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EXPONENTIAL PENALTY FUNCTION WITH MOMA-PLUS FOR THE MULTIOBJECTIVE OPTIMIZATION PROBLEMS

ALEXANDRE SOM, KOUNHINIR SOME, ABDOULAYE COMPAORE, AND BLAISE SOME

Abstract. In this article, we proposed a modified version of MOMA-Plus method in which the Lagrangian penalty function is replaced by the exponential penalty function to transform the constrained multiobjective optimization problem into an unconstrained multiobjective optimization problem. The theoretical results of this new Pareto optimal solutions process are proved and some numerical results are obtained on six common tests problems enabling us to highlight the performance of this new method.

1. Introduction

Generally, the modelling of most decision problems gives the multiobjective optimization formulation. A multiobjective optimization problem is characterized by the presence of several objectives which must be simultaneously optimized through some constrained functions. Practically, the objective functions are in conflict which explains the difficulties in finding a single optimal solution. In addition, the best value of certain functions automatically leads to worst values of other objective functions. Meanwhile, it is almost impossible to find in literature some methods which are able to resolve all kinds of multiobjective optimization problems. In order to find the Pareto optimal solutions of these problems, several methods have been developed [**?**, 6, 7, 11, 14, 19, 22]. One of the fashionable practices used in resolving these problems consists in transforming the multiobjective optimization problem into a single objective optimization problem by using an aggregation function for objective functions and penalization techniques for the constraints. MOMA-Plus method [17, 19, 20] can be seen likened to any of these methods.

In the previous works on the MOMA-Plus method, Tchebychev weighted distance and the Lagrangian function were used to firstly transform successively the initial problem into single objective and then single objective without constraints. The recent works of Liu and Feng [10] on the exponential penalty function enabled us to find out its advantages among which the fact that it can be used for any kind of function with continuous variables whether differentiable or not, see e.g., [2, 5, 8, 10, 12, 13, 16, 21, 23].

In this work, we will replace Lagrangian penalty function by the exponential penalty function to solve some multiobjective optimization problems. Then, the Tchebychev weighted distance will be used to transform the initial multiobjective

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optimization problem into single objective without constraints. Thereafter, the exponential penalty function will be applied to transform the last formulation of the problem into a single objective optimization without constraints problem which is a modification of the MOMA-Plus method and we will obtain a variant of MOMA-Plus. We will demonstrate the theoretical foundation of this MOMA-Plus modified version and thereafter apply it to solve six test problems taken in Zitzler [6].

To better present our work, Section 2 will be used as preliminary. Section 3 will include the main findings of this work and Section 4 will deal with the conclusion.

2. Preliminary

2.1. **Definitions.**

A multiobjective optimization problem can be formulated as follows :

$$\begin{array}{ll} \min & f(x) = (f_1(x), f_2(x), \cdots, f_p(x)) \\ s.t : & \left\{ \begin{array}{l} g(x) \le 0 \\ x \in \mathbb{R}^n \end{array} \right. \end{array} \tag{2.1}$$

where $f = \{f_1, f_2, \cdots, f_p\}$ is the vector whose components are the objective functions; $g = \{g_1, g_2, \cdots, g_m\}$ is also a vector whose components are the constraints for the optimization of f. In order to solve this problem, it is important for us to know the decision space and the objective space which are given by :

$$\chi = \{x \in \mathbb{R}^n,\ g(x) \le 0\} \text{ and } \mathcal{Y} = f(\chi). \tag{2.2}$$

Definition 2.1. For $x^* \in \chi$, x^* is a weakly Pareto optimal for the problem (2.1) if and only if there is no other point $x \in \chi$ such as : $f_j(x) < f_j(x^*)$, $\forall\ j = \overline{1,p}$.

Definition 2.2. For $x^* \in \chi$, x^* is a Pareto optimal for the problem (2.1) if and only if there is no other point $x \in \chi$ such as : $f_j(x) \le f_j(x^*)$, $\forall\ j = \overline{1,p}$ and at least one $j \in \{1, 2, \cdots, p\}$, $f_j(x) < f_j(x^*)$.

Definition 2.3. For $z^* \in \mathcal{Y}$, z^* is called ideal point if and only if the components of z^* are obtained by the following formula : $z^*_k = \min_{x\in\chi} f_k(x), k = \overline{1,p}$.

Definition 2.4. For a objective function $f = (f_1, f_2, \cdots, f_p)$ of a considered optimization problem, the Tchebychev weighted distance is formulated by :

$$\Psi(f(x), \lambda, z^*) = \max_j \big\{\lambda_j |f_j(x) - z^*_j|\big\}, j = 1, 2, \cdots, p \tag{2.3}$$

where $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_p)$ such as $\lambda_j > 0$ and $\sum_{j=1}^{p} \lambda_j = 1$.

Definition 2.5. For a function with several variables, the Alienor transformation allowing to transform it in single variable is formulated as following [3] :

$$x_i = h_i(\theta) = \frac{1}{2}\big[(b_i - a_i)\cos(\omega_i\theta + \phi_i) + a_i + b_i\big], with\ i = 1, \cdots, n. \tag{2.4}$$

where $\forall i = \overline{1,n}$, ω_i and ϕ_i are sequences which are slowly increasing and $\theta \in [0; \theta_{max}]$, with $\theta_{max} = \dfrac{(b-a)\theta^1 + (b+a)}{2}$ and $\theta^1 = \dfrac{2\pi - \phi_1}{\omega_1}$.

Note that $a = \min_{i=\overline{1,n}} a_i$ and $b = \max_{i=\overline{1,n}} b_i$ with $x_i \in [a_i, b_i], i = \overline{1,n}$.

Theorem 2.6. *(See in [3]). All point in $\mathbb{R}^n$ can be approached at least by one point defined by Alienor transformation $x_i = h(\theta)$.*

Proof. The proof can be seen in [3] at page 19. □

Many other Alienor transformation can be found in the literature [1, 4].

Definition 2.7 (See in [10])**.** A penalty function is a function which enables the transformation of the constrained optimization problem into unconstrained optimization problem. A better example is the exponential penalty function which will be used in this work :

$$P_n(x) = \frac{1}{\varrho_n} \sum_{i=1}^{m} \vartheta[\varrho_n g_i(x)] \tag{2.5}$$

where $\vartheta(t) = \exp(t-1)$ and ϱ_n is the penalty coefficient satisfying $\lim_{n\to+\infty} \varrho_n = +\infty$.

2.2. MOMA-Plus method description.

MOMA-Plus method is a method developed by Somé K. [19,20] for the resolution of multiobjective optimization problem. It consists in transforming a given multiobjective optimization problem into the single objective optimization and using Alienor transformation to reduce the number of variable before looking for the solutions. These main steps are :

aggregation **:** they use an aggregative function defined in the Definition 2.4 to transform the initial problem into a single objective optimization problem with constraints;

penalization **:** it is at this stage that the obtained problem from step 1 is transformed into single objective optimization problem without constraints by using the penalty function presented in Definition 2.7;

domain reduction **:** MOMA-Plus authors use the Alienor transformation (see Definition 2.5) to transform the previous problem form several variables into single variable;

resolution **:** for the resolution of the obtained problem which is a single objective and a single variable the authors use the Nelder-Mead simplex algorithm [15];

solution initialization **:** the obtained solution is one dimension solution and the initialization of this solution in the dimension of the initial problem needs to re-use the previous Alienor transformation given by the relation (2.4).

3. Main results

The new method proposed in this work corresponds to the modified version of MOMA-Plus in which the Lagrange penalty function is replaced by a penalty exponential function. The theoretical and numerical results will be presented in this section.

3.1. Theoretical results.

3.1.1. *MOMA-Plus Modified version.*
Each step of our method will be presented in this paragraph.

STEP I : Objective functions aggregation

The use of the aggregation function defined in Definition 2.4 gives the following problem :

$$\begin{array}{ll} \min & \Psi(f(x),\lambda,z^*) \\ s.t: & \left\{ \begin{array}{l} g(x)\leq 0 \\ x\in\mathbb{R}^n. \end{array}\right. \end{array} \tag{3.1}$$

Theorem 3.1. *All optimal solution of the problem (3.1) is the Pareto optimal solution to the problem (2.1) and conversely.*

Proof. Let x^* be a Pareto optimal solution to the problem (2.1) and note that $I=\{1,2,\cdots,p\}$ and $z^*=(z_1^*,z_2^*,\cdots,z_p^*)$ the ideal point. Then, there is no solution $x\in\chi$ such as $f_j(x)\leq f_j(x^*)$, $\forall\, j\in I$ and at least one $k\neq j,\ f_k(x)<f_k(x^*)$.

$$\begin{array}{lll} \Longrightarrow \nexists x \text{ such as} & : & f_j(x)-z_j^*\leq f_j(x^*)-z_j^*,\ \forall\, j\in I \text{ and at} \\ & & \text{least one } k\neq j\text{and } f_k(x)-z_k^*<f_k(x^*)-z_k^*\ . \\ \Longrightarrow \nexists x \text{ such as} & : & |f_j(x)-z_j^*|\leq |f_j(x^*)-z_j^*|,\ \forall\, j\in I \text{ and at} \\ & & \text{least one } k\neq j \text{ and } |f_k(x)-z_k^*|<|f_k(x^*)-z_k^*|. \\ \Longrightarrow \nexists x \text{ such as} & : & \max\limits_{j\in I}\{|f_j(x)-z_j^*|\}\leq \max\limits_{j\in I}\{|f_j(x^*)-z_j^*|\} \text{ and at} \\ & & \text{least one } k\neq j \text{ and} \\ & & \max\limits_{k}\{|f_k(x)-z_k^*|\}<\max\limits_{k}\{|f_k(x^*)-z_k^*|\}. \\ \Longrightarrow \nexists x \text{ such as} & : & \Psi(f(x),\lambda,z^*)<\Psi(f(x^*),\lambda,z^*). \end{array}$$

This implies that x^* is the optimal solution to the problem (3.1).

Conversely, let x^* be an optimal solution to the problem (3.1). Then $\forall x\in\chi$, $\Psi(f(x^*),\lambda,z^*)<\Psi(f(x),\lambda,z^*)$. Assuming that x^* is not a Pareto optimal solution to the problem (2.1). Then $\exists y\in\chi, \forall j\in I$, $f_j(y)\leq f_j(x^*)$ and at least one $k\in I$, $f_k(y)<f_k(x^*)$.

$$\begin{array}{lll} \Longrightarrow \exists y\in\chi,\forall j\in I & : & f_j(y)-z_j^*\leq f_j(x^*)-z_j^* \text{ and at least one} \\ & & k\neq j, f_k(y)-z_k^*<f_k(x^*)-z_k^* \\ \Longrightarrow \exists y\in\chi,\forall j\in I & : & \lambda_j|f_j(y)-z_j^*|\leq\lambda_j|f_j(x^*)-z_j^*| \text{ and at least} \\ & & \text{one } k\neq j, \lambda_j|f_j(y)-z_j^*|<\lambda_j|f_j(x^*)-z_j^*| \\ \Longrightarrow \exists y\in\chi,\forall j\in I & : & \max\limits_{j\in J}\{\lambda_j|f_j(y)-z_j^*|\}\leq\max\limits_{j\in J}\{\lambda_j|f_j(x^*)-z_j^*|\}. \\ \Longrightarrow \exists y\in\chi,\forall j\in I & : & \Psi(f(y),\lambda,z^*)\leq\Psi(f(x^*),\lambda,z^*) \text{ which is} \\ & & \text{absurd.} \end{array}$$

Therefore x^* is a Pareto optimal solution to the problem (2.1). □

STEP II : Penalization of constraints

For the conversion of the problem (3.1) into a single objective optimization problem without constraints, we use the exponential penalty function defined in the relation (2.5). We obtain the following formulation :

$$\min_{x\in\chi} \mathit{\Gamma}(x) = \mathit{\Psi}(f,\lambda,z^*) + \frac{1}{\varrho_n}\sum_{i=1}^{m}\vartheta[\varrho_n g_i(x)]. \tag{3.2}$$

First of all, let us enumerate the important properties about this exponential penalty function :

Lemma 3.2 (See in [10])**.** *Let χ be the set of eligible solutions of the problem (2.1). So :*

(i) *if $x \in \chi$, then* $\lim\limits_{\varrho_n\to+\infty} \dfrac{1}{\varrho_n}\sum\limits_{i=1}^{m}\vartheta[\varrho_n g_i(x)] = 0;$

(ii) *if $x \notin \chi$, then* $\lim\limits_{\varrho_n\to+\infty} \dfrac{1}{\varrho_n}\sum\limits_{i=1}^{m}\vartheta[\varrho_n g_i(x)] = +\infty.$

Definition 3.3 (See in [10])**.** Let us define $S_n \subset \mathbb{R}^k$ and

(i) $\overline{\lim\limits_{n\to+\infty}} S_n = \{x \in \mathbb{R}^k : x \in S_n$ for infinitely many $n \in \mathbb{N}^*, \};$

(ii) $\underline{\lim\limits_{n\to+\infty}} S_n = \{x \in \mathbb{R}^k : x \in S_n$ for all but finitely many $n \in \mathbb{N}^*\}.$

We note χ^* the set of the weakly Pareto optimal solutions to the problem (2.1).

Lemma 3.4 (See in [10])**.** *Let be $\varrho_n > 0$ and $\lim\limits_{\varrho_n\to+\infty}\varrho_n = +\infty$. If $x^* \in \overline{\lim\limits_{n\to+\infty}}\chi_n^*$ then $x^* \in \chi$.*

Proof. See in [10], Lemma 2, page 669. □

Let us designate by $\chi_n^*\backslash\chi^* = \{x : x \in \chi_n^*,$ but $x \notin \chi^*\}$. Let $Y_n^* = \{x^*\}$ be the optimal solution of the problem (3.2).

Theorem 3.5. $\overline{\lim\limits_{n\to+\infty}}(Y_n^*\backslash\chi^*) = \emptyset.$

Proof. Assuming that $\overline{\lim\limits_{n\to+\infty}}(Y_n^*\backslash\chi^*) \neq \emptyset$. Then there exists a subset $\{n_k, k = 1, 2, \cdots\}$ such as $x' \in \overline{\lim\limits_{n_k\to+\infty}}(Y_{n_k}^*\backslash\chi^*)$. Thus, $\exists\, n_0 > 0$ such as for $n_k \geq n_0$ we have $x' \in Y_{n_k}^*\backslash\chi^*$. As $x' \in Y_n^*$ then $\mathit{\Gamma}(x') < \mathit{\Gamma}(y),\ \forall y \in \chi$.
If $x' \in \chi$, as $x' \notin \chi^*$, so there $\exists y \in \chi$ such as $f_j(y) < f_j(x'),\ \forall j \in I$.

$$\begin{aligned}
\Longrightarrow \forall j \in I \quad &: \quad f_j(y) - z_j^* < f_j(x') - z_j^*,\\
\Longrightarrow \forall j \in I \quad &: \quad |f_j(y) - z_j^*| < |f_j(x') - z_j^*|,\\
\Longrightarrow \forall j \in I \quad &: \quad \lambda_j|f_j(y) - z_j^*| < \lambda_j|f_j(x') - z_j^*|,\\
\Longrightarrow \forall j \in I \quad &: \quad \mathit{\Psi}(f(y),\lambda,z^*) < \mathit{\Psi}(f(x'),\lambda,z^*).
\end{aligned}$$

Moreover, as $x' \in \chi$ and also $y \in \chi$, according to the Lemma 3.2 we have these two equations :

$$\begin{cases} \lim\limits_{n_k \to +\infty} (\frac{1}{\varrho_{n_k}} \sum\limits_{i=1}^{m} \vartheta(\varrho_n g_i(x'))) = 0 \\ \lim\limits_{n_k \to +\infty} (\frac{1}{\varrho_{n_k}} \sum\limits_{i=1}^{m} \vartheta(\varrho_{n_k} g_i(y))) = 0. \end{cases}$$

So $\exists\, n_0 \in \mathbb{N}^*$, $n_k > n_0$,
$\Psi(f(y), \lambda, z^*) + \frac{1}{\varrho_{n_k}} \sum\limits_{i=1}^{m} \vartheta(\varrho_{n_k} g_i(y)) < \Psi(f(x'), \lambda, z^*) + \frac{1}{\varrho_{n_k}} \sum\limits_{i=1}^{m} \vartheta(\varrho_{n_k} g_i(x'))$.
That implies that $\Gamma(y) < \Gamma(x')$ which is absurd.

If $x' \notin \chi$, then there exists $y \in \chi$ such as $f_j(y) < f_j(x'), \forall j \in I$. We obtain the same result which is $\Psi(f(y), \lambda, z^*) < \Psi(f(x'), \lambda, z^*)$. But as $x' \notin \chi$ and $y \in \chi$, according to the Lemma 3.2, we have these two equations :

$$\begin{cases} \lim\limits_{n_k \to +\infty} (\frac{1}{\varrho_{n_k}} \sum\limits_{i=1}^{m} \vartheta(\varrho_n g_i(x'))) = +\infty \\ \lim\limits_{n_k \to +\infty} (\frac{1}{\varrho_{n_k}} \sum\limits_{i=1}^{m} \vartheta(\varrho_{n_k} g_i(y))) = 0. \end{cases}$$

So there exists $n_0 \in \mathbb{N}^*$, $n_k \geq n_0$:
$\Psi(f(y), \lambda, z^*) + \frac{1}{\varrho_{n_k}} \sum\limits_{i=1}^{m} \vartheta(\varrho_{n_k} g_i(y)) < \Psi(f(x'), \lambda, z^*) + \frac{1}{\varrho_{n_k}} \sum\limits_{i=1}^{m} \vartheta(\varrho_{n_k} g_i(x'))$.
Thereafter $\Gamma(y) < \Gamma(x')$, which is absurd because $x' \in Y_n^*$, hence $\overline{\lim\limits_{n \to +\infty}} (Y_n^* \backslash \chi^*) = \emptyset$. ☐

Theorem 3.6. $\underline{\lim\limits_{n \to +\infty}} (Y_n^* \backslash \chi^*) = \emptyset$.

Proof. Assuming that $\underline{\lim\limits_{n \to +\infty}} (Y_n^* \backslash \chi^*) \neq \emptyset$.
Then $\exists x' \in \underline{\lim\limits_{n \to +\infty}} (Y_n^* \backslash \chi^*)$ and there exists n_0 such as for $n \geq n_0, x' \in Y_n^* \backslash \chi^*$ which leads to $x' \in Y_n^*$ and $x' \notin \chi^*$ from n_0. So

$$\forall y \in \chi, \Gamma(x') < \Gamma(y). \tag{3.3}$$

If $x' \in \chi$, as $x' \notin \chi^*$ there exists $y \in \chi$ such as $f_j(y) < f_j(x')$. Based on the previous Proof the inequality $\Psi(f(y), \lambda, z^*) < \Psi(f(x'), \lambda, z^*)$ is verified. As $x' \in \chi$ according to the Lemma 3.2,

$$\lim_{n \to +\infty} \frac{1}{\varrho_n} \sum_{i=1}^{m} \vartheta(\varrho_n g_i(x')) = 0 \text{ and } \lim_{n \to +\infty} \frac{1}{\varrho_n} \sum_{i=1}^{m} \vartheta(\varrho_n g_i(y)) = 0.$$

So $\exists n_0 > 0,\ n \geq n_0$:
$\Psi(f(y, \lambda, z^*)) + \frac{1}{\varrho_n} \sum\limits_{i=1}^{m} \vartheta(\varrho_n g_i(y)) < \Psi(f(x'), \lambda, z^*) + \frac{1}{\varrho_n} \sum\limits_{i=1}^{m} \vartheta(\varrho_n g_i(x'))$. It is equivalent to $\Gamma(y) < \Gamma(x')$. This inequality is a contradiction according to the relation (3.3) because x' is an optimal solution to the problem (3.2).

If $x' \notin \chi, \exists y \in \chi,\ f_j(y) < f_j(x'),\ \forall j \in I$. The same reasoning gives $\Psi(f(y), \lambda, z^*) < \Psi(f(x'), \lambda, z^*)$. As $x' \notin \chi$, according to the Lemma 3.2,

$\lim\limits_{n\to+\infty}\dfrac{1}{\varrho_n}\sum\limits_{i=1}^{m}\vartheta(\varrho_n g_i(x')) = +\infty$ and $\lim\limits_{n\to+\infty}\dfrac{1}{\varrho_n}\sum\limits_{i=1}^{m}\vartheta(\varrho_n g_i(y)) = 0$. So $\exists n_0 > 0, n > n_0, \dfrac{1}{\varrho_n}\sum\limits_{i=1}^{m}\vartheta(\varrho_n g_i(y)) < \dfrac{1}{\varrho_n}\sum\limits_{i=1}^{m}\vartheta(\varrho_n g_i(x'))$. Hence $\Psi(f(y),\lambda,z^*) + \dfrac{1}{\varrho_n}\sum\limits_{i=1}^{m}\vartheta(\varrho_n g_i(y)) < \Psi(f(x'),\lambda,z^*) + \dfrac{1}{\varrho_n}\sum\limits_{i=1}^{m}\vartheta(\varrho_n g_i(x'))$. So $\Gamma(y) < \Gamma(x')$ which is absurd because x' is an optimal solution of the problem (3.2), hence $\underline{\lim\limits_{n\to+\infty}}(Y_n^*\backslash\chi^*) = \emptyset$. □

STEP III : Decision space reduction

Alienor transformation presented by relation (2.4) is used here to transform several variables problems into a single variable problem as follows :

$$\begin{cases}\min\ \Gamma(\theta)\\ \theta\in[0;\theta_{max}]\end{cases} \tag{3.4}$$

with $\Gamma(h(\theta)) = (\Gamma\circ h)(\theta)$ where $h(\theta) = \big(h_1(\theta), h_2(\theta), \cdots, h_n(\theta)\big)$.

Theorem 3.7 (See in [3]). *All minimum value of the problem (3.2) can be approached by the minimum value of the problem (3.4).*

Proof. The proof can be found in [3]. □

STEP IV : Resolution in reduced space

As the problem (3.4) is a single objective with one variable then Nelder-Mead algorithm [15] has been used to solve it. This resolution is done after the discretization of research domain $[0;\theta_{\max}]$ as follows (See in [18]) : where

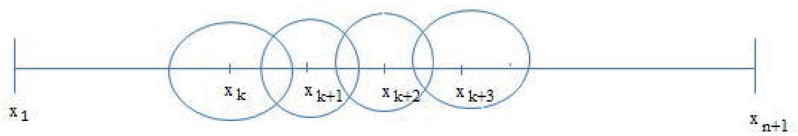

FIGURE 1. Nested domains

$x_1 = 0$ and $x_{n+1} = \theta_{\max}$.

STEP V : Solution configuration

After obtaining the optimal solution of the problem (3.4), we will re-use the Alienor transformation defined above to build a Pareto optimal solution of the initial problem (2.1).

3.1.2. *Algorithm of MOMA-Plus modified version.*

These are the main steps towards the algorithm of our new method :

(1) Choose λ;
(2) For j from 1 to p do
    $\Psi(x) \longleftarrow \max\big[\lambda_j|f_j(x) - z_j^*|\big]$ ("aggregation");
    End for

(3) $P(x) \longleftarrow \vartheta[\varrho_n g_1(x)]$;
For i from 2 to m do
$P(x) \longleftarrow P(x) + \vartheta[\varrho_n g_i(x)]$;
End for
$\Gamma(x) \longleftarrow \Psi(x) + P(x)$; ("penalization")
(4) For i from 1 to n to
$x_i = h_i(\theta)$; ("Alienor transformation")
End for
$F(\theta) \longleftarrow \Gamma(h_1(\theta), h_2(\theta), \cdots, h_n(\theta))$;
(5) $\theta \longleftarrow$ Apply Nelder-Mead algorithm to F(θ); ("resolution")
(6) For i from 1 to n,
$x_i = h_i(\theta)$; ("solution initialization")
End for

3.2. **Numerical results.**

3.2.1. *Problems test presentation.*
All multiobjective problems dealt with in this work are from the Zitzler [6] test problems which are presented in the following table :

Remark 3.8. For the problems whose constraints have the form $x_i \in [a_i, b_i]$, we will transform them into two constraints namely $x_i - b_i \leq 0$ and $a_i - x_i \leq 0, \forall i$.

3.2.2. *Graphic representation of Pareto optimal solutions.*

3.2.3. *Performance analysis.*
A performance study of the obtained solutions is done on the convergence to the Pareto front and the diversification on the Pareto front. There are many formulas in the literature to access the performance of algorithms but we use those proposed by Deb et al [6]. It is γ for the convergence and Δ for the diversification :

$$\gamma = \frac{\sqrt{\sum_{i=1}^{N} d_i^2}}{N} \qquad and \qquad \Delta = \frac{d_f + d_l + \sum_{i=1}^{N-1} |d_i - \overline{d}|}{d_f + d_l + (N-1)\overline{d}} \tag{3.5}$$

where N is the number of the solutions given by the MOMA-Plus modified version; d_f and d_l are respectively the euclidean distances between upper extreme solutions and lower extreme solutions given by MOMA-Plus modified version; d_i and $\overline{d}$ are respectively the euclidean distances between two consecutive obtained solutions and the arithmetic average of d_i. The following table presents the value of convergence and diversification :

For a comparative study of our results, the following table shows the performance index for the same test problem provided by initial version of MOMA-Plus [18] :

According to the table 2 and 3, MOMA-Plus modified version is better than the initial version of MOMA-Plus about the convergence of the obtained solutions in all test problems. However, we have the opposite result for the diversification criterion.

TABLE 1. Multiobjective test problems

Indexes	Mathematical formulation	n	Bounds
T_1	$\begin{cases} \min f_1(x_1,x_2) = x_1 \\ \min f_2(x_1,x_2) = \dfrac{1+x_2}{x_1} \\ \quad 0.1 \leqslant x_1 \leqslant 1 \\ \quad 0 \leqslant x_2 \leqslant 5 \end{cases}$	2	$x_1, x_2 \in [0;1]$
T_2	$\begin{cases} \min f_1(x) = x^2 \\ \min f_2(x) = (x-2)^2 \\ \quad -5 \leqslant x \leqslant 5 \end{cases}$	1	$x \in [0;4]$
T_3	$\begin{cases} \min f_1(x) = x_1 \\ \min f_2(x) = g\left(1 - \sqrt{\frac{f_1(x)}{g}}\right) \\ g(x) = 1 + \dfrac{9}{n-1} \times \sum_{i=2}^{n} x_i \\ x = (x_1, x_2, \cdots, x_n) \in [0.1]^n \end{cases}$	30	$x_i \in [0;1]$
T_4	$\begin{cases} \min f_1(x) = x_1 \\ \min f_2(x) = g(x) \times (1 - (\dfrac{f_1}{g})^2) \\ g(x) = 1 + \dfrac{9}{n-1} \times \sum_{i=2}^{n} x_i \\ x = (x_1, x_2, \cdots, x_n) \in [0.1]^n \end{cases}$	30	$x_i \in [0;1]$
T_5	$\begin{cases} \min f_1(x) = x_1 \\ \min f_2(x) = g(x) \times h(x) \\ g(x) = 1 + \dfrac{9}{n-1} \times \sum_{i=2}^{n} x_i \\ h(x) = 1 - \sqrt{\dfrac{f_1(x)}{g(x)}} - \dfrac{f_1(x)}{g(x)} sin(10\pi f_1(x)) \\ x = (x_1, x_2, \cdots, x_n) \in [0.1]^n \end{cases}$	30	$x_i \in [0;1]$
T_6	$\begin{cases} \min f_1(x) = x_1 \\ \min f_2(x) = g(x) \times \sqrt{1 - \dfrac{f_1}{g}} \\ g(x) = 1 + \dfrac{9}{n-1} \times \sum_{i=2}^{n} x_i \\ x = (x_1, x_2, \cdots, x_n) \in [0.1]^n \end{cases}$	30	$x_i \in [0;1]$

TABLE 2. MOMA-Plus modified version performances

MOMA-Plus modified	T_1	T_2	T_3	T_4	T_5	T_6
γ	0.0077	0.0011	0.0044	0.0018	0.0029	0.0018
Δ	0.9819	0.9843	0.9823	0.9821	0.9823	0.9819

TABLE 3. MOMA-Plus method performances

MOMA-Plus	T_1	T_2	T_3	T_4	$T5$	T_6
γ	0.0691	0.0042	0.0046	0.0137	0.0599	0.1154
Δ	1.1833	0.0309	0.9820	0.3483	0.9835	0.9818

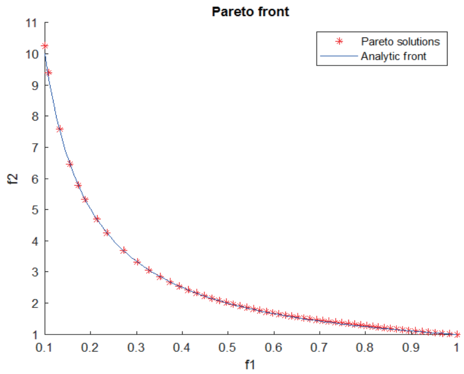

FIGURE 2. Pareto solutions for T1

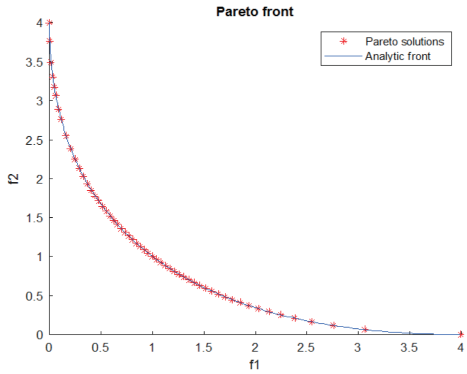

FIGURE 3. Pareto solutions for T2

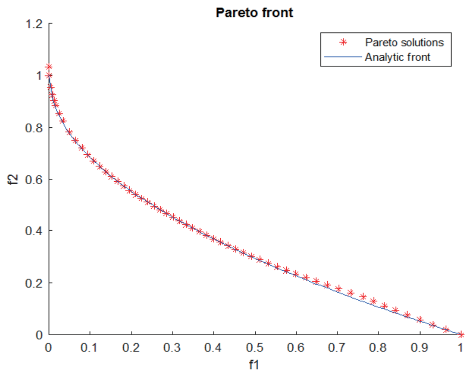

FIGURE 4. Pareto solutions for T3

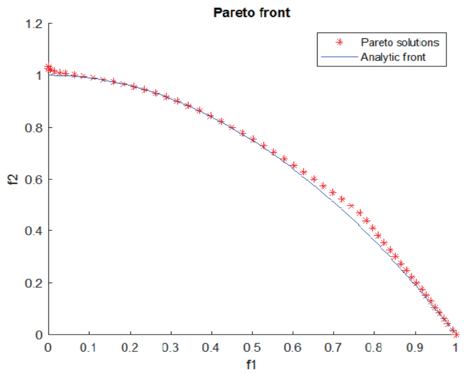

FIGURE 5. Pareto solutions for T4

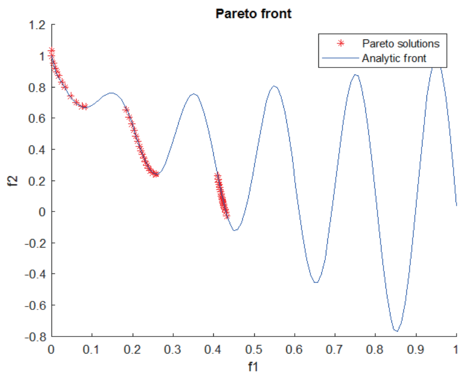

FIGURE 6. Pareto solutions for T5

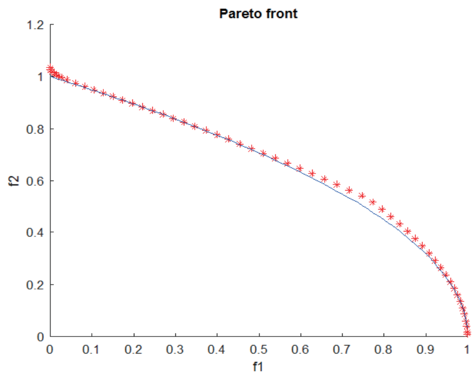

FIGURE 7. Pareto solutions for T6

4. CONCLUSION

Testing the MOMA-Plus modified version on six test problems has indeed been a success. This enables us to obtain solutions with better convergence and diversification. Moreover, our work mainly consisted of an analytical approach at each stage of the MOMA-Plus modified version. Thus, the numerical results have lead us to conclude that MOMA-Plus modified version is the best alternative for solving multiobjective optimization problem when the convergence is the main criterion.

References

[1] T. Benneoula and Y. Cherruault, *Alienor method for Global optimization with a large number of variable*, Kybernetes **34** (2005), 1104–1111.

[2] H. Bernau, *Use of exact penalty functions to determine efficient decisions*, European Journal of Operational Research **49** (1990), 348–355.

[3] Y. Cherruault, *A new method for global optimization(alienor)*, Kybernetes **19** (1989), 19–32.

[4] Y. Cherruault and G. Mora, *Optimization global : Théorie des courbes $\alpha-$denses*, Ed. Economica, 2005.

[5] R. Cominetti and J. P. Dussault, *Stable exponential penalty algorithm with superlinear convergence*, Journal of Optimization Theory and Applications **83** (1994), 285–309.

[6] K. Deb, A. Patrap, S. Agarwal and T. Meyarivan, *A fast and elitist multiobjective genetic algorithm: NSGA-II*, IEEE Transactions on evolutionary computation **6** (2002), 182–197.

[7] M. Ehrgott, *Multicriteria Optimization*, Second Edition, Springer Berlin. Heidelberg 2005.

[8] S.-P. Han, *Exact penalty functions in nonlinear programming*, Mathematical Programming 17 (1979), 251–269.

[9] B. O. Konfe, Y. Cherruault, B. Somé, *Titem Benneouala: Alienor method applied to operational research*, Kybernetes **34** (2005), 1211–1222.

[10] S. Liu and E. Feng, *The exponential penalty function method for multiobjective programming problems*, Optimization Methods and Software **25** (2010), 667–675.

[11] Q. Liu and Z. Lin, *Solving the multiobjective programming problems via homotopy interior point method*, Acta Mathematicae Applicatae Sinica **23** (2000), 188–194.

[12] S. K. Mahato, *Nonlinear constrained optimization using penalty function techniques*, International Journal of Current Trends in Science and Technology **7** (2017), 20379–20394.

[13] Z. Meng, Q. Hu, C. Dang and K. Yang, *An objective penalty function method for nonlinear programming*, Applied Mathematics Letters **17** (2004), 683–689.

[14] K. Miettinen, *Nonlinear Multiobjective optimization*, Kluwer Accademic Publishers, 1999.

[15] J. Nelder and R. Mead, *A simplex method for function minimization*, Computer Journal **7** (1965), 308–313.

[16] V. H. Nguyen and J. J. Strodiot, *On the convergence rate for a penalty function method of exponential type*, Journal of Optimization Theory and Applications **27** (1979), 495–508.

[17] A. Som, K, Somé, A. Compaoré and B. Somé, *Performances assessment of MOMA-Plus method on multiobjective optimization problems.* European Journal of Pure and Applied Mathematics **13**(2020), 48–68.

[18] K. Somé, *Nouvelle métaheuristique basée sur la méthode Alienor pour la résolution des problèmes d'optimisation multiobjectif: Théorie et Applications.* Thèse de doctorat soutenue le 27 juillet 2013 à l'université de Ouagadougou.

[19] K. Somé, B. Ulungu, I. Himidi Mohamed and B. Somé, *A new method for solving nonlinear multiobjective optimization problems*, Journal of Mathematical Sciences **2** (2011), 1–18.

[20] K. Somé, B. Ulungu, W. Olivier Sawadogo and B. Somé, *A theoretical foundation metaheuristic method to solve some multiobjectif optimization problems.* International Journal of Applied Mathematical Research **2** (2013), 464–475.

[21] D. J. White, *Multiobjective programming and penalty functions*, Journal of Optimization Theory and Applications **43** (1998), 583–599.

[22] Q. Yang and G. Chen, *Primal-dual subgradient algorithm for distributed constraint optimization over unbalanced digraphs*, IEEE Access **7** (2019), 85190–85202.

[23] Y. Zheng and Z. Meng, *A new augmented Lagrangian objective penalty function for constrained optimization problems,* Open Journal of Optimization **6** (2017), 39–46.

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Alexandre Som
Joseph KI-ZERBO University, Burkina Faso
E-mail address: irenotesom89@gmail.com

Kounhinir Some
Norbert ZONGO University, Burkina Faso
E-mail address: sokous11@gmail.com

Abdoulaye Compaore
Norbert ZONGO University, Burkina Faso
E-mail address: ab.compaore1@gmail.com

Blaise Some
Joseph KI-ZERBO University, Burkina Faso
E-mail address: blaisesomeouaga1@gmail.com