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SHORT-TIME BEHAVIOR FOR A CLASS OF SEMILINEAR NONLOCAL EVOLUTION EQUATIONS IN HILBERT SPACES

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Abstract. This paper studies the global solvability and finite-time attractivity of solutions to a class of semilinear evolution equations in Hilbert spaces. Our analysis is based on the theory of integral equations with completely positive kernel, the fixed point theory and local estimates of solutions. An application to semilinear integro-differential equations of parabolic type will be shown.

1. Introduction

During the last two decades, there has been an increasing interest in investigating the qualitative behavior of dynamical systems on a bounded time interval. These studies arise from various problems of applications, where one has to analyze transient behavior of the unknown function in differential systems on compact intervals of time, see the introduction of [4, 9] and references therein.

In this work, we employ the concept of the finite-time attractivity which is recently introduced by Giesl and Rasmussen in [6, 15] as a useful recipe in the control theory, to analyze the behavior at terminal time of solutions to the following problem

$$\frac{d}{dt}\big(k_0 u + k * [u - u(0)]\big)(t) + Au(t) = f\big(u(t)\big), t \in (0, T] \tag{1.1}$$

$$u(0) = u_0, \tag{1.2}$$

where the state function $u(\cdot)$ takes values in a separable Hilbert space H, A is a linear operator on H, $f : H \to H$ is a nonlinear function. Here $k * v$, for $v \in L^1_{loc}(\mathbb{R}^+; H)$, denotes the Laplace convolution, i.e., $\big(k * v\big)(t) = \int_0^t k(t-s)v(s)ds$.

Nonlocal differential equations like (1.1) naturally appear in a number of contexts, particularly in the heat transfer processes in memory materials [5] (see also [13] and the references therein) and in the homogenization of an one-phase flow model in a fissured porous medium [1, 8]. In the linear case (f is independent of u), (1.1)-(1.2) was considered in [2, 3, 8] with specific settings, where the authors dealt with the well-posedness.

In order to examine (1.1)-(1.2), we make the following standing assumptions.

- (Hk) $k_0 > 0$ and the kernel $k \in L^1_{loc}(\mathbb{R}^+)$ is a nonnegative and nonincreasing function.
- (Ha) $A : D(A) \to H$ is densely defined, self-adjoint and positively definite operator with a compact resolvent.

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Let ℓ be the solution of the integral equation

$$k_0\ell(t) + \int_0^t k(t-\tau)\ell(\tau)d\tau = 1. \tag{1.3}$$

The hypothesis (Hk) ensures that ℓ is completely positive, according to Clément and Nohel in [5] (see the definition in Section 2). Using (1.3), one can transform (1.1) to an abstract Volterra equation with completely positive kernel. Such equations occur in various applications and have a long history. We refer the readers to monographs of Gripenberg et al. [7] and of Prüss [13] for a full discussion of the Volterra equations in finite dimensional spaces as well as in arbitrary Banach spaces.

Let us now turn to a brief discussion on some related results. In the case $k_0 = 0$ and the kernel $k(t) = g_{1-\alpha}(t) = t^{-\alpha}/\Gamma(1-\alpha), \alpha \in (0,1)$, we proved the finite-time attractivity for problem (1.1)-(1.2) in [9] by using the fractional resolvent theory, a singular Gronwall inequality and local estimates of solutions. The question of finite-time attractivity was also addressed in [11] for a class of tempered fractional equations. Very recently, in [10], the authors established a representation for solutions to problem (1.1)-(1.2) in the case $k_0 = 0$ and derived some regularity and stability results. In the present work, we find some appropriate conditions on k and f ensuring the finite-time attractivity of solutions to (1.1)-(1.2) in the case $k_0 > 0$. As a consequence, we prove the existence of periodic/anti-periodic solution of (1.1), i.e., the solutions satisfying $u(0) = \pm u(T)$.

Our work is arranged as follows. In the next section, using the approach developed in [10], we derive the concept of mild solutions for inhomogeneous problems. In Section 3, we prove the global solvability of problem (1.1)-(1.2) on $[0,T]$ under the assumption that the nonlinearity f is globally/locally Lipschitzian. Section 4 is devoted to showing our main results on the finite-time attractivity and their consequences. In the last section, the obtained results will be demonstrated in a class of partial differential equations.

2. Preliminaries

In this section, we present some preliminary materials which will be used in the sequel. Let the hypothesis (Hk) hold. For each $\mu > 0$, consider the scalar integral Volterra equations

$$s_\mu(t) + \mu\big(\ell * s_\mu\big)(t) = 1, \quad t \geq 0, \tag{2.1}$$

and

$$r_\mu(t) + \mu\big(\ell * r_\mu\big)(t) = \ell(t), \quad t > 0. \tag{2.2}$$

where ℓ is the unique solution of the integral equation

$$k_0\ell + k * \ell = 1 \text{ on } \mathbb{R}^+. \tag{2.3}$$

It is well known (see, e.g. [5, 12]), that ℓ is absolutely continuous and nonnegative function on $[0,T]$ and therefore the equations (2.1) and (2.2) are uniquely solved.

Recall that the kernel function ℓ is completely positive iff $s_\mu(\cdot), r_\mu(\cdot)$ are nonnegative for every $\mu > 0$. Due to [5, Theorem 2.2], the relation (2.3) implies that the kernel function ℓ is completely positive. Some important properties of s_μ and r_μ are gathered in the following proposition, see [5, 13, 17, 18] for more details.

Proposition 2.1. *Assume hypothesis* (Hk)*, then the following claims hold.*

i) *For each $\mu > 0$, s_μ and r_μ belong to $L^1_{loc}(\mathbb{R}^+)$. Moreover $s_\mu \in H^{1,1}_{loc}(\mathbb{R}^+)$, s_μ is nonincreasing and*

$$s_\mu(t) \le \frac{1}{1+\mu(1*\ell)(t)}, \text{ for all } t \ge 0. \tag{2.4}$$

ii) $\mu\big(1*r_\mu\big)(t) = 1 - s_\mu(t), t \ge 0$ *and* $\frac{d}{dt}s_\mu(t) = -\mu r_\mu(t)$ *for a.e.* $t > 0$.

iii) *For each $t > 0$, the mappings*

$$\mu \mapsto s_\mu(t),\ \mu \mapsto r_\mu(t)$$

are nonincreasing.

Proof. The proofs of the assertion i) and ii) can be found in [5, Section 2] or [13, Proposition 4.5]. For the proof of assertion iii) we refer to [10, Proposition 2.1] or [14, Lemm 5.1, Lemma 5.3]. □

For $\mu > 0$, we consider the scalar Volterra equation

$$\frac{d}{dt}\big(k_0 v + k*[v-v_0]\big)(t) = -\mu v(t) + g(t). \tag{2.5}$$

Integrating equation (2.5) over $[0,t]$, and convoluting with the kernel ℓ, one has

$$k_0\ell*[v-v_0] + \ell*k*[v-v_0] = -\mu\big(1*\ell*v\big) + 1*\ell*g.$$

Taking into account (2.3), we get

$$v = v_0 - \mu(\ell*v) + \ell*g. \tag{2.6}$$

In view of properties of $s_\mu(\cdot)$ and $r_\mu(\cdot)$ stated in Proposition 2.1, the Laplace transform of these functions are well defined and given by

$$\widehat{s_\mu}(\lambda) = \frac{1}{\lambda(1+\mu\widehat{\ell}(\lambda))},\ \widehat{r_\mu}(\lambda) = \frac{\widehat{\ell}(\lambda)}{1+\mu\widehat{\ell}(\lambda)}.$$

Applying the Laplace transform to both sides of the equation (2.6), we have

$$\widehat{v} = v_0\lambda^{-1} - \mu\widehat{\ell}\widehat{v} + \widehat{\ell}\widehat{g}. \tag{2.7}$$

It follows that

$$\widehat{v} = \widehat{s_\mu}v_0 + \widehat{r_\mu}\widehat{g}.$$

From the last relation, one finds that the solution of equation (2.5) is

$$v(t) = s_\mu(t)v_0 + \big(r_\mu*g\big)(t).$$

By this observation, we get the following Gronwall type inequality, whose proof can be found in [10, Proposition 2.2].

Lemma 2.2. *Let $\mu > 0$ and let $v : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying the integral inequality*

$$v(t) \le s_\mu(t)v_0 + \int_0^t r_\mu(t-s)\big(a\,v(s) + b(s)\big)ds,$$

where $0 < a < \mu$, $v_0 \geq 0$ *and* $b \in L^1_{loc}(\mathbb{R}^+)$. *Then*

$$v(t) \leq s_{\mu-a}(t)v_0 + \int_0^t r_{\mu-a}(t-s)b(s)\,ds. \tag{2.8}$$

In particular, if b *is constant then*

$$v(t) \leq s_{\mu-a}(t)v_0 + \frac{b}{\mu-a}\big(1 - s_{\mu-a}(t)\big).$$

Consider the linear problem

$$\frac{d}{dt}\big(k_0 u + k * [u - u_0]\big)(t) + Au(t) = f(t), t \in (0,T]. \tag{2.9}$$

$$u(0) = u_0 \tag{2.10}$$

By the assumption (Ha), there exists a nondecreasing sequence $\{\lambda_n\}_{n=1}^\infty$,

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots, \lambda_n \to +\infty \text{ as } n \to +\infty,$$

and a system of vectors $\{e_n\}_{n=1}^\infty \subset D(A)$, which forms an orthonormal basis of H such that $Ae_n = \lambda_n e_n$, for all $n \in \mathbb{N}$. In the sequel, we will use the notations $(\cdot,\cdot)$ and $\|\cdot\|$ for the inner product and the norm in H, respectively.

For $s \in \mathbb{R}$, one can define the fractional power operator A^s of A as follows

$$A^s z := \sum_{n=1}^\infty \lambda_n^s\,(z, e_n)\,e_n, z \in V_s := D(A^s) = \left\{ z \in H : \sum_{n=1}^\infty \lambda_n^{2s}\,|(z,e_n)|^2 < \infty \right\}.$$

It should be noted that V_s is a Banach space with the norm

$$\|z\|_{V_s} = \left(\sum_{n=1}^\infty \lambda_n^{2s}\,|(z,e_n)|^2 \right)^{\frac{1}{2}}, \; z \in D(A^s).$$

We can identify $V_{-s} = D(A^{-s})$ with V_s^*, the dual space of V_s. Then V_{-s} is a Banach space with the norm

$$\|h\|_{V_{-s}} = \left(\sum_{n=1}^\infty \lambda_n^{-2s}\,|\langle h, e_n\rangle|^2 \right)^{\frac{1}{2}},$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between V_{-s} and V_s. Identifying H with its dual H^*, the following relations hold for all $s \geq 0$:

$$V_s \subset H \simeq H^* \subset V_{-s}.$$

It is worth noting that $\langle f, z\rangle = (f,z)$ for all $f \in H, z \in V_s$.

We are in a position to give a representation for the mild solutions in terms of the functions $s_{\lambda_n}(\cdot), r_{\lambda_n}(\cdot)$. We note that equation (2.9) is equivalent to the Volterra equation

$$u + \ell * Au = u_0 + \ell * f. \tag{2.11}$$

Let us define the following family of operators

$$S(t)z = \sum_{n=1}^{\infty} s_{\lambda_n}(t) z_n e_n, t \geq 0, \tag{2.12}$$

$$R(t)z = \sum_{n=1}^{\infty} r_{\lambda_n}(t) z_n e_n, t > 0, z \in H, z_n = (z, e_n). \tag{2.13}$$

It is easily seen that, $\{S(t)\}_{t\geq 0}$ and $\{R(t)\}_{t>0}$ are the bounded linear operators on H. In addition, by using Proposition 2.1(3), one has

$$\|S(t)z\| \leq s_{\lambda_1}(t)\|z\|, \ t \geq 0, z \in H, \tag{2.14}$$

$$\|R(t)z\| \leq r_{\lambda_1}(t)\|z\|, \ t > 0, z \in H. \tag{2.15}$$

Moreover, by reasoning as in [10], for any $f \in C([0,T], H)$, we have $R * f \in C([0,T]; H)$ and the following estimate holds

$$\|(R * f)(t)\| \leq \int_0^t r_{\lambda_1}(t-s)\|f(s)\| \, ds \tag{2.16}$$

here

$$(R * f)(t) = \int_0^t R(t-s) f(s) ds = \sum_{n=1}^{\infty} \big(r_{\lambda_n} * f_n\big)(t) e_n, \ f_n(t) = (f(t), e_n).$$

We have the concept of mild solution to the problem (2.9)-(2.10) as follows.

Definition 2.3. A function $u \in C([0,T]; H)$ is called a mild solution of (2.9)-(2.10) on $[0,T]$ with initial datum u_0 iff

$$u(t) = S(t)u_0 + \int_0^t R(t-s) f(s) \, ds,$$

for any $t \in [0,T]$.

It should be noted that, one can follows the arguments as in [10] to show that, a mild solution u of (2.9)-(2.10) is a weak solution of this problem, i.e., $u \in C([0,T]; H) \cap C((0,T]; V_{\frac{1}{2}})$, $u(0) = u_0$, and u obeys (2.9) in the dual space $V_{-\frac{1}{2}}$.

3. Global solvability

Based on the linear case, we introduce the notion of mild solutions to (1.1)-(1.2) as the following.

Definition 3.1. A function $u \in C([0,T]; H)$ is called a mild solution of problem (1.1)-(1.2) on the interval $[0,T]$ iff

$$u(t) = S(t)u_0 + \int_0^t R(t-s) f\big(u(s)\big) \, ds,$$

for any $t \in [0,T]$.

Let $\Phi : C([0,T];H) \to C([0,T];H)$ be the operator given by

$$\Phi(u)(t) = S(t)u_0 + \int_0^t R(t-s)f\big(u(s)\big)\,ds,\ t \in [0,T]. \tag{3.1}$$

Then a function $u \in C([0,T];H)$ is a mild solution to (1.1)-(1.2) iff it is a fixed point of Φ. As far as the nonlinearity f is concerned, we assume that

(Hf) The nonlinear function $f : H \to H$ is locally Lipschitz, that is

$$\|f(u_1) - f(u_2)\| \le L(r)\|u_1 - u_2\|, \forall u_1, u_2 \in B_r,$$

where B_r is the closed ball centered at origin with radius r in $C([0,T];H)$ and $L(r)$ is a positive number depending on r .

We analyze the first case for global solvability in the following theorem.

Theorem 3.2. *Let the hypotheses* $(Hk), (Ha)$ *and* (Hf) *hold. If* $f(0) = 0$ *and* $\alpha = \limsup_{r\to 0} L(r) < \lambda_1$*, then there exist* $r > 0$ *and* $\delta > 0$ *such that the problem* (1.1)-(1.2) *admits a unique global mild solution* $u \in B_r$ *provided that* $\|u_0\| \le \delta$.

Proof. By assumption on the behaviour of f, for $\theta \in (0, \lambda_1 - \alpha)$, there exists $r^* > 0$ such that, for any $r \in (0, r^*)$ and $\|v\| \le r$, we get

$$\|f(v)\| = \|f(v) - f(0)\| \le L(r)\|v\| \le (\alpha + \theta)\|v\|.$$

Now we consider the solution map $\Phi : B_r \to C([0,T];H)$ defined by (3.1). We see that

$$\begin{aligned}
\|\Phi(u)(t)\| &\le s_{\lambda_1}(t)\|u_0\| + \int_0^t r_{\lambda_1}(t-\tau)(\alpha+\theta)\|u(\tau)\|d\tau \\
&\le s_{\lambda_1}(t)\|u_0\| + (\alpha+\theta)r\lambda_1^{-1}(1 - s_{\lambda_1}(t)) \\
&\le s_{\lambda_1}(t)[\|u_0\| - (\alpha+\theta)\lambda_1^{-1}r] + (\alpha+\theta)\lambda_1^{-1}r \\
&\le r,\ t \in [0,T],
\end{aligned}$$

provided that $\|u_0\| \le \alpha\lambda_1^{-1}r$, thanks to the fact that $(\alpha+\theta)\lambda_1^{-1} < 1$. Fixing an θ and r mentioned above, for $\delta = \alpha\lambda_1^{-1}r$, we have shown that $\Phi(B_r) \subset B_r$ as $\|u_0\| \le \delta$. We now show that $\Phi : B_r \to B_r$ is a contraction mapping. Indeed, we observe that, for $u_1, u_2 \in B_r$,

$$\begin{aligned}
\|\Phi(u_1)(t) - \Phi(u_2)(t)\| &\le \int_0^t r_{\lambda_1}(t-\tau)\|f(u_1(\tau)) - f(u_2(\tau))\|d\tau \\
&\le \int_0^t r_{\lambda_1}(t-\tau)L(r)\|u_1(\tau) - u_2(\tau)\|d\tau \\
&\le \int_0^t r_{\lambda_1}(t-\tau)(\alpha+\theta)\|u_1(\tau) - u_2(\tau)\|d\tau \\
&\le (\alpha+\theta)\lambda_1^{-1}(1 - s_{\lambda_1}(t))\|u_1 - u_2\|_\infty,\ \forall t \in [0,T],
\end{aligned}$$

thanks to Proposition 2.1(ii), here $\|\cdot\|_\infty$ denotes the sup-norm in $C([0,T];H)$. Hence

$$\|\Phi(u_1) - \Phi(u_2)\|_\infty \le (\alpha+\theta)\lambda_1^{-1}\|u_1 - u_2\|_\infty,$$

which completes the proof. $\square$

In the special case the nonlinear function f is globally Lipschitz, we also obtain the global solvability of the problem (1.1)-(1.2) without any restriction on the initial data.

Theorem 3.3. *Let the hypotheses* $(Hk), (Ha)$ *and* (Hf) *hold with* $L(r)$ *being a constant. Then the problem* (1.1)-(1.2) *has a unique global mild solution.*

The proof is similar to the one used in [10, Theorem 4.2].

4. Finite-time attractivity

This section presents our main result on finite-time attractivity of solutions to (1.1).

Definition 4.1. (Finite-time attractivity). Let $u^*(\cdot, u_0)$ be the solution of equation (1.1) corresponding to the initial datum u_0.

(i) u^* is called attractive on $[0, T]$ if there exists an $\eta > 0$ such that

$$\|u(T, \xi) - u^*(T, u_0)\| < \|\xi - u_0\|, \tag{4.1}$$

for all $\xi \in B_\eta(u_0)\backslash\{u_0\}$ and $u(\cdot, \xi)$ being the solution of (1.1) with respect to initial datum ξ.

(ii) u^* is called exponentially attractive on $[0, T]$ if

$$\limsup_{\eta \searrow 0} \frac{1}{\eta} \sup_{\xi \in B_\eta(u_0)} \|u(T, \xi) - u^*(T, u_0)\| < 1, \tag{4.2}$$

where $u(\cdot, \xi)$ is the solution of (1.1) with respect to initial datum ξ.

One can easily verify from the definition that exponential attractivity implies attractivity. The following lemma gives a sufficient condition for exponential attractivity, whose proof can be found in [9, Lemma 3.1].

Lemma 4.2. *Let* $u^*(\cdot, u_0) \in C([0, T]; X)$ *be a solution of* (1.1)*. Then* u^* *is exponentially attractive on* $[0, T]$*, provided that*

$$\limsup_{\|\xi\| \to 0} \frac{\|u(T, u_0 + \xi) - u^*(T, u_0)\|}{\|\xi\|} < 1, \tag{4.3}$$

where $u(\cdot, u_0 + \xi)$ *is the solution of* (1.1) *with respect to initial datum* $u_0 + \xi$*.*

The main result in this section is the following.

Theorem 4.3. *Let the assumptions of Theorem 3.2 hold. Then there exists* $\delta > 0$ *such that, every solution* u *of* (1.1) *with* $\|u(0)\| \leq \delta$ *is exponentially attractive on* $[0, T]$*.*

Proof. Let r, θ and δ be chosen as in the proof of Theorem 3.2, where one has

$$L(r) \leq \alpha + \theta < \lambda_1.$$

Fixed $\xi^* \in B_\delta$ and $u^*(t) = u^*(t, \xi^*)$, we will show the attractivity for u^*. For $\xi \in B_\delta$ and $u(t) = u(t, \xi)$, put

$$\tilde{\xi} = \xi - \xi^*, \ \tilde{u}(t) = u(t) - u^*(t), \ t \in [0, T].$$

Then we see that

$$\begin{aligned}\|\tilde{u}(t)\| &\le s_{\lambda_1}(t)\|\tilde{\xi}\| + \int_0^t r_{\lambda_1}(t-s)\|f\big(u(s)\big) - f\big(u^*(s)\big)\|\,ds \\ &\le s_{\lambda_1}(t)\|\tilde{\xi}\| + \int_0^t r_{\lambda_1}(t-s)L(r)\|\tilde{u}(s)\|\,ds \\ &\le s_{\lambda_1}(t)\|\tilde{\xi}\| + \int_0^t r_{\lambda_1}(t-s)(\alpha+\theta)\|\tilde{u}(s)\|\,ds.\end{aligned}$$

Using Lemma 2.2 again, the above relation implies

$$\|\tilde{u}(t)\| \le s_{\lambda_1-\alpha-\theta}(t)\|\tilde{\xi}\|, \forall t \in [0,T].$$

Thus

$$\limsup_{\|\tilde{\xi}\|\to 0} \frac{\|\tilde{u}(T)\|}{\|\tilde{\xi}\|} < 1,$$

thanks to the fact that $s_\mu(T) < 1$ for every $\mu > 0$. Equivalently,

$$\limsup_{\|\tilde{\xi}\|\to 0} \frac{\|u(T,\xi) - u^*(T,\xi^*)\|}{\|\tilde{\xi}\|} < 1.$$

The proof is complete. □

In the case the nonlinear function f is globally Lipschitzian, we get the following result.

Theorem 4.4. *If the assumptions of Theorem 3.3 hold, then every solution of* (1.1) *is exponentially attractive on* $[0,T]$*, provided that* $L < \lambda_1$.

Proof. The arguments are similar to those in the proof of Theorem 4.3. Let $u^* = u(\cdot,\xi^*)$ and $u = u(\cdot,\xi)$ be solutions of (1.1). Then

$$\begin{aligned}\|u(t) - u^*(t)\| &\le s_{\lambda_1}(t)\|\xi - \xi^*\| + \int_0^t r_{\lambda_1}(t-s)\|f(u(s)) - f(u^*(s))\|ds \\ &\le s_{\lambda_1}(t)\|\xi - \xi^*\| + \int_0^t r_{\lambda_1}(t-s)L\|u(s) - u^*(s)\|ds,\ t \in [0,T].\end{aligned}$$

Employing Lemma 2.2, we get

$$\|u(t) - u^*(t)\| \le s_{\lambda_1-L}(t)\|\xi - \xi^*\|,\ \forall t \in [0,T],$$

which implies the desired conclusion. □

Remark 4.5. We call the set $\mathcal{B} = \{\xi \in H : u(\cdot,\xi)$ is attractive on $[0,T]\}$, where $u(\cdot,\xi)$ is the solution of (1.1) with respect to the initial datum ξ, the basin of attraction for (1.1). Obviously, the basin of attraction for (1.1) under the assumption of Theorem 4.4 is the whole space. By the setting in Theorem 4.3, we know that $\mathcal{B} \supset B_\delta$, with δ small enough. However, the question of determining $\mathcal{B}$ in this case is open.

In the rest of section, by the same conditions ensuring the attractivity, we prove the solvability result of the following problem

$$\frac{d}{dt}\big(k_0 u + k * [u - u(0)]\big)(t) + Au(t) = f\big(u(t)\big), t \in (0,T] \tag{4.4}$$

$$u(0) = g(u), \tag{4.5}$$

where $g : C([0,T]; H) \to H$ satisfies the following assumption.

(Hg) There exists $\tau \in (0,T]$ such that

$$\|g(u_1) - g(u_2)\| \le \sup_{s \in [\tau,T]} \|u_1(s) - u_2(s)\|, \forall u_1, u_2 \in C([0,T]; H).$$

It should be mentioned that assumption (Hg) is satisfied for the following typical cases:

i) $g(u) = u(T)$ (periodic condition)
ii) $g(u) = -u(T)$ (anti-periodic condition)
iii) $g(u) = \sum_{i=1}^{k} \beta_i u(\tau_i)$ where $\sum_{i=1}^{k} |\beta_i| \le 1$ and $0 < \tau_1 < \tau_2 < \cdots < \tau_k \le T$ (multi-point boundary condition).

By mild solution to problem (4.4)-(4.5), we mean a function $u \in C([0,T]; H)$ satisfying

$$u(t) = S(t)g(u) + \int_0^t R(t-s) f\big(u(s)\big)\, ds, \forall t \in [0,T].$$

We are now in a position to prove the solvability of problem (4.4)-(4.5).

Theorem 4.6. *Let the hypotheses of Theorem 3.3 and (*Hg*) hold. Then the problem* (4.4)-(4.5) *has a mild solution.*

Proof. Define the operator

$$J : C([0,T]; H) \to C([0,T]; H)$$
$$v \mapsto J(v) = u$$

where u is the unique solution of the following Cauchy problem

$$\frac{d}{dt}\big(k_0 u + k * [u - u(0)]\big)(t) + Au(t) = f\big(u(t)\big), t \in (0,T]$$
$$u(0) = g(v).$$

It suffices to prove the existence of a fixed point for J. Let $v_1, v_2 \in C([0,T]; H)$ and $u_1 = J(v_1), u_2 = J(v_2)$. Exploiting the estimates as in the proof of Theorem 4.4 and the hypothesis (Hg) we get

$$\begin{aligned}
\|Jv_1(t) - Jv_2(t)\| &= \|u_1(t) - u_2(t)\| \\
&\le s_{\lambda_1 - L}(t)\|u_1(0) - u_2(0)\| \\
&= s_{\lambda_1 - L}(t)\|g(v_1) - g(v_2)\| \\
&\le s_{\lambda_1 - L}(t) \sup_{s \in [\tau,T]} \|v_1(s) - v_2(s)\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|J^2v_1(t)-J^2v_2(t)\| &\le s_{\lambda_1-L}(t)\sup_{[\tau,T]}\|Jv_1(t)-Jv_2(t)\|\\
&\le \sup_{[\tau,T]}\|Jv_1(t)-Jv_2(t)\|\\
&\le \sup_{[\tau,T]}\|v_1(s)-v_2(s)\|\sup_{[\tau,T]}s_{\lambda_1-L}(t)\\
&\le s_{\lambda_1-L}(\tau)\|v_1-v_2\|, \qquad (4.6)
\end{aligned}$$

thanks to the fact that $s_\mu(\cdot)$ is nonincreasing for any $\mu>0$.

Inequality (4.6) guarantees that J^2 is a contraction mapping on $C([0,T];H)$. Let $\bar{v}\in C([0,T];H)$ be such that $J^2(\bar{v})=\bar{v}$ and put $\bar{u}=J(\bar{v})$. It is easily seen that

$$J^2(\bar{u})=J^3(\bar{v})=J(\bar{v})=\bar{u}.$$

This turns out that $\bar{u}$ coincides with $\bar{v}$ and it is the unique fixed point of J. The proof is complete. □

5. An application

Consider the following nonlinear integrodifferential equation

$$\partial_t u(x,t)+\partial_t^\alpha u(x,t)=\partial_x^2 u(x,t)+h\Big(\int_0^1 u^2(x,t)dx\Big)u(x,t), \alpha\in(0,1), \qquad (5.1)$$

for $x\in(0,1), t\in(0,T]$, subject to the boundary condition

$$u(t,0)=u(t,1)=0, \qquad (5.2)$$

and the initial condition

$$u(0,x)=\xi(x),\ x\in[0,1]. \qquad (5.3)$$

In the above model, ∂_t^α stands for the Caputo fractional derivative of order α, ∂_x denotes the generalized derivative in variable x. In this case the kernel k is given by $k(t)=g_{1-\alpha}(t)$.

Let $H=L^2(0,1)$. The inner product and the norm in H are given by

$$(u,v)=\int_0^1 u(x)v(x)dx, \qquad ||v||=\Big(\int_0^1|v(x)|^2dx\Big)^{\frac{1}{2}}.$$

Let $A=-\partial_x^2$ with the domain $D(A)=H^2(0,1)\cap H_0^1(0,1)$. It is known that A is a densely defined, self-adjoint and positive operator with domain $D(A)$ and has a compact resolvent on H (see, e.g [16, Proposition 3.5.1]). Moreover, the eigenvalues of A consists of $\lambda_n=n^2\pi^2, n=1,2,\ldots$, with corresponding eigenvectors $e_n=\sqrt{2}\sin(nx), n\ge 1$, which form an orthonormal basis in H. So the hypothesis (Ha) is verified.

Let

$$f(v)(x)=h\Big(\int_0^1 v^2(x)\,dx\Big)v(x), v\in L^2(0,1).$$

Clearly, the problem (5.1)-(5.3) is a model of (1.1)-(1.2) with $k_0 = 1$. A simple computation shows that $(-1)^n k^{(n)}(t) \ge 0, \forall n \in \mathbb{N}_0, t > 0$ and hence the kernel k is completely monotonic. Consequently, the hypothesis (Hk) is satisfied.

Regarding the nonlinearity in equation (5.1), we assume that the function h belongs to $C^1\big(\mathbb{R}\big)$ and $|h(r)| \le a + b|r|^\beta$, for some nonnegative constants a, b, β. One can check that

- f maps $L^2(0,1)$ into itself since for all $v \in L^2(0,1)$

$$\begin{aligned}\|f(v)\| &= h\bigg(\int_0^1 v^2(x)\,dx\bigg)\bigg(\int_0^1 v^2(x)\,dx\bigg)^{1/2}\\ &= h(\|v\|^2)\|v\| \le (a + b\|v\|^{2\beta})\|v\|.\end{aligned}$$

- For all $v_1, v_2 \in L^2(0,1)$ such that $\|v_1\|, \|v_2\| \le r$, by the mean value theorem, one has

$$\begin{aligned}\|f(v_1) - f(v_2)\| &\le |h(\|v_1\|^2) - h(\|v_2\|^2)|\|v_1\| + h(\|v_2\|^2)\|v_1 - v_2\|\\ &\le r|\|v_1\|^2 - \|v_2\|^2||h'\big(\theta\|v_1\|^2 + (1-\theta)\|v_2\|^2\big) + h(\|v_2\|^2)\|v_1 - v_2\|\\ &\le \big(2r^2 \sup_{z\in[0,r^2]} |h'(z)| + a + br^{2\beta}\big)\|v_1 - v_2\|.\end{aligned}$$

Hence, the hypothesis (Hf) is fulfilled with $L(r) = 2r^2 \sup_{z\in[0,r^2]} |h'(z)| + a + br^{2\beta}$. Obviously, $\lim\limits_{r\to 0} L(r) = a$. Therefore, if $a < \pi^2$, then every solution of (5.1)-(5.3) with ξ small enough is attractive on $[0,T]$.

References

[1] B. Amaziane, L. Pankratov and A. Piatnitski, *Homogenization of a single phase flow through a porous medium in a thin layer*, Math. Models Methods Appl. Sci. **17** (2007), 1317–1349.

[2] A. Ashyralyev, *Well-posedness of the Basset problem in spaces of smooth functions*, Appl. Math. Lett. **24** (2011), 1176–1180.

[3] E. Bazhlekova and I. Dimovski, *Exact solution of two-term time-fractional Thornley's problem by operational method*, Integral Transforms Spec. Funct. **25** (2014), 61–74.

[4] A. Berger, D.T. Son and S. Siegmund, *Nonautonomous finite-time dynamics*, Discrete Contin. Dyn. Syst. Ser. B **9** (2008), 463–492.

[5] P. Clément and J. A. Nohel, *Asymptotic behavior of solutions of nonlinear Volterra equations with completely positive kernels*, SIAM J. Math. Anal. **12** (1981), 514–535.

[6] P. Giesl and M. Rasmussen, *Areas of attraction for nonautonomous differential equations on finite time intervals*, J. Math. Anal. Appl. **390** (2012), 27–46.

[7] G. Gripenberg, S.-O. Londen and O. Staffans, *Volterra Integral and Functional Equations*, Encycl. Math. Appl., vol. 34, Cambridge University Press, Cambridge, 1990.

[8] U. Hornung and R. Showalter, *Diffusion models for fractured media*, J. Math. Anal. Appl. **147** (1990), 69–80.

[9] T. D. Ke and T. V. Tuan, *Finite-time attractivity for semilinear fractional differential equations*, Results Math. **73** (2018): 73.

[10] T. D. Ke, N. N. Thang and L. T. P. Thuy, *Regularity and stability analysis for a class of semilinear nonlocal differential equations in Hilbert spaces*, J. Math. Anal. Appl. **483** (2020): 123655.

[11] T. D. Ke and N. N. Quan, *Finite-time attractivity for semilinear tempered fractional wave equations*, Fract. Calc. Appl. Anal. **21** (2018), 1471–1492.

[12] R. K. Miller, *On Volterra integral equations with nonnegative integrable resolvents*, J. Math. Anal. Appl. **22** (1968), 319–340.

[13] J. Prüss, *Evolutionary Integral Equations and Applications*, Modern Birkhäuser Classics, Birkhäuser/Springer Basel AG, Basel, 1993.

[14] J. C. Pozo and V. Vergara, *Fundamental solutions and decay of fully non-local problems*, Discrete Contin. Dyn. Syst. **39** (2019), 639–666.

[15] M. Rasmussen, Attractivity and Bifurcation for Nonautonomous Dynamical Systems, Lecture Notes in Mathematics 1907. Springer, Berlin, 2007.

[16] M. Tucsnak and G. Weiss, *Observation and Control for Operator Semigroups*, Springer, New York, 2009.

[17] V. Vergara and R. Zacher, *Optimal decay estimates for time-fractional and other nonlocal subdiffusion equations via energy methods*, SIAM J. Math. Anal. **47** (2015), 210–239.

[18] R. Zacher, *Maximal regularity of type L_p for abstract parabolic Volterra equations*, J. Evol. Equ. **5** (2005), 79–103.

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