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SYSTEM OF GENERALIZED RESOLVENT EQUATIONS INVOLVING XOR-OPERATION IN q-UNIFORMLY SMOOTH BANACH SPACES

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Abstract. In this paper, we study a system of generalized resolvent equations involving XOR-operation in q-uniformly smooth Banach spaces. We have shown the equivalence of system of generalized resolvent equations involving XOR-operation with a system of generalized variational inclusions involving XOR-operation. Some iterative algorithms are proposed to approximate the solution for system of generalized resolvent equations involving XOR-operation.The convergence criteria is also discussed.

1. Introduction

It is worth to mention that variational inequalities and their generalizations are extended in various directions after their existence since early sixties. Variational inclusions are powerful tools to solve many problems of real life, for example, to solve problems related to mechanics, optimization and control, elasticity, basic and applied sciences etc., see for example [?,6,10,12,13,20,21,23] and references therein. System of variational inequalities are considered and studied by Pang [22], Cohen and Chaplais [8], Bianchi [7], etc..Pang have shown that the traffic equilibrium problem, the Nash equilibrium, and the general equilibrium programing problem can be modelled as a system of variational inequalities over product of sets. Agarwal et al. [2] studied the sensitivity analysis of solutions for a system of generalized nonlinear mixed quasi-variational inclusions, Pang and Zhu [24] studied a system of mixed quasi-variational inclusions with (H,η)-monotone operators and Lan et al. [14] studied a system of nonlinear A-monotone multivalued variational inclusions. Ahmad and Yao [1] studied a system of generalized resolvent equations with corresponding system of variational inclusions in real Banach spaces.

XOR is a binary operation, it stands for "exclusive or", that is to say the resulting bit evaluates to one if only exactly one of the bits is a set. This operation is commutative, associative and self-inverse. It is also same as addition modulo 2 in Boolean algebra. XOR(A,B) represents the logical exclusive disjunction and XOR(A,B) is true when either A or B are true.If both A and B are true or false, XOR(A,B) is false. As an application of XOR-terminology, we mention an example: Consider a light bulb to two 3-ways switches. The light goes on if only one switch is in the "up" position and the other switch is in the "down" position. If both are in the "up" position or both are in the "down" position, the light is off. The lights state (on,off) is the XOR of the state of the two switches. One can find its applications in many branchs of science, for example, to generate random pseudo numbers,

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to detect error in digital communications, to implement multilayer perception in neutral network, cryptography, etc.. Several inclusion problems involving XOR-operation are introduced and studied by Li and his co-author [15–19], Ahmad et al [3, 4, 15] and others. The above mentioned facts motivated us to extend the problem studied by Ahmad and Yao [1] with XOR-operation in q-uniformly smooth Banach spaces. That is, a system of generalized resolvent equations involving XOR-operation in q-uniformly smooth Banach spaces is considered and studied.

2. Preliminaries

Let E be a real Banach space with its norm $\|\cdot\|$, E^* be the topological dual of E, $\langle\cdot,\cdot\rangle$ be the duality pairing between E and E^*, d be the metric induced by the norm $\|\cdot\|$, 2^E (respectively CB(E)) be the family of non-empty (respectively, nonempty closed and bounded) subsets of E, and $D(\cdot,\cdot)$ be the Hausdorff metric on CB(E) defined by

$$D(P,Q) = max\{\underset{x\in P}{Sup} d(x,Q), \underset{y\in Q}{Sup} d(P,y)\},$$

where

$$d(x,Q) = \underset{y\in Q}{Inf} d(x,y) \; and \; d(P,y) = \underset{x\in P}{Inf} d(x,y).$$

The following concepts are required for the presentation of this paper.

Definition 2.1. The generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^q \; and \; \|f\| = \|x\|^{q-1}\}, \; for \; all \; x \in E,$$

where $q > 1$ is a constant. It is well known that

$$J_q(x) = \|x\|^{q-2}J(x),$$

for all $x \in E$. If E is a Hilbert space, then J is the identity mapping. We mention some properties of generalized duality mapping J_q below:

(i) $J_q(x) = \|x\|^{q-2}J_2(x), for \; all \; x \in E, x \neq 0,$
(ii) $J_q(tx) = t^{q-1}J_q(x)$, for all $x \in E$ and $t \in [0,\infty)$,
(iii) $J_q(x) = -J_q(x), for \; all \; x \in E.$

The modulus of smoothness of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(t) = Sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| \leq 1, \|y\| \leq t\right\}.$$

A Banach space E is called uniformly smooth if

$$\lim_{t\to 0}\frac{\rho_E(t)}{t} = 0,$$

and is called q-uniformly smooth, if there exist a constant $C > 0$ such that

$$\rho_E(t) \leq Ct^q, q > 1.$$

Lemma 2.2 ([25]). *Let E be a real uniformly smooth Banach space. Then, E is q-uniformly smooth if and only if there exists $C_q > 0$ such that for all $x, y \in E$,*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + C_q\|y\|^q.$$

Throughout the paper, we take E to be real ordered Banach space, unless otherwise specified.

Definition 2.3. A nonempty closed convex subset P of E is said to be cone, if

(i) for any $x \in P$ and $\lambda > 0$, then $\lambda x \in P$,
(ii) for any $x \in P$ and $-x \in P$, then $x = 0$.

Definition 2.4. Let P be a cone. For arbitrary elements $x, y \in E, x \leq y$ holds if and only if $x - y \in P$. Then, the relation " $\leq$ " in E is called partial order relation.

The following concepts and results can be found in [9, 15–19].

Definition 2.5. For arbitrary elements $x, y \in E$, if $x \leq y$(*or* $y \leq x$) holds, then x and y are said to be comparable to each other (denoted by $x \propto y$).

Definition 2.6. For arbitrary elements $x, y \in E$, $lub\{x, y\}$ and $glb\{x, y\}$ means the least upper bound and the greatest lower bound for the set $\{x, y\}$. Suppose $lub\{x, y\}$ and $glb\{x, y\}$ exist, then some binary operations are defined as follows:

(i) $x \vee y = lub\{x, y\}$,
(ii) $x \wedge y = glb\{x, y\}$,
(iii) $x \oplus y = (x - y) \vee (y - x)$,
(iv) $x \odot y = (x - y) \wedge (y - x)$.

The operators $\wedge, \vee, \oplus$ *and* $\odot$ are called OR, AND, XOR and XNOR operations, respectively.

Proposition 2.7. *Let $\oplus$ be an XOR-operation and $\odot$ be an XNOR-operation. Then the following assertions hold:*

(i) $x \odot x = 0, x \odot y = y \odot x = -(x \oplus y) = -(y \oplus x)$,
(ii) *if* $x \propto 0$, $-x \oplus 0 \leq x \leq x \oplus 0$,
(iii) $(\lambda x) \oplus (\lambda y) = |\lambda|(x \oplus y)$,
(iv) $0 \leq x \oplus y, if\ x \propto y$,
(v) *if* $x \propto y$, *then* $x \oplus y = 0$, *if and only if* $x = y$.

Proposition 2.8. *Let P be a cone in E, then for each $x, y \in E$, the following relations hold:*

(i) $\|0 \oplus 0\| = \|0\| = 0$,
(ii) $\|x \vee y\| \leq \|x\| \vee \|y\| \leq \|x\| + \|y\|$,
(iii) $\|x \oplus y\| \leq \|x - y\|$,
(iv) *if* $x \propto y$, *then* $\|x \oplus y\| = \|x - y\|$.

Definition 2.9. Let $M : E \to 2^E$ be a set-valued mapping and $M(x)$ be a closed subset in E. Then

(i) The set-valued mapping M is said to be a comparison mapping, if for any $v_x \in M(x), x \propto v_x$, and if $x \propto y$, then for any $v_x \in M(x)$ and any $v_y \in M(y), v_x \propto v_y$, for all $x, y \in E$,
(ii) The set-valued comparison mapping M is said to be α-non-ordinary difference mapping if for each $x, y \in E, v_x \in M(x)$ and $v_y \in M(y)$, there exist a constant $\alpha > 0$ such that $(v_x \oplus v_y) \oplus \alpha(x \oplus y) = 0$.

(*iii*) The set-valued comparison mapping M is said to be ρ-order monotone mapping, if there exists a constant $\rho > 0$ such that

$$\rho(v_x - v_y) \geq (x - y), for\ all\ x, y \in E, v_x \in M(x)\ and\ v_y \in M(y).$$

(*iv*) The set-valued comparison mapping M is said to be an (α, ρ)-NODM mapping, if M is an α-non-ordinary difference and ρ-order monotone mapping, and

$$(I + \rho M)(E) = E, for\ all\ \alpha, \rho > 0.$$

Lemma 2.10 ([15]). *If M is an α-non-ordinary difference mapping, then an inverse mapping $J_M^\rho = (I + \rho M)^{-1} : E \to 2^E$ of $(I + \rho M)$ is a single-valued mapping, where I is the identity mapping, $\alpha > 0$ and $\rho > 0$ are constants.*

Definition 2.11. Let E be an real ordered Banach space. Let $P \subset E$ be a cone and M be an α-non-ordinary difference mapping. The resolvent operator $J_M^\rho : E \to E$ is defined as

$$J_M^\rho(x) = (I + \rho M)^{-1}(x),\ for\ all\ x \in E, where\ \rho > 0\ is\ a\ constant.$$

Theorem 2.12 ([15]). *Let the set-valued comparison mapping M be (α, ρ)-NODM mapping, then*

(*i*) *The resolvent operator $J_M^\rho : E \to E$ is a comparison mapping,*

(*ii*) *The resolvent operator $J_M^\rho : E \to E$ is Lipschitz-type continuous, i.e*

$$J_M^\rho(x) \oplus J_M^\rho(y) \leq \frac{1}{(\alpha\rho - 1)}(x \oplus y), where\ \alpha > \frac{1}{\rho}.$$

3. Formulation of the problem and iterative algorithm

Let E_1 and E_2 be two closed subspaces of an ordered real q-uniformly smooth Banach space E such that they preserve partial ordering " $\leq$ " of the norm $\|\cdot\|$ of E, $S : E_1 \times E_2 \to E_1$, $T : E_1 \times E_2 \to E_2$, $p, f : E_1 \to E_1$ and $g, Q : E_2 \to E_2$ be the single-valued mappings. Let $A : E_1 \to CB(E_1)$, $B : E_2 \to CB(E_2)$, $M : E_1 \to 2^{E_1}$ and $N : E_2 \to 2^{E_2}$ be the set-valued mappings such that $f(E_1) \cap D(M) \neq 0$ and $g(E_2) \cap D(N) \neq 0$, where $D(M)$ means domain of M. We consider the following problem:

Find $(x, y) \in E_1 \times E_2, u \in A(x), v \in B(y), z' \in E_1, z'' \in E_2$ such that

$$\begin{aligned} S(p(x), v) \oplus \rho^{-1} R_M^\rho(z') = 0,\ \rho > 0, \\ T(u, Q(y)) \oplus \gamma^{-1} R_N^\gamma(z'') = 0,\ \gamma > 0, \end{aligned} \tag{3.1}$$

where $R_M^\rho = (I \oplus J_M^\rho)$, $R_N^\gamma = (I \oplus J_N^\gamma)$ and J_M^ρ, J_N^γ are the resolvent operators associated with M and N, respectively, $\rho > 0$ and $\gamma > 0$ are constants.

The corresponding system of generalized variational inclusions with XOR-operations for (3.1) is the following:

Find $(x, y) \in E_1 \times E_2$, $u \in A(x)$, $v \in B(y)$ such that

$$\begin{aligned} 0 \in S(p(x), v) \oplus M(f(x)), \\ 0 \in T(u, Q(y)) \oplus N(g(y)). \end{aligned} \tag{3.2}$$

If we relax the condition of ordered q-uniformly smooth on E_1 and E_2, we take E_1 and E_2 to be uniformly smooth Banach spaces and if we replace $\oplus$ by $+$, then problem (3.1) reduces to the problem studied by Ahmad and Yao [1]. If we take E_1 and E_2 to be Hilbert spaces and replacing $\oplus$ by $+$, then problem (3.2) reduces to the problem studied by Lan et al. [14].

The following Lemma is a fixed point formulation of problem (3.2).

Lemma 3.1. *The set of elements $(x, y) \in E_1 \times E_2$, $u \in A(x)$, $v \in B(y)$ constitute a solution of system of generalized variational inclusions involving XOR-operation* (3.2) *if and only if the following equations are satisfied:*

$$\begin{aligned} f(x) &= J_M^{\rho}(f(x) \oplus \rho S(p(x), v)),\ \rho > 0, \\ g(y) &= J_N^{\gamma}(g(y) \oplus \gamma T(u, Q(y)),\ \gamma > 0. \end{aligned}$$

where $\rho > 0, \gamma > 0$ are constants and J_M^{ρ}, J_N^{γ} are the resolvent operators associated with M and N, respectively.

Proof. The proof is a direct consequence of the definition of resolvent operators $J_M^{\rho}$and J_N^{γ} and hence omitted. □

4. Iterative algorithms

We first establish an equivalence relation between system of generalized resolvent equations involving XOR-operation (3.1) and system of generalized variational inclusions involving XOR-operation (3.2). On the basis of this equivalence, we suggest some iterative algorithms for solving system of generalized resolvent equations involving XOR-operation (3.1).

Proposition 4.1. *The system of generalized variational inclusions involving XOR-operation* (3.2) *has a solution (x, y, u, v), where $(x, y) \in E_1 \times E_2, u \in A(x), v \in B(y)$, if and only if the system of generalized resolvent equations involving XOR-operation* (3.1) *has a solution (z', z'', x, y, u, v), where $(x, y), (z', z'') \in E_1 \times E_2, u \in A(x), v \in B(y)$ and*

$$f(x) = J_M^{\rho}(z'), \tag{4.1}$$

$$g(y) = J_N^{\gamma}(z''), \tag{4.2}$$

where

$$z' = f(x) \oplus \rho S(p(x), v) \text{ and } z'' = g(y) \oplus \gamma T(u, Q(y)).$$

Assume that $S(p(x), v) \propto R_M^{\rho}(z')$ and $T(u, Q(y)) \propto R_N^{\gamma}(z'')$.

Proof. Let (x, y, u, v) be a solution of system of generalized variational inclusions involving XOR-operation (3.2). Then by Lemma 3.1, the following equations are satisfied:

$$\begin{aligned} f(x) &= J_M^{\rho}(f(x) \oplus \rho S(p(x), v)), \\ g(y) &= J_N^{\gamma}(g(y) \oplus \gamma T(u, Q(y))). \end{aligned}$$

Let

$$z' = f(x) \oplus \rho S(p(x), v) \text{ and } z'' = g(y) \oplus \gamma T(u, Q(y)),$$

then we have

$$f(x) = J_M^{\rho}(z'),$$
$$g(y) = J_N^{\gamma}(z'').$$

Thus

$$z' = J_M^{\rho}(z') \oplus \rho S(p(x), v), \tag{4.3}$$

and

$$z'' = J_N^{\gamma}(z'') \oplus \gamma T(u, Q(y)). \tag{4.4}$$

It follows from (4.3) and (4.4) that

$$\begin{aligned} z' \oplus J_M^{\rho}(z') &= J_M^{\rho}(z') \oplus J_M^{\rho}(z') \oplus \rho S(p(x), u), \\ i.e\ (I + J_M^{\rho})(z') &= \rho S(p(x), v), \\ and\ z'' \oplus J_N^{\gamma}(z'') &= J_N^{\gamma}(z'') \oplus J_N^{\gamma}(z'') \oplus \gamma T(u, Q(y)), \\ i.e\ (I + J_N^{\gamma})(z'') &= \gamma T(u, Q(y)). \end{aligned}$$

As $R_M^{\rho}(z') = (I \oplus J_M^{\rho})(z')$ and $R_N^{\gamma}(z'') = (I \oplus J_N^{\gamma})(z'')$, we have

$$R_M^{\rho}(z') = \rho S(p(x), v),$$
$$R_N^{\gamma}(z'') = \gamma T(u, Q(y)).$$

Since $S(p(x), v) \propto R_M^{\rho}(z')$ and $T(u, Q(y)) \propto R_N^{\gamma}(z'')$, using (v) of Proposition 2.7, we have

$$S(p(x), v) \oplus \rho^{-1} R_M^{\rho}(z'') = 0,$$
$$T(u, Q(y)) \oplus \gamma^{-1} R_N^{\gamma}(z'') = 0.$$

i.e

(z', z'', x, y, u, v) is a solution of system of generalized resolvent equations involving XOR-operation (3.1).

Conversely, let (z', z'', x, y, u, v) be a solution of system of generalized resolvent equations involving XOR-operation (3.1) and using the above mentioned facts, we have

$$\rho S(p(x), v) = R_M^{\rho}(z'), \tag{4.5}$$

$$\gamma T(u, Q(y)) = R_N^{\gamma}(z''). \tag{4.6}$$

Now,

$$\begin{aligned} \rho S(p(x), v) &= R_M^{\rho}(z') \\ &= (I \oplus J_M^{\rho})(z') \\ &= z' \oplus J_M^{\rho}(z') \\ &= f(x) \oplus \rho S(p(x), v) \oplus J_M^{\rho}(f(x) \oplus \rho S(p(x), v)), \end{aligned}$$

that is,

$$f(x) = J_M^{\rho}(f(x) \oplus \rho S(p(x), v)).$$

Similarly, it follows from (4.6) that

$$g(y) = J_N^{\gamma}(g(y) \oplus \gamma T(u, Q(y)).$$

Applying Lemma 3.1, we conclude that (x, y, u, v) is a solution of system of generalized variational inclusions involving XOR-operation (3.2). $\square$

Now we present an alternative proof of Proposition 4.1.
Alternative Proof.

Proof. Let $z^{'} = f(x) \oplus \rho S(p(x), v)$ and $z^{''} = g(y) \oplus \gamma T(u, Q(y))$.
Using Lemma 3.1, we can write

$$\begin{aligned} z^{'} &= J_M^{\rho}(z^{'}) \oplus \rho S(p(x), v), \\ \text{and } z^{''} &= J_N^{\gamma}(z^{''}) \oplus T(u, Q(y)), \end{aligned}$$

which implies that

$$\begin{aligned} S(p(x), v) \oplus R_M^{\rho}(z^{'}) &= 0, \\ \text{and } T(u, Q(y)) \oplus R_N^{\gamma}(z^{''}) &= 0. \end{aligned}$$

which is the required system of generalized resolvent equations involving XOR-operation (3.1). $\square$

Now, we define some iterative algorithms for solving system of generalized resolvent equations involving XOR-operation (3.1).

Algorithm 4.1. For given $(x_0, y_0) \in E_1 \times E_2, u_0 \in A(x_0), v_0 \in B(y_0), z_0^{'} \in E_1,$ *and* $z_0^{''} \in E_2$, compute the sequences $\{x_n\}, \{y_n\}, \{u_n\}, \{v_n\}, \{z_n^{'}\}$ *and* $\{z_n^{''}\}$ by the following scheme:

$$f(x_n) = J_M^{\rho}(z_n^{'}), \tag{4.7}$$

$$g(y_n) = J_N^{\gamma}(z_n^{''}), \tag{4.8}$$

$$u_n \in A(x_n) : \|u_{n+1} \oplus u_n\| \leq \|u_{n+1} - u_n\| \leq D(A(x_{n+1}), A(x_n)), \tag{4.9}$$

$$v_n \in B(y_n) : \|v_{n+1} \oplus v_n\| \leq \|v_{n+1} - v_n\| \leq D(B(y_{n+1}), B(y_n)), \tag{4.10}$$

$$z_{n+1}^{'} = f(x_n) \oplus \rho S(p(x_n), v_n), \tag{4.11}$$

$$z_{n+1}^{''} = g(y_n) \oplus \gamma T(u_n, Q(y_n)), \tag{4.12}$$

where $n = 0, 1, 2, ...,$ $\rho > 0$ *and* $\gamma > 0$ are constants.

The system of generalized resolvent equations involving XOR-operation (3.1) can also be written as

$$\begin{aligned} z^{'} &= f(x) \oplus S(p(x), v) + (I - \rho^{-1})R_M^{\rho}(z^{'}), \\ z^{''} &= g(y) \oplus T(u, Q(y)) + (I - \gamma^{-1})R_N^{\gamma}(z^{''}). \end{aligned}$$

Based on above formulation, we suggest the following iterative Algorithm.

Algorithm 4.2. For given $(x_0, y_0) \in E_1 \times E_2, u_0 \in A(x_0), v_0 \in B(y_0), z_0^{'} \in E_1$ and $z_0^{''} \in E_2$, compute the sequences $\{x_n\}, \{y_n\}, \{u_n\}, \{v_n\}, \{z_n^{'}\}$ and $\{z_n^{''}\}$ by the following scheme:

$$\begin{aligned} f(x_n) &= J_M^{\rho}(z_n^{'}), \\ g(y_n) &= J_N^{\gamma}(z_n^{''}), \end{aligned}$$

$$\begin{aligned} u_n \in A(x_n) : \|u_{n+1} \oplus u_n\| &\leq \|u_{n+1} - u_n\| \leq D(A(x_{n+1}), A(x_n)), \\ v_n \in B(y_n) : \|v_{n+1} \oplus v_n\| &\leq \|v_{n+1} - v_n\| \leq D(B(y_{n+1}), B(y_n)), \end{aligned}$$

$$z_{n+1}^{'} = f(x_n) \oplus S(p(x_n), v_n) + (I - \rho^{-1}) R_M^{\rho}(z_n^{'}), \tag{4.13}$$

$$z_{n+1}^{''} = g(y_n) \oplus T(u_n, Q(y_n)) + (I - \gamma^{-1}) R_N^{\gamma}(z_n^{''}), \tag{4.14}$$

where $n = 0, 1, 2, ...,\ \rho > 0$ *and* $\gamma > 0$ are constants.

For positive stepsize $\delta^{'}$ and $\delta^{''}$, the system of generalized resolvent equations involving XOR-operation (3.1) can also be expressed as:

$$\begin{aligned} f(x, z^{'}) &= f(x, z^{'}) \oplus \delta^{'} \big[\{z^{'} \oplus J_M^{\rho}(z^{'})\} \oplus \rho S(p(x), v)\big], \\ g(y, z^{''}) &= g(y, z^{''}) \oplus \delta^{''} \big[\{z^{''} \oplus J_N^{\gamma}(z^{''})\} \oplus \gamma T(u, Q(y))\big]. \end{aligned}$$

This fixed point formulation enables us to propose the following iterative Algorithm.

Algorithm 4.3. For given $(x_0, y_0) \in E_1 \times E_2, u_0 \in A(x_0), v_0 \in B(y_0), z_0^{'} \in E_1$ and $z_0^{''} \in E_2$, compute the sequence $\{x_n\}, \{y_n\}, \{u_n\}, \{v_n\}, \{z_n^{'}\}$ and $\{z_n^{''}\}$ by the following iterative schemes:

$$\begin{aligned} u_n \in A(x_n) : \|u_{n+1} \oplus u_n\| &\leq \|u_{n+1} - u_n\| \leq D(A(x_{n+1}), A(x_n)), \\ v_n \in B(y_n) : \|v_{n+1} \oplus v_n\| &\leq \|v_{n+1} - v_n\| \leq D(B(y_{n+1}), B(y_n)), \end{aligned}$$

$$f(x_{n+1}, z_{n+1}^{'}) = f(x_n, z_n^{'}) \oplus \delta^{'} \big[\{z_n^{'} \oplus J_M^{\rho}(z_n^{'})\} \oplus \rho S(p(x_n), v_n)\big], \tag{4.15}$$

$$g(y_{n+1}, z_{n+1}{''}) = g(y_n, z_n^{''}) \oplus \delta^{''} \big[\{z_n^{''} \oplus J_N^{\gamma}(z^{''})\} \oplus \gamma T(u_n, Q(y_n))\big], \tag{4.16}$$

where $n = 0, 1, 2...,\ \rho > 0$ *and* $\gamma > 0$ are constants.

It is to be noted that for $\delta^{'} = \delta^{''} = 1, f(x_n, z_n^{'}) = f(x_n), g(y_n, z_n^{''}) = g(y_n)$, Algorithm 4.3 reduces to the following algorithm which is applicable to solve system of generalized variational inclusions involving XOR-operation (3.2).

Algorithm 4.4. For any given $(x_0, y_0) \in E_1 \times E_2, u_0 \in A(x_0)$ and $v_0 \in B(y_0)$, compute the sequences $\{x_n\}, \{y_n\}, \{u_n\}$ and $\{v_n\}$ by the following iterative schemes:

$$\begin{aligned} f(x_{n+1}) &= J_M^{\rho}(f(x_n) \oplus \rho S(p(x_n), v_n)), \\ g(y_{n+1}) &= J_N^{\gamma}(g(y_n) \oplus \gamma T(u, Q(y_n)), \\ u_n \in A(x_n) : \|u_{n+1} \oplus u_n\| &\leq \|u_{n+1} - u_n\| \leq D(A(x_{n+1}), A(x_n)), \\ v_n \in B(y_n) : \|v_{n+1} \oplus v_n\| &\leq \|v_{n+1} - v_n\| \leq D(B(y_{n+1}), B(y_n)). \end{aligned}$$

where $n = 0, 1, 2...,\ \rho > 0$ *and* $\gamma > 0$ are constants.

We prove the following existence and convergence result for the system of generalized resolvent equations involving XOR-operation (3.1).

Theorem 4.2. *Let E_1 and E_2 be two closed subspaces of an ordered real q-uniformly smooth Banach space E and $P \subset E$ be a cone. Let $A : E_1 \to CB(E_1)$ and $B : E_2 \to CB(E_2)$ be D-Lipschitz continuous mappings with constants λ_{D_A} and λ_{D_B}, respectively. Let the set-valued mapping $M : E_1 \to 2^{E_1}$ be α-non-ordinary difference and ρ-ordered monotone mapping; the set-valued mapping $N : E_2 \to 2^{E_2}$ be $\alpha^{'}$-non-ordinary and $\rho^{'}$-ordered monotone mapping. Let $f : E_1 \to E_1$, $g : E_2 \to E_2$ be Lipschitz continuous mappings with constants λ_f and λ_g, respectively; $p : E_1 \to E_1$, $Q : E_2 \to E_2$ be Lipschitz continuous mappings with constants λ_p and λ_Q, respectively. Let $S : E_1 \times E_2 \to E_1$, $T : E_1 \times E_2 \to E_2$ be Lipschitz continuous mappings in both the arguments with constants λ_{S_1}, λ_{S_2} and λ_{T_1}, λ_{T_2}, respectively. Suppose that $z_n^{'} \propto z_{n+1}^{''}, z_n^{''} \propto z_{n+1}^{''}, S(p(x_n), v_n) \propto R_M^{\rho}(z_n^{'}), T(u_n, Q(y_n)) \propto R_N^{\gamma}(z_n^{''})$, for $n = 0, 1, 2...$ and if there exist constants $\gamma > 0$; $\alpha, \alpha^{'} > 0$; $\rho > 0$ and $\rho^{'} > 0$ such that*

$$0 < \frac{(\theta_1 + \theta_2 + \theta_6)\theta_7}{1 - \theta_1} < 1,\ 0 < \frac{(\theta_3 + \theta_4 + \theta_5)\theta_8}{1 - \theta_4} < 1, \tag{4.17}$$

where $\theta_1 = \sqrt[q]{1 + q\lambda_f + C_q\lambda_f^q}$, $\theta_2 = \sqrt[q]{1 + \rho q\lambda_{S_1}\lambda_p} + \sqrt[q]{C_q}\rho\lambda_{S_1}\lambda_p$, $\theta_3 = \sqrt[q]{C_q}\rho\lambda_{S_2}\lambda_{D_B}$, $\theta_4 = \sqrt[q]{1 + q\lambda_g + C_q\lambda_g^q}$, $\theta_5 = \sqrt[q]{1 + q\gamma\lambda_{T_2}\lambda_Q} + \sqrt[q]{C_q}$, $\theta_6 = \sqrt[q]{C_q}\gamma\lambda_{T_1}\lambda_{D_A}$, $\theta_7 = \frac{1}{(\alpha\rho - 1)}$ and $\theta_8 = \frac{1}{(\alpha^{'}\rho^{'} - 1)}$, then there exists $(x, y) \in E_1 \times E_2, z^{'} \in E_1, z^{''} \in E_2, u \in A(x)$ and $v \in B(y)$ satisfying the system of generalized resolvent equations involving XOR-operation (3.1). Moreover, the iterative sequences $\{x_n\}, \{y_n\}, \{z_n^{'}\}, \{z_n^{''}\}, \{u_n\}$ and $\{v_n\}$ generated by Algorithm 4.1 converge strongly to $x, y, z^{'}, z^{''}, u$ and v, respectively.

Proof. From Algorithm 4.1, we have

$$\begin{aligned}
\|z_{n+1}^{'} \oplus z_n^{'}\| &= \|[f(x_n) + \rho S(p(x_n), v_n)] \oplus [f(x_{n-1}) + \rho S(p(x_{n-1}), v_{n-1})]\| \\
&= \|[f(x_n) \oplus f(x_{n-1})] \oplus \rho[S(p(x_n), v_n) \oplus S(p(x_{n-1}), v_{n-1})]\| \\
&\leq \|(x_n - x_{n-1}) + (f(x_n) \oplus f(x_{n-1}))\| \\
&\quad + \|(x_{n-1} - x_n) + \rho[S(p(x_n), v_n) \oplus S(p(x_{n-1}), v_{n-1})]\|.
\end{aligned} \tag{4.18}$$

Using Lipschitz continuity of f with constant λ_f, (iii) of Proposition 2.8 and Lemma 2.2, we have

$$\begin{aligned}
&\|(x_n - x_{n-1}) + (f(x_n) \oplus f(x_{n-1}))\|^q \\
&\quad \leq \|x_n - x_{n-1}\|^q + q\langle f(x_n) \oplus f(x_{n-1}), J_q(x_n - x_{n-1})\rangle \\
&\qquad + C_q\|f(x_n) \oplus f(x_{n-1})\|^q \\
&\quad \leq \|x_n - x_{n-1}\|^q + q\|f(x_n) \oplus f(x_{n-1})\|\|x_n - x_{n-1}\|^{q-1} \\
&\qquad + C_q\|f(x_n) \oplus f(x_{n-1})\|^q \\
&\quad \leq \|x_n - x_{n-1}\|^q + q\|f(x_n) - f(x_{n-1})\|\|x_n - x_{n-1}\|^{q-1} \\
&\qquad + C_q\|f(x_n) - f(x_{n-1})\|^q \\
&\quad \leq \|x_n - x_{n-1}\|^q + q\lambda_f\|x_n - x_{n-1}\|^q + C_q\lambda_f^q\|x_n - x_{n-1}\| \\
&\quad \leq (1 + q\lambda_f + C_q\lambda_f^q)\|x_n - x_{n-1}\|^q,
\end{aligned}$$

which implies that

$$\begin{aligned} \|(x_n - x_{n-1}) + (f(x_n) \oplus f(x_{n-1}))\| &\leq \sqrt[q]{(1 + q\lambda_f + C_q\lambda_f^q)}\|x_n - x_{n-1}\| \\ &= \theta_1\|x_n - x_{n-1}\|, \end{aligned} \tag{4.19}$$

where $\theta_1 = \sqrt[q]{1 + q\lambda_f + C_q\lambda_f^q}$.

As S is Lipschitz continuous in both arguments with constants λ_{S_1} and λ_{S_2}, respectively, p is Lipschitz continuous with constant λ_p, B is D-Lipschitz continuous, using (iii) of Proposition 2.8 and Algorithm 4.1 we obtain $\|S(p(x_n), v_n) \oplus S(p(x_{n-1}), v_{n-1})\|$

$$\begin{aligned} &\leq \|S(p(x_n), v_n) - S(p(x_{n-1}), v_{n-1})\| \\ &= \|S(p(x_n), v_n) + S(p(x_{n-1}), v_n) \\ &\quad -S(p(x_{n-1}), v_n) - S(p(x_{n-1}), v_{n-1})\| \\ &= \|S(p(x_n), v_n) - S(p(x_{n-1}), v_n) \\ &\quad +S(p(x_{n-1}), v_n) - S(p(x_{n-1}), v_{n-1})\| \\ &\leq \|S(p(x_n), v_n) - S(p(x_{n-1}), v_n)\| \\ &\quad +\|S(p(x_{n-1}), v_n) - S(p(x_{n-1}), v_{n-1})\| \\ &\leq \lambda_{S_1}\lambda_p\|x_n - x_{n-1}\| + \lambda_{S_2}\|v_n - v_{n-1}\| \\ &\leq \lambda_{S_1}\lambda_p\|x_n - x_{n-1}\| + \lambda_{S_2}D(B(y_n), B(y_{n-1})) \\ &\leq \lambda_{S_1}\lambda_p\|x_n - x_{n-1}\| + \lambda_{S_2}\lambda_{D_B}\|y_n - y_{n-1}\|. \end{aligned} \tag{4.20}$$

Using (4.20), Lemma 2.2, we obtain
$\|(x_{n-1} - x_n) + \rho[S(p(x_n), v_n) \oplus S(p(x_{n-1}), v_{n-1})]\|^q$

$$\begin{aligned} &\leq \|x_n - x_{n-1}\|^q + \rho q\langle S(p(x_n), v_n) \oplus S(p(x_{n-1}), v_{n-1}), J_q(x_{n-1} - x_n)\rangle \\ &\quad +C_q\rho^q\|S(p(x_n), v_n) \oplus S(p(x_{n-1}), v_{n-1})\|^q \\ &\leq \|x_n - x_{n-1}\|^q + \rho q\|S(p(x_n), v_n) \oplus S(p(x_{n-1}), v_{n-1})\|\|x_n - x_{n-1}\|^{q-1} \\ &\quad +C_q\rho^q\|S(p(x_n), v_n) \oplus S(p(x_{n-1}), v_{n-1})\|^q \\ &\leq \|x_n - x_{n-1}\|^q + \rho q[\lambda_{S_1}\lambda_p\|x_n - x_{n-1}\| \\ &\quad +\lambda_{S_2}\lambda_{D_B}\|y_n - y_{n-1}\|](\|x_n - x_{n-1}\|^{q-1}) + C_q\rho^q[\lambda_{S_1}\lambda_p\|x_n - x_{n-1}\| \\ &\quad +\lambda_{S_2}\lambda_{D_B}\|y_n - y_{n-1}\|]^q \\ &= (1 + \rho q\lambda_{S_1}\lambda_p)\|x_n - x_{n-1}\|^q + \rho q\lambda_{S_2}\lambda_{D_B}\|y_n - y_{n-1}\|\|x_n - x_{n-1}\|^{q-1} \\ &\quad +C_q\rho^q(\lambda_{S_1}\lambda_p\|x_n - x_{n-1}\| + \lambda_{S_2}\lambda_{D_B}\|y_n - y_{n-1}\|)^q \\ &= (\sqrt[q]{1 + \rho q\lambda_{S_1}\lambda_p}\|x_n - x_{n-1}\|)^q + \rho q\lambda_{S_2}\lambda_{D_B}\|y_n - y_{n-1}\|\|x_n - x_{n-1}\|^{q-1} \\ &\quad +(\sqrt[q]{C_q}\rho\lambda_{S_1}\lambda_p\|x_n - x_{n-1}\| + \sqrt[q]{C_q}\rho\lambda_{S_2}\lambda_{D_B}\|y_n - y_{n-1}\|)^q \\ &\leq \Big[(\sqrt[q]{1 + \rho q\lambda_{S_1}\lambda_p} + \sqrt[q]{C_q}\rho\lambda_{S_1}\lambda_p)\|x_n - x_{n-1}\| + \sqrt[q]{C_q}\rho\lambda_{S_2}\lambda_{D_B}\|y_n - y_{n-1}\|\Big]^q \end{aligned}$$

$$= \big[\theta_2\|x_n - x_{n-1}\| + \theta_3\|y_n - y_{n-1}\|\big]^q, \tag{4.21}$$

where $\theta_2 = (\sqrt[q]{1 + \rho q\lambda_{S_1}\lambda_p} + \sqrt[q]{C_q}\rho\lambda_{S_1}\lambda_p)$ *and* $\theta_3 = \sqrt[q]{C_q}\rho\lambda_{S_2}\lambda_{D_B}$.

Thus from (4.21), it follows that
$\|(x_{n-1}-x_n)+\rho[S(p(x_n),v_n)\oplus S(p(x_{n-1}),v_{n-1})]\|$

$$\leq \theta_2\|x_n-x_{n-1}\|+\theta_3\|y_n-y_{n-1}\|. \tag{4.22}$$

Combining (4.19) and (4.22), (4.18) becomes

$$\|z'_{n+1}\oplus z_n'\|\leq(\theta_1+\theta_2)\|x_n-x_{n-1}\|+\theta_3\|y_n-y_{n-1}\|. \tag{4.23}$$

Since $z'_n\propto z'_{n+1}$, we have

$$\|z'_{n+1}\oplus z'_n\|=\|z'_{n+1}-z'_n\|\leq(\theta_1+\theta_2)\|x_n-x_{n-1}\|+\theta_3\|y_n-y_{n-1}\|. \tag{4.24}$$

Again using Algorithm 4.1, we have

$$\begin{aligned}\|z''_{n+1}\oplus z''_n\| &= \|[g(y_n)+\gamma T(u_n,Q(y_n))]\oplus[g(y_{n-1})+\gamma T(u_{n-1},Q(y_{n-1}))]\|\\ &= \|[g(y_n)\oplus g(y_{n-1})]\oplus\gamma[T(u_n,Q(y_n)\oplus T(u_{n-1},Q(y_{n-1}))]\|\\ &\leq \|(y_n-y_{n-1})+(g(y_n)\oplus g(y_{n-1}))\|\\ &\quad+\|(y_n-y_{n-1})+\gamma[T(u_n,Q(y_n))\oplus T(u_{n-1},Q(y_{n-1}))]\|.\end{aligned} \tag{4.25}$$

Using the Lipschitz continuity of g with constant λ_g and using the same arguments as for (4.19), we have

$$\begin{aligned}\|(y_n-y_{n-1})+(g(y_n)\oplus g(y_{n-1}))\| &= \sqrt[q]{1+q\lambda_g+C_q\lambda_g^q}\|y_n-y_{n-1}\|\\ &= \theta_4\|y_n-y_{n-1}\|.\end{aligned} \tag{4.26}$$

where $\theta_4=\sqrt[q]{1+q\lambda_g+C_q\lambda_g^q}$.
As T is Lipschitz continuous in both arguments with constants λ_{T_1} and λ_{T_2}, respectively, Q is Lipschitz continuous with constant λ_Q, A is D-Lipschitz continuous and using the same arguments as for (4.22), we have
$\|(y_n-y_{n-1})+\gamma[T(u_n,Q(y_n))\oplus T(u_{n-1},Q(y_{n-1}))]\|$

$$\leq\theta_5\|y_n-y_{n-1}\|+\theta_6\|x_n-x_{n-1}\|, \tag{4.27}$$

where $\theta_5=\sqrt[q]{1+q\gamma\lambda_{T_2}\lambda_Q}+\sqrt[q]{C_q}$ and $\theta_6=\sqrt[q]{C_q}\gamma\lambda_{T_1}\lambda_{D_A}$.
Thus,

$$\|z''_{n+1}\oplus z''_n\|\leq(\theta_4+\theta_5)\|y_n-y_{n-1}\|+\theta_6\|x_n-x_{n-1}\|. \tag{4.28}$$

Since $z''_n\propto z''_{n+1}$, we have

$$\|z''_{n+1}\oplus z''_n\|=\|z''_{n+1}-z''_n\|\leq(\theta_4+\theta_5)\|y_n-y_{n-1}\|+\theta_6\|x_n-x_{n-1}\|. \tag{4.29}$$

Combining (4.24) and (4.29), we have

$$\begin{aligned}\|z'_{n+1}\oplus z'_n\|+\|z''_{n+1}\oplus z''_n\| &= \|z'_{n+1}-z'_n\|+\|z''_{n+1}-z''_n\|\\ &\leq (\theta_1+\theta_2)\|x_n-x_{n-1}\|+\theta_3\|y_n-y_{n-1}\|\\ &\quad+(\theta_4+\theta_5)\|y_n-y_{n-1}\|+\theta_6\|x_n-x_{n-1}\|\\ &= (\theta_1+\theta_2+\theta_6)\|x_n-x_{n-1}\|\\ &\quad+(\theta_3+\theta_4+\theta_5)\|y_n-y_{n-1}\|.\end{aligned} \tag{4.30}$$

Using (4.19) and Lipschitz-type continuity of the resolvent operator J^{ρ}_{M}, we have

$$\begin{aligned}\|x_n - x_{n-1}\| &= \|x_n - x_{n-1} + ((f(x_n) \oplus f(x_{n-1})) - (J^{\rho}_{M}(z_n{}') \oplus J^{\rho}_{M}(z^{'}_{n-1})))\| \\ &\leq \|(x_n - x_{n-1}) + (f(x_n) \oplus f(x_{n-1})\| + \|J^{\rho}_{M}(z^{'}_n) \oplus J^{\rho}_{M}(z^{'}_{n-1})\| \\ &\leq \theta_1\|x_n - x_{n-1}\| + \frac{1}{(\alpha\rho - 1)}\|z^{'}_n - z^{'}_{n-1}\| \\ &\leq \theta_1\|x_n - x_{n-1}\| + \theta_7\|z^{'}_n - z^{'}_{n-1}\|,\end{aligned}$$

which implies that

$$\|x_n - x_{n-1}\| \leq \frac{\theta_7}{1-\theta_1}\|z^{'}_n - z^{'}_{n-1}\|, \ \textit{where}\ \theta_7 = \frac{1}{(\alpha\rho - 1)}, \alpha > \frac{1}{\rho}. \tag{4.31}$$

Using (4.26) and Lipschitz-type continuity of the resolvent operator J^{γ}_{N}, we have

$$\begin{aligned}\|y_n - y_{n-1}\| &= \|(y_n - y_{n-1}) + ((g(y_n) \oplus g(y_{n-1})) - (J^{\gamma}_{N}(z_n{}'') \oplus J^{\gamma}_{N}(z^{''}_{n-1})))\| \\ &\leq \|(y_n - y_{n-1}) + (g(y_n) \oplus g(y_{n-1}))\| + \|J^{\gamma}_{N}(z_n{}'') \oplus J^{\gamma}_{N}(z^{''}_{n-1})\| \\ &= \theta_4\|y_n - y_{n-1}\| + \frac{1}{(\alpha^{'}\rho^{'} - 1)}\|z^{''}_n - z^{''}_{n-1}\| \\ &\leq \theta_4\|y_n - y_{n-1}\| + \theta_8\|z^{''}_n - z^{''}_{n-1}\|,\end{aligned}$$

which implies that

$$\|y_n - y_{n-1}\| \leq \frac{\theta_8}{1-\theta_4}\|z^{''}_n - z^{''}_{n-1}\|, \textit{where}\ \theta_8 = \frac{1}{(\alpha^{'}\rho^{'} - 1)}, \alpha^{'} > \frac{1}{\rho^{'}}. \tag{4.32}$$

Combining (4.31), (4.32) with (4.30), we have

$$\begin{aligned}\|z^{'}_{n+1} - z^{'}_n\| + \|z^{''}_{n+1} - z^{''}_n\| &\leq \frac{(\theta_1+\theta_2+\theta_6)\theta_7}{1-\theta_1}\|z^{'}_n - z^{'}_{n-1}\| \\ &\quad + \frac{(\theta_3+\theta_4+\theta_5)\theta_8}{1-\theta_4}\|z^{''}_n - z^{''}_{n-1}\|, \\ &\leq \zeta(\theta)[\|z^{'}_n - z^{'}_{n-1}\| + \|z^{''}_n - z^{''}_{n-1}\|], \qquad (4.33)\end{aligned}$$

where

$$\zeta(\theta) = max\left\{\frac{(\theta_1+\theta_2+\theta_6)\theta_7}{1-\theta_1}, \frac{(\theta_3+\theta_4+\theta_5)\theta_8}{1-\theta_4}\right\}. \tag{4.34}$$

By (4.17), we know that $0 < \zeta(\theta) < 1$, and so (4.33) implies that $\{z^{'}_n\}$ and $\{z^{''}_n\}$ are both cauchy sequences. Thus, there exists $z^{'} \in E_1$ and $z^{''} \in E_2$ such that $z^{'}_n \to z^{'}$ and $z^{''}_n \to z^{''}$ as $n \to \infty$. From (4.31) and (4.32), it follows that $\{x_n\}$ and $\{y_n\}$ are also cauchy sequences in E_1 and E_2, respectively, that is, there exist $x \in E_1$, $y \in E_2$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. □

From Algorithm 4.1 and D-Lipschitz continuity of A and B, we have

$$\begin{aligned}\|u_{n+1} \oplus u_n\| &\leq \|u_{n+1} - u_n\| \leq D(A(x_{n+1}), A(x_n)) \leq \lambda_{D_A}\|x_{n+1} - x_n\|, \\ \|v_{n+1} \oplus v_n\| &\leq \|v_{n+1} - v_n\| \leq D(B(x_{n+1}), B(y_n)) \leq \lambda_{D_B}\|y_{n+1} - y_n\|.\end{aligned}$$

and hence, $\{u_n\}$ *and* $\{v_n\}$ are also Cauchy sequences, let $u_n \to u$ and $v_n \to v$, respectively. By using the techniques of Ahmad and Yao [1], it is easy to show that

$u \in A(x), v \in B(y)$. By continuity of $f, g, p, Q, A, B, s, T, J_M^{\rho}, J_N^{\gamma}$ and Algorithm 4.1, we have

$$\begin{aligned} z' &= f(x) \oplus \rho S(p(x), v) = J_M^{\rho}(z') \oplus \rho S(p(x), v) \in E_1, \\ z'' &= g(y) \oplus \gamma T(u, Q(y)) = J_N^{\gamma}(z'') \oplus \gamma T(u, Q(y)) \in E_2. \end{aligned}$$

By Proposition 4.1, the required result follows.

5. Conclusion

This paper is devoted to the study of a system of generalized resolvent equations involving XOR-operation in q-uniformly smooth Banach spaces with its corresponding system of generalized variational inclusions involving XOR-operation. It is shown that both the problems are equivalent and a fixed point formulation is also established. Some iterative algorithms are suggested and finally an existence and convergence result is proved.

We remark that our results are useful for other researchers of related domain and further can be extended in different directions.

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