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SOLITON WAVE SOLUTIONS OF THE OSKOLKOV EQUATION ARISING IN INCOMPRESSIBLE VISCO-ELASTIC KELVIN-VOIGT FLUID

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Abstract. A nonlinear model that defines the dynamics of an incompressible viscoelastic Kelvin-Voigt fluid is observed in the instant investigation. The proposed object is performed by manipulating computational results, and we deliberate especially kink wave and interaction between lump wave, rough wave and periodic wave solutions of the model applying the -expansion method. This analysis has manipulated this process to seek novel soliton wave results of the Oskolkov equation. The dynamics of obtained wave solutions are analyzed and illustrated in figures by selecting appropriate parameters. With 2D, 3D and contour graphical representation, mathematical effects explicitly show the recommended algorithm's complete responsibility and high play in physics, mathematics and engineering.

1. Introduction

Recently, the nonlinear dynamics have been continuously improved for various innovative support, and outstanding development has been manufactured in the contribution of the soliton solutions for nonlinear evolution equations (NLEEs), which have been necessary inspection for researchers. Accordingly, the examinations of the soliton effects for NLEEs have vast importance in examining nonlinear natural impacts. The NLEEs have a meaningful impact on various sectors, for example, fluid mechanics, nonlinear optics, signal processing, plasma physics, optical fiber, water wave mechanics, and so numerous. Due to the reconstructed features in multiple applications, the soliton solutions to NLEEs have attracted various investigations' offerings and play a vital function in investigating nonlinear physical issues. Researchers have designed numerous ways in their various studies, which as the novel (G'/G)-expansion method [3, 4], simple equation method [25], the modified (G'/G)-expansion method [5, 6, 8, 12], the power index method [27], Legendre-Galerkin spectral method [22], Sine-Gordon expansion method [26], extended Jacobian elliptic function expansion method [1], the Jacobi elliptic ansatz method [14], Natural transform method [18], Kudryashov method [15], Jacobi elliptic function method [17], Exp-function method [16], Fokas method [29], Generalized Exp-Function method [20], Residual power series method [21], transformed rational function method [24], Cole-Hopf transformation [23], the variation of (G'/G)-expansion method [7] and so many, see ,for example [2,9–11,13,19,28] and the biography therein. The goal of this letter is to provide the (Φ, Ψ)-expansion method [7] to

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find new soliton wave solutions for the (1+1)-dimensional Oskolkov equation [2,13]. We are considering the (1+1)-dimensional Oskolkov equation:

$$\hbar_t - \sigma\hbar_{xxt} - p_2\hbar_{xx} + \hbar\hbar_x = 0, \tag{1.1}$$

where $\hbar$ is a function of x and t and σ, p_2 are constants. Equation (1) presented the dynamics of an incompressible visco-elastic Kelvin-Voigt fluid and fluid dynamics. Various kinds of Oskolkov equation are determined through numerous techniques [9,19] to build a closed-form wave solution. The modified simple equation approach [19] is executed to discover the closed-form wave answer from Oskolkov equation. The (G'/G)-expansion approach [2] is implemented to determine the exact wave answer from Oskolkov equation.

2. Glimpse of the (Φ, Ψ)-expansion method

We are considering:

$$P(\hbar, \hbar_x, \hbar_{xx}, \hbar_t, \hbar_{tt}, \hbar_{xt}, ...) = 0, \tag{2.1}$$

where P is a polynomial in $\hbar$ and its derivatives.

Step 1: Use the traveling variable:

$$\hbar = \hbar(x,t) = \hbar(\Gamma), \Gamma = p_3(x - Vt), \tag{2.2}$$

where p_3 and V are constants to be determined later. Using (3) into (2), we get:

$$R(\hbar, p_3\hbar', p_3^2\hbar'', -p_3V\hbar', p_3^2V^2\hbar'', -p_3^2V^2\hbar'', ...) = 0. \tag{2.3}$$

Step 2: Calculate N through rule of the homogeneous balance in equation (4).

Step 3: Considering the solving form:

$$\hbar = \sum_{i=0}^{M} S_i\Psi^i + \sum_{i=1}^{M} T_i\Psi^{i-1}\Phi, \tag{2.4}$$

where $\Psi = (\frac{\ominus'}{\ominus})$ and $\Phi = (\frac{\Omega'}{\Omega})$. And $\ominus = \ominus(\Gamma)$ and $\Omega = \Omega(\Gamma)$ represent the solution of the coupled Riccati equations

$$\ominus'(\Gamma) = - \ominus(\Gamma)\Omega(\Gamma),$$
$$\Omega'(\Gamma) = \{1 - \Omega(\Gamma)\}^2.$$

These coupled Riccati equations give us four types of hyperbolic function solutions including sech, tanh, csch and coth such as

$$\ominus(\Gamma) = \pm sech(\Gamma), \Omega(\Gamma) = tanh(\Gamma),$$
$$\ominus(\Gamma) = \pm csch(\Gamma), \Omega(\Gamma) = coth(\Gamma).$$

Step 4: A polynomial in Ψ or Φ is accomplished plugging equation (5) into equation (4). Determining the coefficients of the equivalent power of Ψ or Φ produces a system of algebraic equations, which can be determined to construct the values of S_i or T_i using MAPLE. Turning the over measured values of S_i or T_i in equation

(5), the general solution of the studied equation completes the calculations of the result of the proposed model.

3. Mathematical analysis

We are considering the Oskolkov equation:

$$\hbar_t - \sigma\hbar_{xxt} - p_2\hbar_{xx} + \hbar\hbar_x = 0. \tag{3.1}$$

Using $\hbar = \hbar(x,t) = \hbar(\Gamma)$ and $\Gamma = p_3x - p_4t$, then (6) becomes:

$$2p_3^2p_4p_1\hbar'' - 2p_2p_3^2\hbar' - 2p_4\hbar + p_3\hbar^2 = 0. \tag{3.2}$$

Now implementing the method of homogeneous balance between $\hbar''$ and $\hbar^2$ in (7), then we find:

$$\begin{aligned}\hbar(\Gamma) &= S_0\Psi^0 + S_1\Psi^1 + S_2\Psi^2 + T_1\Psi^0\Phi + T_2\Psi^1\Phi\\ &= (S_0 - T_2) + \frac{T_1}{\Omega} - (S_1 + T_1)\Omega + (S_2 + T_2)\Omega^2.\end{aligned} \tag{3.3}$$

Collecting the coefficient of Ψ and Φ and solving the resultant system, then we find:

Cluster I:

$$p_3 = \pm\sqrt{\frac{-1}{24p_1}}, \quad p_4 = \frac{-p_2}{10p_1}, \quad S_0 = \frac{6}{5}p_3p_2 + T_2,$$

$$S_1 = \frac{12}{5}p_3p_2, \quad S_2 = \frac{6}{5}p_3p_2 - T_2, \quad T_1 = 0.$$

Substituting the above values of (8), we get:

$$\hbar_1(\Gamma) = \frac{6}{5}p_3p_2\{1 - 2tanh(\Gamma) + tanh^2(\Gamma)\}.$$

$$\hbar_2(\Gamma) = \frac{6}{5}p_3p_2\{1 - 2coth(\Gamma) + coth^2(\Gamma)\}.$$

Cluster II:

$$p_3 = \pm\sqrt{\frac{1}{24p_1}}, \quad p_4 = \frac{p_2}{10p_1}, \quad S_0 = \frac{18}{5}p_3p_2 + T_2,$$

$$S_1 = \frac{12}{5}p_3p_2, \quad S_2 = -\frac{6}{5}p_3p_2 - T_2, \quad T_1 = 0.$$

Similarly, we get:

$$\hbar_3(\Gamma) = \frac{6}{5}p_3p_2\{3 - 2tanh(\Gamma) - tanh^2(\Gamma)\}.$$

$$\hbar_4(\Gamma) = \frac{6}{5}p_3p_2\{3 - 2coth(\Gamma) - coth^2(\Gamma)\}.$$

Cluster III:

$$p_3 = \pm\sqrt{\frac{-1}{24p_1}}, \quad p_4 = \frac{p_2}{10p_1}, \quad S_0 = -\frac{6}{5}p_3p_2 + T_2,$$

$$S_1 = \frac{12}{5}p_3p_2, \quad S_2 = -\frac{6}{5}p_3p_2 - T_2, \quad T_1 = 0.$$

Similarly, we get:

$$\hbar_5(\Gamma) = -\frac{6}{5}p_3p_2\{1 + 2tanh(\Gamma) + tanh^2(\Gamma)\}.$$

$$\hbar_6(\Gamma) = -\frac{6}{5}p_3p_2\{1 + 2coth(\Gamma) + coth^2(\Gamma)\}.$$

Cluster IV:

$$p_3 = \pm\sqrt{\frac{1}{24p_1}}, \quad p_4 = \frac{-p_2}{10p_1}, \quad S_0 = -\frac{18}{5}p_3p_2 + T_2,$$

$$S_1 = \frac{12}{5}p_3p_2, \quad S_2 = \frac{6}{5}p_3p_2 - T_2, \quad T_1 = 0.$$

Similarly, we get:

$$\hbar_7(\Gamma) = \frac{6}{5}p_3p_2\{-3 - 2tanh(\Gamma) + tanh^2(\Gamma)\}.$$

$$\hbar_8(\Gamma) = \frac{6}{5}p_3p_2\{-3 - 2coth(\Gamma) + coth^2(\Gamma)\}.$$

4. Numerical simulations

In this part, we found some new soliton wave solutions that represented as hyperbolic function solutions through the (Φ, Ψ)-expansion method. After applying the (Φ, Ψ)-expansion method, we got six new soliton wave answers: Representation kink-type shape and interaction between lump shape and rough wave shapes. Using the (Φ, Ψ)-expansion method to the Oskolkov equation has not been published earlier to our knowledge's skilled. We will provide a few graphical depictions of the nonlinear model's above-defined new soliton wave answers arising in incompressible visco-elastic Kelvin-Voigt fluid received employing the (Φ, Ψ)-expansion method. Figures 1 to 4 illustrate the graphical depictions of some selected computational results of the problem received utilizing the studied method. They are pictured below. In Figure 1, we show the kink-type wave shape of the solution $\hbar_1$ using the following parameter values $\lambda = 3$, $\mu = 1$, $p_1 = 0.5$ and $p_2 = 1$. In particular, Figure 1 shows the three-dimensional shape and contour shape of the solution $\hbar_1$. Using the same parameter values as shown three-dimensional shape and the contour shape of the solution $\hbar_2$, $\hbar_3$ and $\hbar_4$ represented by the lump shape, rough wave shape and interaction between lump shape and rough wave shapes are plotted in Figures 2, 3 and 4.

5. Conclusion

This study successfully performed the (Φ, Ψ)-expansion method on the Oskolkov equation that defines the dynamics of an incompressible visco-elastic Kelvin-Voigt fluid. Utilizing the (Φ, Ψ)-expansion method, we make some new soliton wave solutions such as hyperbolic function solutions which represent as kink-type shape, the interaction between lump shape and rough wave shapes. The applied method's play reveals that this method's effectiveness and attraction and its power to perform other nonlinear models arising physics, mathematics and engineering and deserves future research.

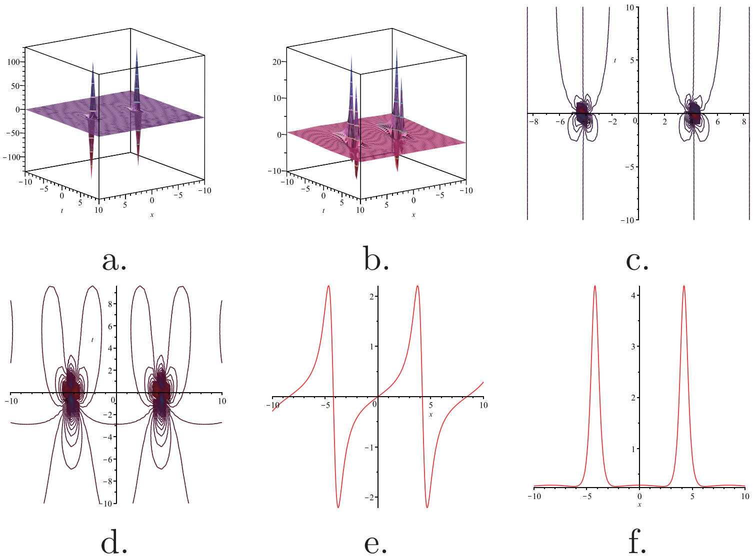


FIGURE 1. The graphical representation of the solution $\hbar_1$: (a) Real three dimensional shape, (b) Imaginary three dimensional shape, (c) Real contour plot, (d) Imaginary contour plot, (e) Real two dimensional plot and (f) Imaginary two dimensional plot.

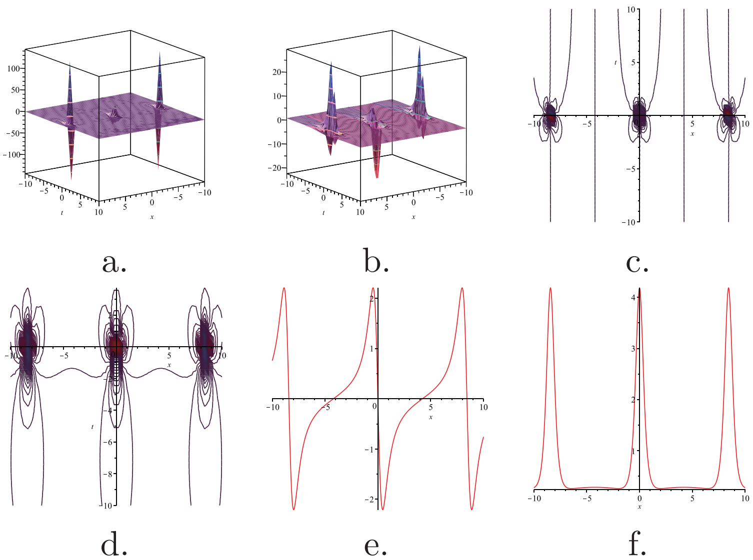
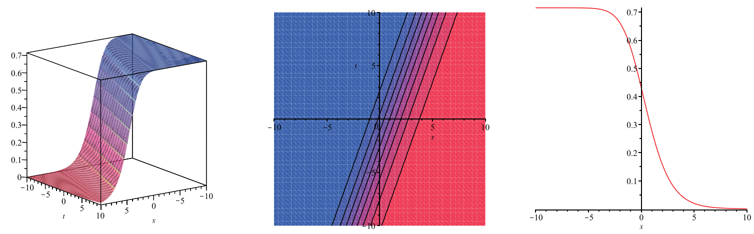
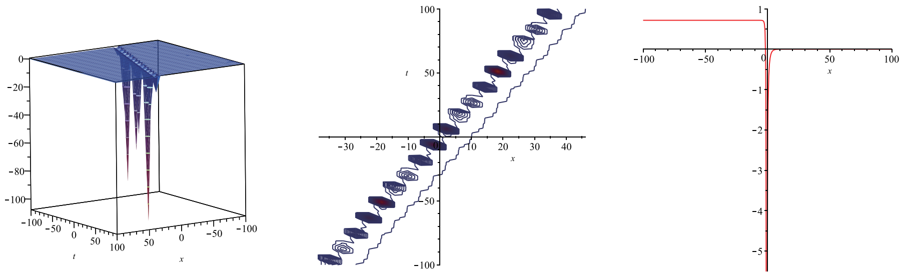


FIGURE 2. The graphical representation of the solution $\hbar_2$: (a) Real three dimensional shape, (b) Imaginary three dimensional shape, (c) Real contour plot, (d) Imaginary contour plot, (e) Real two dimensional plot and (f) Imaginary two dimensional plot.



a. b. c.

FIGURE 3. The graphical representation of the solution $\hbar_3$: (a) Three dimensional shape, (b) contour plot and (c) Two dimensional plot.



a. b. c.

FIGURE 4. The graphical representation of the solution $\hbar_4$: (a) Three dimensional shape, (b) contour plot and (c) Two dimensional plot.

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