any $f \in \mathcal{B}$ the minimization problem of f on X is well-posed with respect to $\mathcal{M}_T(\phi)$.*

5. Proofs of Theorems 4.1 and 4.2

The next result follows from Lemmas 4.9 and 4.10 of [14].

Lemma 5.1. *Let $f \in \mathcal{M}$ and let δ be a positive number. Then there exists a neighborhood U of f in $\mathcal{M}$ with the weak topology such that for each $g \in U$*

$$|\inf\{g(x) : x \in X\} - \inf\{f(x) : x \in X\}| \le \delta.$$

Denote by $\mathcal{E}_T$ the set of all $f \in \mathcal{M}_T$ for which there exists $x_f \in X$ such that

$$f(x_f) = \inf\{f(x) : x \in X\}. \tag{5.1}$$

Lemma 5.2. *Let*

$$f \in \mathcal{M}_T. \tag{5.2}$$

Then there exists a sequence $f_n \in \mathcal{E}_T$, $n = 1, 2 \dots$ such that $f_n(x) \ge f(x)$ for all $x \in X$ and $n = 1, 2, \dots$ and $f_n \to f$ as $n \to \infty$ in the strong topology.

Proof. For each natural number n there exists $x_n \in X$ such that

$$f(x_n) \le \inf\{f(x) : x \in X\} + 1/n. \tag{5.3}$$

For $n = 1, 2, \dots$ define

$$f_n(x) = \max\{f(x), f(x_n)\} \text{ for all } x \in X. \tag{5.4}$$

By (5.3) and (5.4), for $n = 1, 2 \dots$ and all $x \in X$,

$$f_n(T(x)) = \max\{f(T(x)),\ f(x_n)\} = \max\{f(x), f(x_n)\} = f_n(x). \tag{5.5}$$

By (5.1) and (5.3)-(5.5),

$$f_n \in \mathcal{E}_T,\ n = 1, 2, \dots.$$

By (5.3) and (5.4),

$$\lim_{n\to\infty} f_n = f$$

in the strong topology. Lemma 5.2 is proved. □

Lemma 5.3. *Let $f \in \mathcal{E}_T$, $x_f \in X$,*

$$f(x_f) = \inf\{f(x) : x \in X\},$$

$$\epsilon \in (0, 1),\ \epsilon_1 \in (0, \epsilon),\ \delta \in (0, 16^{-1}\epsilon_1^2), \tag{5.6}$$

$$\rho(T(x), x_f) \le \epsilon/8 \text{ for all } x \in B(T(x_f), 2\epsilon_1),$$

$$\rho(T(x), T(x_f)) \le \epsilon_1 \text{ for all } x \in B(x_f, 6\delta\epsilon^{-1}). \tag{5.7}$$

Define a function $\bar{f}$ by

$$\bar{f}(x) = f(x) + 2^{-1}\epsilon \min\{\rho(T(x), x_f), \rho(x, x_f), 1\} \text{ for all } x \in X. \tag{5.8}$$

Then there exists a neighborhood U of $\bar{f}$ in $\mathcal{M}$ with the weak topology such that for each $g \in U$ and each $x \in X$ satisfying $g(x) \le \inf(g) + \delta$,

$$|g(x) - f(x_f)| \le \epsilon,$$

$$\min\{\rho(x, x_f),\ \rho(T(x), x_f)\} \le \epsilon.$$

Proof. By (5.8),

$$\bar{f} \in \mathcal{E}_T.$$

Lemma 5.1 implies that there exists a neighborhood U_0 of $\bar{f}$ in $\mathcal{M}$ with the weak topology such that

$$|\inf(g) - \inf(\bar{f})| < \delta/2 \text{ for all } g \in U_0. \tag{5.9}$$

Fix an integer

$$n_0 > |\inf(f)| + 4 + 4\delta^{-1} \tag{5.10}$$

and put

$$U = U_0 \cap \{g \in \mathcal{M} :\ (\bar{f}, g) \in E_w(n_0)\}. \tag{5.11}$$

Let $g \in U$ and let $x \in X$ satisfy

$$g(x) \leq \inf(g) + \delta. \tag{5.12}$$

In view of (5.9), (5.11) and (5.12),

$$g(x) \leq \inf(\bar{f}) + 2\delta. \tag{5.13}$$

It follows from C(ii), (5.10), (5.11) and (5.13) that there exist

$$(y, \alpha) \in \mathrm{epi}(\bar{f}) \tag{5.14}$$

such that

$$\rho(y, x) \leq 2n_0^{-1} \tag{5.15}$$

and

$$|\alpha - (\inf(\bar{f}) + 2\delta)| \leq 2n_0^{-1}. \tag{5.16}$$

By (5.6), (5.8), (5.10), (5.14), (5.16) and the definition of $\bar{f}$,

$$\begin{aligned} f(y) + 2^{-1}\epsilon \min\{\rho(y, x_f),\ \rho(T(y), x_f), 1\} = \bar{f}(y) &\leq \alpha \\ \leq \inf(\bar{f}) + 2\delta + 2n_0^{-1} &\leq \inf(\bar{f}) + 3\delta \\ = f(x_f) + 3\delta &\leq f(y) + \delta. \end{aligned}$$

Together with (5.7) this implies that

$$\min\{\rho(y, x_f),\ \rho(T(y), x_f), 1\} \leq 6\delta\epsilon^{-1}$$

and

$$\min\{\rho(y, x_f),\ \rho(T(y), x_f)\} \leq 6\delta\epsilon^{-1}. \tag{5.17}$$

By (5.9) and (5.13),

$$|g(x) - f(x_f)| \leq |g(x) - \inf(g)| + |\inf(g) - \inf(\bar{f})| \leq 2\delta < \epsilon.$$

In view of (5.17), there are two cases:

$$\rho(y, x_f) \leq 6\delta\epsilon^{-1}, \tag{5.18}$$

$$\rho(T(y), x_f)\} \leq 6\delta\epsilon^{-1}. \tag{5.19}$$

If (5.18) is true, then together with (5.6), (5.10) and (5.15) this implies that

$$\rho(x, x_f) \leq 2n_0^{-1} + 6\delta\epsilon^{-1} < \epsilon.$$

Assume that (5.19) holds. Then (5.7), (5.19) and the equality $T^2(y) = y$,

$$\rho(y, T(x_f)) \le \epsilon_1.$$

Together with (5.10) and (5.15) this implies that

$$\rho(x, T(x_f)) \le 2n_0^{-1} + \epsilon_1 \le 2\epsilon_1.$$

Together with (5.8) this implies that

$$\rho(T(x), x_f) \le \epsilon/8.$$

Thus in both cases

$$\min\{\rho(x, x_f),\ \rho(T(x), x_f)\} \le \epsilon.$$

Lemma 5.3 is proved. □

By Theorem 2.1, in order to prove Theorem 4.1 (Theorem 4.2 respectively) it is sufficient to show that (H) holds. The hypothesis (H) follows from Lemmas 5.2 and 5.3.

6. The fourth result

Denote by $\mathcal{M}_b$ the set of all bounded from below lower semicontinuous functions $f : X \to R^1 \cup \{\infty\}$ which are not identically infinity. We equip the set $\mathcal{M}_b$ with a weak and a strong topology.

For each natural number n denote by $G_w(n)$ the set of all $(f, g) \in \mathcal{M}_b \times \mathcal{M}_b$ such that

$$\sup\{\Delta_f(x, \alpha) :\ (x, \alpha) \in \ \mathrm{epi}(g)\},\ \sup\{\Delta_g(x, \alpha) :\ (x, \alpha) \in \ \mathrm{epi}(f)\} \le n^{-1}.$$

(Note that the equation above means that the Hausdorff distance between epigraphs does not exceed $1/n$.) For the set $\mathcal{M}_b$ we consider the complete metrizable uniformity determined by the base $G_w(n)$, $n = 1, 2, \dots$. We equip the space $\mathcal{M}_b$ with a topology τ_w induced by this uniformity. The topology τ_w is called a weak topology.

Also for the set $\mathcal{M}_b$ we consider the complete metrizable uniformity determined by the following base:

$$U(n) = \{(f, g) \in \mathcal{M}_b \times \mathcal{M}_b :$$
$$f(x) \le g(x) + n^{-1} \text{ and } g(x) \le f(x) + n^{-1},\ x \in X\}.$$

The space $\mathcal{M}_b$ is endowed with the topology τ_s induced by this uniformity. The topology τ_s is called a strong topology.

Set $\mathcal{A} = \mathcal{M}_b$ and $f_a = a$ for each $a \in \mathcal{A}$. Assume that a mapping $T : X \to X$ is continuous and the mapping $T^2 = T \circ T$ is an identity mapping in X:

$$T^2(x) = x \text{ for all } x \in X.$$

Set

$$\mathcal{A}_T = \{f \in \mathcal{M}_b :\ f \circ T = f\}.$$

Clearly, $\mathcal{A}_T$ is a closed set in $\mathcal{A}$ with the strong topology. We prove the following result.

Theorem 6.1. *There exists an everywhere dense (in the strong topology) set $\mathcal{B} \subset \mathcal{A}_T$ which is a countable intersection of open (in the weak topology) subsets of $\mathcal{A}_T$ such that for any $f \in \mathcal{B}$ the minimization problem of f on X is well-posed with respect to $\mathcal{A}$.*

In order to prove Theorem 6.1 we need the following two lemmas. The first lemma is proved in a straightforward manner.

Lemma 6.2. *Assume that $f \in \mathcal{M}_b$ and that ϵ is a positive number. Then there exists a neighborhood U of f in $\mathcal{M}_b$ with the weak topology such that*

$$|\inf(g) - \inf(f)| \le \epsilon \text{ for all } g \in U.$$

Denote by $\mathcal{E}$ the set of all $f \in \mathcal{A}_T$ such that there exists $x_f \in X$ satisfying

$$f(x_f) = \inf\{f(x) : x \in X\}.$$

Analogously to Lemma 5.2 we can show that $\mathcal{E}$ is everywhere dense in $\mathcal{A}_T$ with the strong topology.

Lemma 6.3. *Let $f \in \mathcal{E}_T$, $x_f \in X$,*

$$f(x_f) = \inf\{f(x) : x \in X\}, \tag{6.1}$$

$$\epsilon \in (0,1),\ \epsilon_1 \in (0,\epsilon),\ \delta \in (0, 16^{-1}\epsilon_1^2), \tag{6.2}$$

$$\rho(T(x), x_f) \le \epsilon/8 \text{ for all } x \in B(T(x_f), 2\epsilon_1), \tag{6.3}$$

$$\rho(T(x), T(x_f)) \le \epsilon_1 \text{ for all } x \in B(x_f, 6\delta\epsilon^{-1}). \tag{6.4}$$

Define a function $\bar{f}$ by

$$\bar{f}(x) = f(x) + 2^{-1}\epsilon \min\{\rho(T(x), x_f), \rho(x, x_f), 1\} \text{ for all } x \in X. \tag{6.5}$$

Then there exists a neighborhood U of $\bar{f}$ in $\mathcal{A}$ with the weak topology such that for each $g \in U$ and each $x \in X$ satisfying $g(x) \le \inf(g) + \delta$,

$$|g(x) - f(x_f)| \le \epsilon$$

$$\min\{\rho(x, x_f),\ \rho(T(x), x_f)\} \le \epsilon.$$

Proof. Clearly,

$$\bar{f} \in \mathcal{A}_T.$$

Lemma 6.2 implies that there exists a neighborhood U of $\bar{f}$ in $\mathcal{M}_b$ with the weak topology such that for each $g \in U$,

$$\Delta_g(x, \alpha) \le 16^{-1}\delta,\ (x, \alpha) \in \text{ epi}(\bar{f}), \tag{6.6}$$

$$\Delta_{\bar{f}}(x, \alpha) \le 16^{-1}\delta,\ (x, \alpha) \in \text{ epi}(g), \tag{6.7}$$

$$|\inf(g) - \inf(\bar{f})| \le \delta/16. \tag{6.8}$$

Let $g \in U$ and let $x \in X$ satisfy

$$g(x) \le \inf(g) + \delta. \tag{6.9}$$

In view of (6.8) and (6.9),

$$|g(x) - \inf(\bar{f})| \le 2\delta. \tag{6.10}$$

It follows from (6.8) and (6.10) that there exist

$$(y, \alpha) \in \text{epi}(\bar{f}) \tag{6.11}$$

such that

$$\rho(y, x) \le 8^{-1}\delta \tag{6.12}$$

and

$$|\alpha - (\inf(\bar{f}) + 2\delta)| \le 8^{-1}\delta. \tag{6.13}$$

By (6.1), (6.5), (6.11) and (6.13),

$$f(y) + 2^{-1}\epsilon \min\{\rho(y, x_f),\ \rho(T(y), x_f), 1\} = \bar{f}(y) \le \alpha$$

$$\le \inf(\bar{f}) + 2\delta + 8^{-1}\delta \le f(x_f) + 3\delta \le f(y) + \delta$$

and

$$\min\{\rho(y, x_f),\ \rho(T(y), x_f)\} \le 6\delta\epsilon^{-1}. \tag{6.14}$$

By (6.1), (6.2), (6.5) and (6.10),

$$|g(x) - f(x_f)| \le |g(x) - \inf(\bar{f})| \le 2\delta < \epsilon.$$

In view of (6.14), there are two cases:

$$\rho(y, x_f) \le 6\delta\epsilon^{-1}, \tag{6.15}$$

$$\rho(T(y), x_f) \le 6\delta\epsilon^{-1}. \tag{6.16}$$

If (6.15) is true, then together with (6.2) and (6.12) this implies that

$$\rho(x, x_f) \le 6\delta\epsilon^{-1} + 8^{-1}\delta < \epsilon.$$

Assume that (6.16) holds. Then by (6.4) and (6.16),

$$\rho(y, T(x_f)) = \rho(T^2(y), T(x_f)) \le \epsilon_1.$$

Together with (6.2) and (6.12) this implies that

$$\rho(x, T(x_f)) \le \epsilon_1 + 8^{-1}\delta \le 2\epsilon_1.$$

Together with (6.3) this implies that

$$\rho(T(x), x_f) \le \epsilon/8.$$

Thus in both cases

$$\min\{\rho(x, x_f),\ \rho(T(x), x_f)\} \le \epsilon.$$

Lemma 6.3 is proved. □

By Theorem 2.1, in order to prove Theorem 6.1 it is sufficient to show that (H) holds. The hypotheses (H) follows from Lemma 6.3.

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Manuscript received May 1 2021
revised June 23 2021

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