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A CLASS OF IMPULSIVE HISTORY-DEPENDENT EVOLUTION INCLUSIONS WITH APPLICATIONS

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Abstract. In the note we consider nonlinear first order impulsive evolution inclusions with a history-dependent operator. We provide results on the existence of solution to the Cauchy problem and the compactness of the solution set. Then, a result on existence of optimal solution to a control problem is presented. The control is a distributed one and in the initial condition. An application to a time-dependent semipermeability problem with impulses and history-dependent operator illustrates the abstract results.

1. Introduction

Let $V \subset H \subset V^*$ be an evolution triple of spaces and $I = [0, T]$ with $0 < T < +\infty$. In this paper we study first order evolution subdifferential inclusions with impulses of the form

$$w'(t) + A(w(t)) + (Rw)(t) + M^*\partial J(t, Mw(t)) + N^*\partial\varphi(t, Nw(t)) \ni f(t) \quad \text{a.e. } t \in I \setminus D, \tag{1.1}$$

$$w(0) = w_0, \tag{1.2}$$

$$w(t_i^+) \in w(t_i^-) + G_i(w(t_i^-)) \quad \text{for} \quad i = 1, \dots, m. \tag{1.3}$$

Here, $A \colon V \to V^*$ is a nonlinear operator, ∂J stands for the generalized gradient of a locally Lipschitz function $J(t, \cdot)$, $\partial\varphi$ denotes the convex subdifferential of a convex and lower semicontinuous function $\varphi(t, \cdot)$, $w_0 \in V$, the nonlinear operator R is assumed to be history-dependent. Further, $D = \{t_1, \dots t_m\}$, $G_i \colon H \to 2^H \setminus \{\emptyset\}$ is a multivalued map and $w(t_i^+)$, $w(t_i^-)$ denote the right and left limits of $w(t)$ at $t = t_i$, $i = 1, \dots, m$. The difference $w(t_i^+) - w(t_i^-)$ represents the jump in the state w at time $t = t_i$, where G_i determines the size of the jump at time t_i.

The paper is a continuation of our two previous works [16, 18]. In [16], we have investigated the well-posedness of history-dependent evolution inclusions without impulses. On the other hand, in [18], we have considered impulsive inclusions and derived a result of nonemptiness and compactness of the solution set to a hyperbolic

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(second order in time) hemivariational inequality. The current work is a generalization of the former results in the following aspects. First, we prove existence of solution to the problem (1.1)–(1.3) which is a variational-hemivariational inequality involving both convex and nonconvex potentials φ and J, respectively, as well as history-dependent operators. Second, we provide a new application of the problem (1.1)–(1.3) to a time-dependent semipermeability problem involving both monotone and nonmonotone subdifferential boundary conditions. The main feature of the semipermeability boundary conditions is that they describe behavior of various types of membranes, natural and artificial ones and arise in models of heat conduction, electrostatics, hydraulics and in the description of flow of Bingham's fluids, where the solution represents temperature, electric potential and pressure. These boundary conditions were first examined in [8] in the convex setting, where semipermeability relations were assumed to be monotone and they led to variational inequalities. More generally, nonmonotone semipermeability conditions can be modeled by the Clarke generalized gradient, see, e.g., [9, 17, 24, 25].

The notion of a hemivariational inequality has been introduced in 1980s by Panagiotopoulos [24, 25] to investigate mathematical models in solid mechanics. We refer to [10, 20, 21, 27, 28] for various related results on history-dependent inequality problems, and to a recent monograph [29] for a comprehensive research. Note that inclusion (1.1) has an equivalent formulation as a history-dependent variational-hemivariational inequality

$$\begin{aligned}\langle w'(t) + A(w(t)) + (Rw)(t) - f(t), v - w(t)\rangle \\ + J^0(t, Mw(t); Mv - Mw(t)) + \varphi(t, Nv) - \varphi(t, Nw(t)) \geq 0\end{aligned} \tag{1.4}$$

for all $v \in V$, a.e. $t \in (0, T) \setminus D$. Existence and uniqueness results for a particular form of inequality (1.4), under more restrictive hypotheses, have been recently proved in [23] where $J \equiv 0$. We refer to [14] for optimal control of history-dependent evolution inclusions with applications to frictional contact problems.

The impulsive evolution equations and inclusions for the first order problems have been studied in [12] by using the semigroup approach, in [4] where the impulsive functional differential inclusions were treated, and in [26] where strongly nonlinear impulsive evolution equations and optimal control have been examined in the framework of an evolution triple of spaces. For other works on nonlinear impulsive systems on infinite dimensional spaces, we refer to [1, 2, 11, 13, 18].

The main motivation to study problem (1.1)–(1.3) is to combine two phenomena. On one hand, the impulses naturally introduce the discontinuities and jumps in the problem. On the other hand, the presence of locally Lipschitz potentials allows, in many interesting cases, to include in the system "fully filled jumps", see [24, p.18] and [22, Section 7.4], generated by the set-valued and nonmonotone relations described by the Clarke subgradient. This combination of phenomena in one system represents the main novel feature of the paper. The variational-hemivariational inequalities with impulses have not been studied in the literature. Further, we provide a new application of the existence result and study a semipermeability model for parabolic inclusion with impulses and the subdifferential boundary condition.

The paper is organized as follows. In Section 2 we review some preliminary material on nonlinear analysis. In Section 3 we prove the nonemptiness and compactness of the solution set to the impulsive evolution inclusion. The Bolza type optimal control problem for the impulsive controlled subdifferential inclusion is treated in Section 4. Finally, in Section 5 we apply our results to study a class of semipermeability problems.

2. Preliminary material

This section is devoted to recall notation, definitions and preliminary material which will be needed in the sequel. For more details, we refer to [6, 7, 29].

A triple of spaces (X, H, X^*) is said to be an evolution triple, if X is a separable reflexive Banach space, X^* is its topological dual, H is a separable Hilbert space identified with its dual $H^* \simeq H$, X is embedded continuously in H, denoted $X \subset H$, and densely in H. We denote by $\mathcal{L}(E, F)$ the space of linear and bounded operators from a Banach space E to a Banach space F endowed with the usual norm $\|\cdot\|_{\mathcal{L}(E,F)}$. A Banach space E equipped with the weak topology is denoted by E_w. For a subset S of a Banach space $(E, \|\cdot\|_E)$, we write $\|S\|_E = \sup\{\|s\|_E \mid s \in S\}$.

Let E be a reflexive Banach space. An operator $A\colon E \to E^*$ is called bounded if its maps bounded set into bounded sets. It is weakly continuous if it is continuous from E_w to E^*_w. Let $B\colon (0, T) \times E \to E^*$ be an operator. Then, the operator $\mathcal{B}\colon L^2(0, T; E) \to L^2(0, T; E^*)$ given by $(\mathcal{B}v)(t) = B(t, v(t))$ for $v \in L^2(0, T; E)$ and a.e. $t \in (0, T)$ is the Nemytski operator corresponding to B.

Given a locally Lipschitz function $J\colon E \to \mathbb{R}$ on a Banach space E, we denote by $J^0(u; v)$ the generalized (Clarke) directional derivative of J at the point $u \in E$ in the direction $v \in E$ defined by

$$J^0(u; v) = \limsup_{\lambda \to 0^+,\, w \to u} \frac{J(w + \lambda v) - J(w)}{\lambda}.$$

The generalized gradient of J at $u \in E$ is defined by

$$\partial J(u) = \{\, \xi \in E^* \mid J^0(u; v) \geq \langle \xi, v \rangle \ \text{ for all } \ v \in E \,\}.$$

In this paper, all subgradients are taken with respect to the last variable of a function.

For $I = [0, T]$, $0 \leq \tau_1 < \tau_2 \leq T$ and an evolution triple (V, H, V^*) with the compact embedding $V \subset H$, we define

$$W(\tau_1, \tau_2) = \{v \in L^2(\tau_1, \tau_2; V) \mid v' \in L^2(\tau_1, \tau_2; V^*)\},$$

where the time derivative is understood in the sense of vector-valued distributions. We denote by $\|\cdot\|_V$, $\|\cdot\|_H$ and $\|\cdot\|_{V^*}$ the norms in V, H and V^*, respectively, and by $\langle \cdot, \cdot \rangle_{V^* \times V}$ the duality brackets for the pair (V, V^*). Let $\mathcal{V} = L^2(I; V)$ and $\mathcal{V}^* = L^2(I; V^*)$. The space $W(\tau_1, \tau_2)$ endowed with the norm $\|v\|_{W(\tau_1,\tau_2)} = \|v\|_{L^2(\tau_1,\tau_2;V)} + \|v'\|_{L^2(\tau_1,\tau_2;V^*)}$ becomes a separable, reflexive Banach space. It is known, see, e.g., [7, Proposition 8.4.14], that $W(\tau_1, \tau_2)$ is embedded continuously in $C(\tau_1, \tau_2; H)$ (the space of continuous functions on $[\tau_1, \tau_2]$ with values in H), i.e., every element of $W(\tau_1, \tau_2)$, after a possible modification on a set of measure zero, has a unique continuous representative in $C(\tau_1, \tau_2; H)$. Recall also that the embedding $W(\tau_1, \tau_2)$ into $L^2(\tau_1, \tau_2; H)$ is compact, see [7, Proposition 3.4.14].

Let $D = \{t_1, \dots, t_m\}$ be a finite set of (impulsive) points such that

$$0 = t_0 < t_1 < t_2 < \ldots < t_m < t_{m+1} = T.$$

Given $\sigma_i = (t_i, t_{i+1})$ for $i = 0, 1, \dots, m$, we set

$$PW(I) = \{v\colon I \to V \text{ such that } v|_{\sigma_i} \in W(\sigma_i) \text{ for } i = 0, 1, \dots, m\}$$

which becomes a Banach space with norm $\|v\|_{PW(I)} = \sum_{i=0}^{m} \|v|_{\sigma_i}\|_{W(\sigma_i)}$. We say that a sequence $\{v_n\} \subset PW(I)$ converges weakly in $PW(I)$ to v if $v_n|_{\sigma_i} \to v|_{\sigma_i}$ weakly in $W(\sigma_i)$ for all $i = 0, 1, \dots, m$. We need also the following space of piecewise continuous functions

$$\begin{aligned} PC(I; H) = \{v\colon I \to H \text{ such that } v \text{ is continuous at } t \in I \setminus D, v \text{ is left} \\ \text{continuous at } t \in D, \text{ the right limits } v(t_i^+) \text{ exist for } i = 1, \dots, m\}. \end{aligned}$$

Then, $PC(I; H)$ is a Banach space with the usual supremum norm.

We begin with a result on the unique solvability to an inclusion without impulses. Let X and Y be Banach spaces. We need the following hypotheses.

<u>$H(A)$</u> : $A\colon V \to V^*$ is such that

(i) A is weakly continuous.

(ii) $\|Av\|_{V^*} \le a_0 + a_1\|v\|_V$ for all $v \in V$ with $a_0,\ a_1 \ge 0$.

(iii) A is strongly monotone with constant $m_A > 0$, i.e.,

$$\langle Av_1 - Av_2, v_1 - v_2\rangle_{V^* \times V} \ge m_A\|v_1 - v_2\|_V^2 \quad \text{for all} \quad v_1, v_2 \in V.$$

<u>$H(J)$</u> : $J\colon (0, T) \times X \to \mathbb{R}$ is such that

(i) $J(\cdot, v)$ is measurable on $(0, T)$ for all $v \in X$.

(ii) $J(t, \cdot)$ is locally Lipschitz on X for a.e. $t \in (0, T)$.

(iii) $\|\partial J(t, v)\|_{X^*} \le c_{0J}(t) + c_{1J}\|v\|_X$ for all $v \in X$, a.e. $t \in (0, T)$ with $c_{0J} \in L^2(0, T)$, $c_{0J},\ c_{1J} \ge 0$.

(iv) ∂J is relaxed monotone in the following sense

$$\langle \partial J(t, v_1) - \partial J(t, v_2), v_1 - v_2\rangle_{X^* \times X} \ge -m_J\|v_1 - v_2\|_X^2$$

for all $v_1,\ v_2 \in X$, a.e. $t \in (0, T)$ with $m_J \ge 0$.

<u>$H(\varphi)$</u> : $\varphi\colon (0, T) \times Y \to \mathbb{R}$ is such that

(i) $\varphi(\cdot, u)$ is measurable on $(0, T)$ for all $u \in Y$.

(ii) $\varphi(t, \cdot)$ is convex and lower semicontinuous on Y for a.e. $t \in (0, T)$.

(iii) $\|\partial\varphi(t, u)\|_{Y^*} \le c_{0\varphi}(t) + c_{1\varphi}\|u\|_Y$ for all $u \in Y$, a.e. $t \in (0, T)$ with $c_{0\varphi} \in L^2(0, T)$, $c_{0\varphi},\ c_{1\varphi} \ge 0$.

<u>$H(M, N)$</u> : $M \in \mathcal{L}(V, X)$, $N \in \mathcal{L}(V, Y)$, and their Nemytski operators

$$\mathcal{M}\colon W(0, T) \subset \mathcal{V} \to L^2(0, T; X) \quad \text{and} \quad \mathcal{N}\colon W(0, T) \subset \mathcal{V} \to L^2(0, T; Y)$$

are compact.

$\underline{H(R)}$: $R\colon \mathcal{V} \to \mathcal{V}^*$ is such that

$$\|(Rv_1)(t) - (Rv_2)(t)\|_{V^*} \le c_R \int_0^t \|v_1(s) - v_2(s)\|_V \, ds \tag{2.1}$$

for all v_1, $v_2 \in \mathcal{V}$, a.e. $t \in (0,T)$ with $c_R > 0$, it is weakly continuous, and $R0$ remains in a bounded subset of $\mathcal{V}^*$.

$\underline{(H_0)}$: $f \in \mathcal{V}^*$, $w_0 \in H$.

$\underline{(H_1)}$: $m_A > m_J \|M\|^2_{\mathcal{L}(V,X)}$.

The condition (2.1) defines a history-dependent operator R. This notion has been recently introduced and extensively studied in modeling of contact problems in mechanics. The history-dependent operators in mechanics appear in the constitutive laws and boundary conditions, see [14, 15, 21, 27–29] and the references therein.

We consider the following auxiliary Cauchy problem without impulses which will be used in proving the main results of the paper. Let $0 \le \tau_1 < \tau_2 \le T$. Find $w \in W(\tau_1, \tau_2)$ such that

$$\begin{cases} w'(t) + A(w(t)) + (Rw)(t) + M^*\partial J(t, Mw(t)) \\ \qquad + N^*\partial\varphi(t, Nw(t)) \ni f(t) \ \text{ for a.e. } \ t \in (\tau_1, \tau_2), \\ w(\tau_1) = w_0. \end{cases} \tag{2.2}$$

In order to indicate an interval and an initial value, problem (2.2) is denoted by $\mathcal{P}(\tau_1, \tau_2; w_0)$. The unique solvability of problem (2.2) has been analyzed in [16, Theorem 5] under the hypothesis $w_0 \in V$ and with a more general condition on the operator A. The following results shows that existence and uniqueness holds under more general hypothesis on the initial condition which belongs to H.

Theorem 2.1. *Let $0 \le \tau_1 < \tau_2 \le T$. Under hypotheses $H(A)$, $H(J)$, $H(\varphi)$, $H(M,N)$, $H(R)$, (H_0), and (H_1), we have*

(a) *problem* (2.2) *has a unique solution in $W(\tau_1, \tau_2)$.*

(b) *if $w \in W(\tau_1, \tau_2)$ solves* (2.2)*, then there is a constant $c > 0$ such that*

$$\|w\|_{C(\tau_1,\tau_2;H)} + \|w\|_{W(\tau_1,\tau_2)} \le c\left(1 + \|w_0\|_H + \|f\|_{\mathcal{V}^*} + \|R0\|_{\mathcal{V}^*}\right).$$

(c) *if $f \in \mathcal{V}^*$, $\{f_n\} \subset \mathcal{V}^*$, $f_n \to f$ weakly in $\mathcal{V}^*$, $\{w_0^n\} \subset H$, $w_0^n \to w_0$ weakly in H, and $w \in W(\tau_1, \tau_2)$, $\{w_n\} \subset W(\tau_1, \tau_2)$ are the unique solutions of problem* (2.2) *corresponding to (f, w_0) and $\{(f_n, w_0^n)\}$, respectively, then $w_n \to w$ weakly in $W(\tau_1, \tau_2)$, as $n \to \infty$.*

Proof. (a) The existence of solution to problem (2.2) under the hypothesis $w_0 \in V$ is a consequence of [16, Theorem 5]. Now, suppose that $w_0 \in H$. Then, by the density of V in H, there exists $\{w_0^n\} \subset V$ such that $w_0^n \to w_0$ in H, as $n \to \infty$. We denote by $w_n \in W(\tau_1, \tau_2)$ the unique solution to $\mathcal{P}(\tau_1, \tau_2; w_0^n)$, i.e.,

$$\begin{cases} w_n'(t) + A(w_n(t)) + (Rw_n)(t) + M^*\partial J(t, Mw_n(t)) \\ \qquad + N^*\partial\varphi(t, Nw_n(t)) \ni f(t) \ \text{ for a.e. } \ t \in (\tau_1, \tau_2), \\ w_n(\tau_1) = w_0^n. \end{cases}$$

Exploiting [16, Theorem 7], we deduce that the sequence $\{w_n\}$ converges weakly in $W(\tau_1, \tau_2)$ to an element which is the unique solution to problem (2.2).

(b) The estimate is a consequence of [16, Proposition 6] and the density of the embedding $V \subset H$.

(c) It follows by arguments similar to the ones used in a continuous dependence result of [16, Theorem 7]. $\square$

3. Impulsive inclusion

We pass to the main result on existence of solution to the impulsive evolution inclusion (1.1)–(1.3).

We give a definition of a solution to problem (1.1)–(1.3).

Definition 3.1. A function $w \in PW(I) \subset PC(I;H)$ is called a solution to (1.1)–(1.3) if there exist w^*, $\eta^* \in \mathcal{V}^*$ such that

$$\begin{cases} w'(t) + A(w(t)) + (Rw)(t) + w^*(t) + \eta^*(t) = f(t) \ \text{ a.e. } \ t \in I \setminus D, \\ w(0) = w_0, \\ w(t_i^+) = w(t_i^-) + \xi_i \ \text{ for } \ i = 1, \dots, m \end{cases}$$

with $w^*(t) \in M^*\partial J(t, Mw(t))$, $\eta^*(t) \in N^*\partial\varphi(t, Nw(t))$ for a.e. $t \in I \setminus D$, and $\xi_i \in G_i(w(t_i^-))$ for $i = 1, \dots, m$.

We denote the solution map to problem (1.1)–(1.3) by

$$\mathbb{S} = \{w \in PW(I) \subset PC(I;H) \mid w \text{ is a solution to (1.1)–(1.3)}\}.$$

In what follows we establish properties of this set. We need one more hypothesis.

$\underline{H(G)}$: $G_i \colon H \to 2^H \setminus \{\emptyset\}$ is such that

(i) G_i is a bounded map for $i = 1, \dots, m$.

(ii) G_i has a sequentially closed graph in $H_w \times H_w$ topology for $i = 1, \dots, m$.

The following result for problem (1.1)–(1.3) is the main result of the paper.

Theorem 3.2. *Assume the hypotheses of Theorem 2.1. We have*

(a) *if* $H(G)$(i) *holds, then the solution set* $\mathbb{S}$ *is nonempty.*

(b) *if, in addition,* $H(G)$(ii) *holds, then the solution set* $\mathbb{S}$ *is a weakly compact subset of* $PW(I)$.

Proof. (a) A solution to (1.1)–(1.3) is constructed in several steps. First, we solve the problem on the interval $\sigma_0 = (0, t_1)$, then on the interval $\sigma_1 = (t_1, t_2)$ and so on until the final interval $\sigma_m = (t_m, T)$. More precisely, we proceed in the following way.

(i) First, we consider the problem without impulses $\mathcal{P}(\sigma_0; w_0)$. From Theorem 2.1(a), this problem admits the unique solution $w^{(0)} \in W(\sigma_0) \subset C(\overline{\sigma}_0; H)$. Thus, the left limit $w^{(0)}(t_1^-)$ exists in H, and we define $w^{(0)}(t_1) = w^{(0)}(t_1^-) \in H$. By assumption $H(G)$, $w^{(0)}(t_1^+)$ is defined and it is given by

$$w_1 := w^{(0)}(t_1^+) = w^{(0)}(t_1) + \xi_1, \quad w_1 \in H,$$

where $\xi_1 \in G_1(w^{(0)}(t_1))$.

(ii) Next, we consider the problem without impulses $\mathcal{P}(\sigma_1; w_1)$. Using Theorem 2.1(a), we obtain the unique solution $w^{(1)} \in W(\sigma_1) \subset C(\overline{\sigma}_1; H)$. Analogously as in Step (i), we set

$$w_2 := w^{(1)}(t_2^+) = w^{(1)}(t_2) + \xi_2, \quad w_2 \in H,$$

where $\xi_2 \in G_2(w^{(1)}(t_2))$.

(iii) Further, we continue the process and for $i = 0, 1, \ldots, m$, we obtain $w^{(i)} \in W(\sigma_i) \subset C(\overline{\sigma}_i; H)$ which is the unique solution to problem $\mathcal{P}(\sigma_i; w_i)$, where

$$w_i := w^{(i-1)}(t_i) + \xi_i, \quad w_i \in H,$$

with $\xi_i \in G_i(w^{(i-1)}(t_i))$ for $i = 1, \ldots, m$. Now we define the function $w \colon I \to V$ by

$$w(t) = \begin{cases} w^{(0)}(t), & t \in [0, t_1] \\ w^{(1)}(t), & t \in (t_1, t_2] \\ \ldots & \ldots \\ w^{(m)}(t), & t \in (t_m, T]. \end{cases}$$

It is easy to see that $w \in PW(I) \subset PC(I; H)$ is a solution to (1.1)–(1.3).

(b) Let $\{w_n\} \subset \mathbb{S}$. In the following m steps we find a subsequence of $\{w_n\}$ which converges weakly in $PW(I)$ to an element of $\mathbb{S}$.

(i) Let $w_n^{(0)} := w_n|_{\sigma_0}$. By Theorem 2.1(a), we know that $w_n^{(0)} \in W(\sigma_0) \subset C(\overline{\sigma}_0; H)$ is the unique solution to problem $\mathcal{P}(\sigma_0; w_0)$. Hence, $w_n^{(0)} = w^{(0)}$ for all $n \in \mathbb{N}$, where $w^{(0)}$ solves $\mathcal{P}(\sigma_0; w_0)$, and obviously $w_n^{(0)} \to w^{(0)}$ weakly in $W(\sigma_0)$, as $n \to \infty$. Next, we define $\{w_1^n\} \subset H$ by

$$w_1^n := w^{(0)}(t_1) + \xi_n^1 \ \text{ with } \ \xi_n^1 \in G_1(w^{(0)}(t_1)).$$

Since G_1 is a bounded map, $\{\xi_n^1\}$ is bounded in H uniformly with respect to n. By the reflexivity of H, passing to a subsequence, if necessary, we may assume

$$\xi_{n_k}^1 \to \xi^1 \ \text{ weakly in } H.$$

Using the closedness of the graph of G_1 in the weak topology, we deduce $\xi^1 \in G_1(w^{(0)}(t_1))$. Hence, putting $w_1 := w^{(0)}(t_1) + \xi^1$, we have

$$w_1^{n_k} \to w_1 \ \text{ weakly in } H, \text{ as } k \to \infty. \tag{3.1}$$

Now, we consider the corresponding subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $w_{n_k}(t_1) = w_1^{n_k}$ and denote it again by $\{w_n\}$.

(ii) We define $w_n^{(1)} = w_n|_{\sigma_1}$. Then $w_n^{(1)} \in W(\sigma_1) \subset C(\overline{\sigma}_1; H)$ solves problem without impulses $\mathcal{P}(\sigma_1; w_1^n)$. From Theorem 2.1(c) and (3.1), we deduce $w_n^{(1)} \to w^{(1)}$ weakly in $W(\sigma_1)$ with $w^{(1)} \in W(\sigma_1) \subset C(\overline{\sigma}_1; H)$ being the solution to $\mathcal{P}(\sigma_1; w_1)$. Next, we define $\{w_2^n\} \subset H$ by

$$w_2^n := w_n^{(1)}(t_2) + \xi_n^2 \ \text{ with } \ \xi_n^2 \in G_2(w_n^{(1)}(t_2)).$$

Since $\{w_n^{(1)}\}$ remains in a bounded subset of $W(\sigma_1) \subset C(\overline{\sigma}_1; H)$, see Theorem 2.1(b), the sequence $\{w_n^{(1)}\}$ converges weakly in $W(\sigma_1)$ to $w^{(1)}$. By [19, Lemma 4], we know that $w_n^{(1)}(t) \to w^{(1)}(t)$ weakly in H for all $t \in \overline{\sigma}_1$. Hence $w_n^{(1)}(t_2) \to w^{(1)}(t_2)$ weakly in H. Again, by the boundedness of G_2, $\{\xi_n^2\}$ is bounded in H uniformly with respect to n. Thus, at least for a subsequence, we obtain

$$\xi_{n_k}^2 \to \xi^2 \text{ weakly in } H \text{ with } \xi^2 \in H.$$

Exploiting the closedness of the graph of G_2 in $H_w \times H_w$ topology, we have $\xi^2 \in G_2(w^{(1)}(t_2))$. Now, we set $w_2 := w^{(1)}(t_2) + \xi^2 \in H$ and we obtain $w_2^{n_k} \to w_2$ weakly in H, as $k \to \infty$.

(iii) We continue the process for $i = 0, 1, \dots, m$. We denote by $\{w_n\}$, the subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $w_{n_k}(t_i) = w_i^{n_k}$. In this way, we get a family of functions $\{w^{(i)}\}_{i=0}^m$, where $w^{(i)} \in W(\sigma_i) \subset C(\overline{\sigma}_i; H)$ is the solution to $\mathcal{P}(\sigma_i; w_i)$ for $i = 0, 1, \dots, m$. Similarly, as in part (a) of the proof, these functions allow one to construct the function $w \in PW(I) \subset PC(I; H)$ which satisfies (1.1)–(1.3). Hence $w \in \mathbb{S}$. We also see that the subsequence of $\{w_n\}$ converges weakly in $PW(I)$ to w. The proof is complete. $\square$

4. Optimal control problem

In this section we consider the following Bolza type optimal control problem for the following impulsive controlled subdifferential inclusion

$$\tag{4.1} \begin{cases} w'(t) + A(w(t)) + (Rw)(t) + M^*\partial J(t, Mw(t)) \\ \qquad + N^*\partial\varphi(t, Nw(t)) \ni f(t) + B(t)u_1(t) \text{ for a.e. } t \in I \setminus D, \\ w(0) = w_0 + Cu_2, \\ w(t_i^+) \in w(t_i^-) + G_i(w(t_i^-)) \text{ for } i = 1, \dots, m, \end{cases}$$

where $B(\cdot)$ and C are given operators. The space of controls is represented by $L^2(0, T; Y_1) \times Y_2$, where Y_1 and Y_2 are separable reflexive Banach spaces.

We need the following hypotheses on the operators B and C.

$\underline{H(B, C)}$: $B \in L^\infty(0, T; \mathcal{L}(Y_1, V^*))$, $C \in \mathcal{L}(Y_2, H)$.

Observe that the state of system (4.1) is not uniquely determined for a given control (u_1, u_2). Therefore, the crucial property for optimal control problems for (4.1) is the dependence of the solution set of (4.1) on the control. Let $\mathbb{S}\colon L^2(0, T; Y_1) \times Y_2 \ni (u_1, u_2) \mapsto \mathbb{S}(u_1, u_2) \subset PW(I)$ be the solution map of (4.1), where $\mathbb{S}(u_1, u_2)$ denotes a set of states $w = w(u_1, u_2)$ of the controlled system (4.1).

Lemma 4.1. *Under hypotheses $H(A)$, $H(J)$, $H(\varphi)$, $H(M, N)$, $H(R)$, (H_0), (H_1), $H(G)$, and $H(B, C)$, the solution map*

$$\mathbb{S}\colon L^2(0, T; Y_1) \times Y_2 \to 2^{PW(I)}$$

has a closed graph in $L^2(0, T; Y_1)_w \times Y_{2w} \times PW(I)_w$ topology.

Proof. From Theorem 3.2, it follows that $\mathbb{S}(u_1, u_2) \neq \emptyset$ for all $(u_1, u_2) \in L^2(0, T; Y_1) \times Y_2$. Let $(u_1^n, u_2^n) \in L^2(0, T; Y_1) \times Y_2$, $u_1^n \to u_1$ weakly in $L^2(0, T; Y_1)$, $u_2^n \to u_2$ weakly in Y_2, and $\{w_n\} \subset \mathbb{S}(u_1^n, u_2^n)$ be the solutions to (4.1) corresponding to (u_1^n, u_2^n) such that $w_n \to w$ weakly in $PW(I)$. We show that $w \in \mathbb{S}(u_1, u_2)$.

We observe that $w_n|_{\sigma_0} \to w|_{\sigma_0}$ weakly in $W(\sigma_0)$. As before, by [19, Lemma 4], we infer that $w_n(t) \to w(t)$ weakly in H for all $t \in \overline{\sigma}_0$. It follows from [22, Proposition 1.41] and hypothesis $H(B, C)$ that

$$w_n(0) = w_0 + Cu_2^n \to w_0 + Cu_2 \text{ weakly in } H.$$

Thus, we deduce that w satisfies the initial condition $w(0) = w_0 + Cu_2$.

On the other hand, since the Nemytski operator $\mathcal{B}\colon L^2(0, T; Y_1) \to \mathcal{V}^*$ for B satisfies $\mathcal{B} \in \mathcal{L}(L^2(0, T; Y_1), \mathcal{V}^*)$, it is clear that $\mathcal{B}u_1^n \to \mathcal{B}u_1$ weakly in $\mathcal{V}^*$. Now, it is enough to apply Theorem 2.1(c) with $f_n = f + \mathcal{B}u_1^n$ and $w_0^n = w_0 + Cu_2^n$ in order to obtain that the limit $w \in PW(I)$ satisfies the inclusion in problem (4.1).

Finally, we verify the jump condition in (4.1). We know that

$$w_n(t_i^+) = w_n(t_i^-) + \xi_n^i \text{ with } \xi_n^i \in G_i(w_n(t_i^-))$$

for all $i = 1, \dots, m$. From the weak convergence of $\{w_n\}$ in $W(\sigma_i)$ on each subinterval σ_i, we deduce that $w_n(t_i^-) \to w(t_i^-)$ and $w_n(t_i^+) \to w(t_i^+)$ both weakly in H. Since G_i is a bounded map with a weakly closed graph, we have that $\xi_n^i \to \xi^i$ weakly in H and $\xi^i \in G_i(w(t_i^-))$ at least for a subsequence. Hence, in the limit, we obtain

$$w(t_i^+) = w(t_i^-) + \xi^i \text{ with } \xi^i \in G_i(w(t_i^-))$$

for all $i = 1, \dots, m$. This proves that $w \in \mathbb{S}(u_1, u_2)$. $\square$

The cost criterion $F\colon L^2(0, T; Y_1) \times Y_2 \times PW(I) \to \mathbb{R}$ for the optimal control problem reads as follows

$$F(u_1, u_2, w) = l(w(T), u_2) + \int_0^T L(t, w(t), u_1(t))\, dt \to \inf =: \overline{m}, \tag{4.2}$$

where $w \in \mathbb{S}(u_1, u_2)$, $u_1(t) \in U_1(t) \subset Y_1$ a.e. $t \in (0, T)$, and $u_2 \in U_2 \subset Y_2$.

We need the following hypotheses.

$\underline{H(F)}$: $l\colon H \times Y_2 \to \mathbb{R}$ is lower semicontinuous on $H_w \times Y_{2w}$,
$L\colon (0, T) \times H \times Y_1 \to \mathbb{R}$ is a measurable function such that

(i) $L(t, \cdot, \cdot)$ is lower semicontinuous on $H \times Y_1$, a.e. $t \in (0, T)$,

(ii) $L(t, w, \cdot)$ is convex on Y_1, for all $w \in H$, a.e. $t \in (0, T)$,

(iii) there exist $c_L > 0$ and $\psi \in L^1(0, T)$ such that

$$L(t, w, u) \geq \psi(t) - c_L \left(\|w\|_H + \|u\|_{Y_1}\right)$$

for all $w \in H$, $u \in Y_1$ and a.e. $t \in (0, T)$.

$\underline{H(U_1)}$: $U_1\colon (0, T) \to 2^{Y_1}$ with nonempty, closed, convex values, and
$t \to \|U_1(t)\|_{Y_1} := \sup\{\|u\|_{Y_1} \mid u \in U_1(t)\}$ belongs to $L^\infty(0, T)$.

$\underline{H(U_2)}$: U_2 is a nonempty, bounded, closed, convex subset of Y_2.

We recall that under hypothesis $H(U_1)$, the set of selectors of the set-valued map U_1 defined by $S_{U_1} = \{z \in L^2(0, T; Y_1) \mid z(t) \in U_1(t) \text{ a.e. } t \in (0, T)\}$ is nonempty.

By an admissible control-state triple (u_1, u_2, w) for problem (4.2) we mean a control pair $(u_1, u_2) \in S_{U_1} \times U_2$ and a state $w \in \mathbb{S}(u_1, u_2) \subset PW(I)$. An admissible triple (u_1, u_2, w) is called an optimal solution to (4.2) if $F(u_1, u_2, w) = \overline{m}$.

Theorem 4.2. *If the hypotheses* $H(A)$, $H(J)$, $H(\varphi)$, $H(M,N)$, $H(R)$, (H_0), (H_1), $H(G)$, $H(B,C)$, $H(F)$, $H(U_1)$ *and* $H(U_2)$ *hold, then problem* (4.2) *has an optimal solution.*

Proof. From Theorem 3.2, it follows that $\mathbb{S}(u_1, u_2) \neq \emptyset$ for all $(u_1, u_2) \in S_{U_1} \times U_2$. Let $\{(u_1^n, u_2^n, w_n)\} \subseteq L^2(0,T;Y_1) \times U_2 \times PW(I)$ be a minimizing sequence for (4.2), i.e., $w_n \in \mathbb{S}(u_1^n, u_2^n)$, $u_1^n(t) \in U_1(t)$ for a.e. $t \in (0,T)$, $u_2^n \in U_2$ and $\lim F(u_1^n, u_2^n, w_n) = \overline{m}$. From hypothesis $H(U_1)$, we obtain that $\{u_1^n\}$ is uniformly bounded in $L^2(0,T;Y_1)$ while by $H(U_2)$ we easy infer that $\{u_2^n\}$ remains in a bounded subset of Y_2. The latter is weakly compact subset of Y_2. Hence, we may suppose, passing to a subsequence, if necessary, that

$$u_1^n \to u_1 \ \text{ weakly in } \ L^2(0,T;Y_1) \ \text{ and } \ u_2^n \to u_2 \ \text{ weakly in } \ Y_2$$

with $u_1 \in L^2(0,T;Y_1)$ and $u_2 \in Y_2$. Further, since $U_1(t)$ is weakly compact subset of Y_1 for a.e. $t \in (0,T)$, we are in a position to apply [6, Proposition 4.7.44] to obtain

$$u_1(t) \in \overline{\text{co}}\text{–}K(Y_{1w}) \limsup \{u_n(t)\}_{n\geq 1} \subseteq \overline{\text{co}}\, U_1(t) \ \text{ a.e. } t \in (0,T),$$

which together with the fact that $U_1(t)$ is a closed and convex subset of Y_1, implies

$$u_1(t) \in U_1(t) \ \text{ for a.e. } \ t \in (0,T).$$

By the weak closedness of the set U_2, we have $u_2 \in U_2$. Next, since $\{u_1^n\}$ is bounded in $L^2(0,T;Y_1)$, $\mathcal{C} \in \mathcal{L}(L^2(0,T;Y_1), \mathcal{V}^*)$ and $\{u_2^n\}$ is bounded in Y_2, by Theorem 2.1, it follows that $\{w_n\}$ is bounded in $PW(I)$. Thus, at least for a subsequence, we have $w_n \to w$ weakly in $PW(I)$, and Lemma 4.1 entails $w \in \mathbb{S}(u_1, u_2)$. Hence, (u_1, u_2, w) is an admissible triple for problem (4.2).

To conclude, it is enough to show that (u_1, u_2, w) is an optimal solution. Exploiting the continuity of the embedding $W(0,T) \subset C(0,T;H)$ and the compactness of $W(0,T)$ into $L^2(0,T;H)$, we get

$$w_n(t) \to w(t) \ \text{ weakly in } H \ \text{ for all } \ t \in [0,T], \ \text{ and } w_n \to w \ \text{ in } \ L^2(0,T;H).$$

We use the weak lower semicontinuity of l on $H \times Y_2$ to obtain

$$l(w(T), u_2) \leq \liminf l(w_n(T), u_2^n). \tag{4.3}$$

Finally, for the functional $\Phi\colon PW(I) \times L^2(0,T;Y_1) \to \mathbb{R}$ defined by

$$\Phi(w, u_1) = \int_0^T L(t, w(t), u_1(t))\, dt,$$

we invoke [3, Theorem 2.1] and deduce that Φ is lower semicontinuous on $L^2(0,T;H) \times L^2(0,T;Y_1)_w$. Therefore, it yields

$$\begin{aligned} \overline{m} &\leq F(u_1, u_2, w) = l(w(T), u_2) + \Phi(w, u_1) \\ &\leq \liminf l(w_n(T), u_2^n) + \liminf \Phi(w_n, u_1^n) \leq \liminf F(u_1^n, u_2^2, w_n) = \overline{m}. \end{aligned}$$

This implies that (u_1, u_2, w) is an optimal solution to problem (4.2). This completes the proof. ☐

5. Application to semipermeability model

In this section we illustrate the applicability of results of Sections 3 and 4 in analysis of a semipermeability problem. Our aim is to provide the weak formulation of the problem which will be an impulsive variational-hemivariational inequality with history-dependent operators, to establish its solvability, and treat an optimal control problem.

Let $I = [0, T]$, $0 < T < \infty$ and $D = \{t_1, \ldots, t_m\}$ be such that $0 = t_0 < t_1 < t_2 < \ldots < t_m < t_{m+1} = T$. Let Ω be a bounded domain of $\mathbb{R}^d$ with Lipschitz continuous boundary $\Gamma = \partial\Omega$. The boundary is decomposed into two disjoint and relatively open subsets Γ_1 and Γ_2 such that $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$ and $m(\Gamma_2) > 0$. We set $Q = \Omega \times (I \setminus D)$, $\Sigma_1 = \Gamma_1 \times (I \setminus D)$ and $\Sigma_2 = \Gamma_2 \times (I \setminus D)$.

Consider the following initial-boundary value problem.

Problem 5.1. Find $w \colon \Omega \times I \to \mathbb{R}$ such that

$$\begin{aligned}
\frac{\partial w(t)}{\partial t} + A(w(t)) + S\Big(\int_0^t w(s)\,ds + z_0\Big) + \partial g(w(t)) \ni f_0(t) \quad &\text{in} \quad Q \\
\frac{\partial w(t)}{\partial \nu_A} + \partial j(w(t)) \ni f_1(t) \quad &\text{on} \quad \Sigma_1 \\
w(t) = 0 \quad &\text{on} \quad \Sigma_2 \\
w(0) = w_0 \quad &\text{in} \quad \Omega \\
w(t_i^+) \in w(t_i^-) + B_H(0, r_i(\|Kw(t_i^-)\|_H)) \quad &\text{in} \quad \Omega
\end{aligned}$$

for $i = 1, \ldots, m$, where A represents a linear operator $A \colon V \to V^*$, $\frac{\partial w}{\partial \nu_A}$ denotes the conormal derivative with respect to operator A, ν stands for the unit outward normal on the boundary, and $B_H(0, r)$ is a closed ball in H with radius $r > 0$.

To provide the weak formulation of Problem 5.1, let

$$V = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_2\} \quad \text{and} \quad H = L^2(\Omega).$$

We denote by $i \colon V \to H$ the embedding operator and by $\gamma \colon V \to L^2(\Gamma)$ the trace operator. For $v \in H^1(\Omega)$, we always write v instead of iv and γv.

We need the following hypotheses on the data.

$\underline{H(A)_1}$: $A \colon V \to V^*$ is such that $A = -\sum_{i,j=1}^d \frac{\partial}{\partial x_i}\big(a_{ij}(x)\frac{\partial}{\partial x_j}\big)$, and

(i) $a_{ij} \in L^\infty(\Omega)$ for $i, j = 1, \ldots, d$.

(ii) $\sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \geq \alpha\|\xi\|^2_{\mathbb{R}^d}$ for all $\xi \in \mathbb{R}^d$, a.e. $x \in \Omega$ with $\alpha > 0$.

$\underline{H(S)}$: $S \in \mathcal{L}(V, V)$.

$\underline{H(j)}$: $j \colon \mathbb{R} \to \mathbb{R}$ is such that

(i) j is locally Lipschitz.

(ii) $|\partial j(r)| \leq c_{1j} + c_{2j}|r|$ for all $r \in \mathbb{R}$ with c_{1j}, $c_{2j} \geq 0$.

(iii) $(\partial j(r_1) - \partial j(r_2))(r_1 - r_2) \geq -\beta_j|r_1 - r_2|^2$ for all r_1, $r_2 \in \mathbb{R}$ with $\beta_j \geq 0$.

$\underline{H(g)}$: $g \colon \mathbb{R} \to \mathbb{R}$ is such that

(i) g is convex and lower semicontinuous.

(ii) $|\partial g(r)| \leq c_{0g} + c_{1g}|r|$ for all $r \in \mathbb{R}$ with c_{0g}, $c_{1g} \geq 0$.

$\underline{H(K,r)}$: $K\colon H \to H$ is a compact operator, $r_i\colon \mathbb{R}_+ \to \mathbb{R}_+$ is an upper semicontinuous and bounded function.

$\underline{H(f)}$: $f_0 \in L^2(Q),\ f_1 \in L^2(\Sigma_1),\ z_0 \in V,\ w_0 \in H.$

$\underline{(H_2)}$: $\alpha > \beta_j \|\gamma\|^2_{\mathcal{L}(V,L^2(\Gamma))}$.

By a standard procedure, we obtain the following weak formulation of Problem 5.1.

Problem 5.2. Find $w \in PW(I) \subset PC(I;H)$ such that

$$\begin{cases} \langle w'(t) + A(w(t)) + (Rw)(t) - f(t), v - w(t)\rangle_{V^*\times V} \\ \qquad + \displaystyle\int_{\Gamma_1} j^0(w(t); v - w(t))\, d\Gamma + \int_\Omega (g(v) - g(w(t)))\, dx \geq 0, \\ \qquad\qquad\qquad \text{for all } v \in V, \ \text{a.e. } t \in I \setminus D, \\ w(0) = w_0, \\ w(t_i^+) \in w(t_i^-) + B_H(0, r_i(\|Kw(t_i^-)\|_H)) \ \text{ for } \ i = 1, \ldots, m. \end{cases}$$

Here $f\colon (0,T) \to V^*$ is defined by

$$\langle f(t), v\rangle_{V^*\times V} = \int_\Omega f_0(t) v\, dx + \int_{\Gamma_1} f_1(t) v\, d\Gamma \tag{5.1}$$

for $v \in V$, a.e. $t \in (0,T)$, and $R\colon \mathcal{V} \to \mathcal{V} \subset \mathcal{V}^*$ is given by

$$(Rw)(t) = S\Big(\int_0^t w(s)\, ds + z_0\Big) \ \text{ for } \ w \in \mathcal{V},\ t \in I. \tag{5.2}$$

Theorem 5.3. *If hypotheses $H(A)_1$, $H(S)$, $H(j)$, $H(g)$, $H(K,r)$, $H(f)$, and (H_2) hold, then Problem 5.2 has a solution $w \in PW(I) \subset PC(I;H)$.*

Proof. We will apply Theorem 3.2 with $X = L^2(\Gamma_1)$ and $Y = H$, and the following data $J\colon X \to \mathbb{R}$ and $\varphi\colon H \to \mathbb{R}$ are defined by

$$J(v) = \int_{\Gamma_1} j(v(x))\, d\Gamma \ \text{ for } v \in L^2(\Gamma_1),$$
$$\varphi(v) = \int_\Omega g(v(x))\, dx \ \text{ for } v \in H,$$

respectively, the operator $A\colon V \to V^*$ is defined in $H(A)_1$ and $R\colon \mathcal{V} \to \mathcal{V}$ is defined by (5.2), $G_i\colon H \to 2^H$ is given by

$$G_i(v) = B_H(0, r_i(\|Kv\|_H)) \ \text{ for } \ v \in H,\ i = 1, \ldots, m, \tag{5.3}$$

and operators $M = \gamma \in \mathcal{L}(V, L^2(\Gamma))$, $N = i \in \mathcal{L}(V,H)$. With this notation we consider the following impulsive evolution problem.

Problem 5.4. Find $w \in PW(I) \subset PC(I;H)$ such that

$$\begin{cases} w'(t) + A(w(t)) + (Rw)(t) + M^*\partial J(Mw(t)) + N^*\partial\varphi(Nw(t)) \ni f(t) \\ \qquad\qquad\qquad\qquad \text{a.e. } \ t \in I \setminus D, \\ w(0) = w_0, \\ w(t_i^+) \in w(t_i^-) + G_i(w(t_i^-)) \ \text{ for } \ i = 1, \ldots, m. \end{cases}$$

It is clear by definitions of the convex and Clarke subdifferentials, and properties of the generalized directional derivative, see [22, Theorem 3.47], that any solution to Problem 5.4 is also a solution to Problem 5.2. Therefore, to complete the proof, it is enough to verity that Problem 5.4 has a solution.

We will check hypotheses $H(A)$, $H(J)$, $H(\varphi)$, $H(M,N)$, $H(R)$, $H(G)$, (H_0), and (H_1) of Theorem 3.2.

Since the operator $A \in \mathcal{L}(V,V^*)$ is coercive, it satisfies condition $H(A)$ with $m_A = \alpha$. Next, the functional J is locally Lipschitz by [5, Theorem 2.7.5] and it satisfies $H(J)$(iii) due to $H(j)$(ii). Further, $H(J)$(iv) holds with $m_J = \beta_j$ by the relaxed monotonicity condition in $H(j)$(iii). This means that $H(J)$ is satisfied. Moreover, we easily verify that φ satisfies condition $H(\varphi)$. Also, it is known that the trace and the embedding operators are linear, bounded, and compact. The latter allows to deduce that the Nemytski operators corresponding to M and N are compact, see [9, Examples 5.2 and 5.3]. Hence $H(M,N)$ holds.

The operator R defined by (5.2) is a history-dependent operator, see [29, Example 5, p.36]. Let $\{v_n\} \subset \mathcal{V}$ be such that $v_n \to v$ weakly in $\mathcal{V}$. Then, for all $\psi \in V^*$, all $t \in I$, we obtain

$$\begin{aligned}\langle \int_0^t v_n(s)\,ds, \psi\rangle_{V^*\times V} &= \int_0^t \langle v_n(s), \psi\rangle_{V^*\times V}\,ds = \langle v_n, \psi\rangle_{\mathcal{V}^*\times\mathcal{V}} \\ &\to \langle v, \psi\rangle_{\mathcal{V}^*\times\mathcal{V}} = \int_0^t \langle v(s), \psi\rangle_{V^*\times V}\,ds = \langle \int_0^t v(s)\,ds, \psi\rangle_{V^*\times V},\end{aligned}$$

which means

$$\int_0^t v_n(s)\,ds + z_0 \to \int_0^t v(s)\,ds + z_0 \ \text{ weakly in } V, \text{ for all } t \in I.$$

Since S is linear and bounded, we deduce $Rv_n \to Rv$ weakly in $\mathcal{V}^*$. We conclude that R is weakly continuous, history-dependent, and clearly, $R0$ is bounded in $\mathcal{V}^*$. Hence, $H(R)$ is verified.

We use the hypothesis $H(K,r)$ to show that the operator G_i defined by (5.3) satisfies $H(G)$. Indeed, let $\{v_n\} \subset H$, $v_n \to v$ weakly in H, $\zeta_n \in G_i(v_n)$ for $i = 1,\ldots,m$ and $\zeta_n \to \zeta$ weakly in H. Then $Kv_n \to Kv$ in H, and $\|Kv_n\|_H \to \|Kv\|_H$, which by $H(K,r)$ implies

$$\|\zeta\|_H \le \liminf \|\zeta_n\|_H \le \limsup r_i(\|Kv_n\|_H) \le r_i(\|Kv\|_H).$$

Therefore, $\zeta \in G_i(v)$ and $H(G)$ follows.

Condition (H_0) holds automatically by (5.1). Moreover, condition $\alpha > \beta_j\|\gamma\|^2_{\mathcal{L}(V,L^2(\Gamma))}$ implies the smallness assumption (H_1). We are now in a position to apply Theorems 3.2 to Problem 5.4 to obtain its solvability. This completes the proof. $\square$

Note that if G_i are single-valued operators such that $G_i \in \mathcal{L}(H,H)$ for $i = 1,\ldots,m$, then $H(G)$ is also fulfilled.

Consider the following example of optimal control problem.

Problem 5.5.

$$F(u_1, u_2, w) = \int_\Omega |w(x,T) - w_T|^2 + |u_2(x)|^2\, dx + \int_0^T \int_{\Gamma_1} |u_3(x,t)|^2\, d\Gamma dt$$
$$+ \int_0^T \int_\Omega |w(x,t) - \overline{w}(x)|^2 + |u_1(x,t)|^2\, dx dt \to \inf = \overline{m}$$

such that

$$\begin{aligned} \frac{\partial w(t)}{\partial t} + A(w(t)) + (Rw)(t) + \partial g(w(t)) \ni f_0(t) + b(t)u_1(t) \quad &\text{in} \quad Q \\ \frac{\partial w(t)}{\partial \nu_A} + \partial j(w(t)) \ni f_1(t) + u_3(t) \quad &\text{on} \quad \Sigma_1 \\ w(t) = 0 \quad &\text{on} \quad \Sigma_2 \\ w(0) = w_0 + u_2 \quad &\text{in} \quad \Omega \\ w(t_i^+) \in w(t_i^-) + B_H(0, r_i(\|Kw(t_i^-)\|_H)) \quad &\text{in} \quad \Omega \end{aligned}$$

for $i = 1, \ldots, m$, where $|u_1(x,t)| \leq \rho_1(x,t)$ a.e. in Q, $|u_2(x)| \leq \rho_2(x)$ a.e in Ω, and $|u_3(x,t)| \leq \rho_3(x,t)$ a.e. on Σ_1.

Here $b \in L^\infty(I)$, $\rho_1 \in L^\infty(Q)$, $\rho_2 \in L^\infty(\Omega)$, $\rho_3 \in L^\infty(\Sigma_1)$, and $\overline{w}$, $w_T \in H$. The weak formulation of the above inclusion leads to Problem 5.2 with $f \in \mathcal{V}^*$ replaced by

$$\langle f(t), v\rangle_{V^* \times V} = \int_\Omega (f_0(t) + b(t)u_1(t))v\, dx + \int_{\Gamma_1} (f_1(t) + u_3(t))v\, d\Gamma$$

for $v \in V$, a.e. $t \in (0,T)$, and $w_0 \in H$ replaced by the element $w_0 + u_2$. We set $Y_1 = L^2(\Omega) \times L^2(\Gamma_1)$, $Y_2 = L^2(\Omega)$,

$$\begin{aligned} U_1(t) = &\{v \in H \mid \|v\|_H \leq \|\rho_1(\cdot,t)\|_{L^\infty(Q)}\} \\ &\times \{z \in L^2(\Gamma_1) \mid \|z\|_{L^2(\Gamma_1)} \leq \|\rho_3(\cdot,t)\|_{L^1(\Gamma_1)}\}, \end{aligned}$$

for a.e. $t \in I$, $U_2 = \{v \in H \mid \|v\|_H \leq \|\rho_2\|_{L^\infty(\Omega)}\}$, $B(t)(v,z) = b(t)v$ for $(v,z) \in Y_1$, and $C = \mathrm{Id} \in \mathcal{L}(H,H)$. We choose $l\colon H \times H \to \mathbb{R}$ and $L\colon (0,T) \times H \times H \times L^2(\Gamma_1) \to \mathbb{R}$ by

$$\begin{aligned} &l(z, u_2) = \|z - w_T\|_H^2 + \|u_2\|_H^2, \\ &L(t, w, u_1, u_3) = \|u_1\|_H^2 + \|u_3\|_{L^2(\Gamma_1)}^2 + \|w - \overline{w}\|_H^2 \end{aligned}$$

for u_1, u_2, w, $z \in H$, $u_3 \in L^2(\Gamma_1)$ and a.e. $t \in (0,T)$. Under these definitions, it is easy to see that Problem 5.5 satisfies the hypotheses of Theorem 4.2. Hence Problem 5.5 has at least one optimal solution.

6. Final comments

(i) The results of Sections 3 and 4 can be readily extended to systems described by the following generalization of problem (1.1)–(1.3) involving history-dependent operators R_1, R_2 and R_3: find $w \in PW(I)$ such that

$$\begin{aligned}
&w'(t) + A(w(t)) + (R_1 w)(t) + M^* \partial J(t, (R_2 w)(t), Mw(t)) \\
&\qquad + N^* \partial \varphi(t, (R_3 w)(t), Nw(t)) \ni f(t) \quad \text{a.e. } t \in I \setminus D, \\
&w(0) = w_0, \\
&w(t_i^+) \in w(t_i^-) + G_i(w(t_i^-)) \quad \text{for} \quad i = 1, \dots, m.
\end{aligned}$$

The dependence of the nonconvex potential J and the convex potential φ on history-dependent operators is essential in many applications in contact mechanics, see, for instance, a dynamic viscoelastic contact problem with friction in solid mechanics [16, 22].

(ii) It is an interesting open problem to study the continuous dependence of solution on the data in the space $PC(I;H)$ with the usual supremum norm. Such a result will allow to extend the current results to a more general form of the impulsive operators G_i.

(iii) Furthermore, another open problem is to prove existence of solution to problem (1.1)–(1.3) in the space $PC(I;V)$. Such an extension will open a way to investigate other optimal control problems: maximum stay control problem, time optimal control problem, and problems with control in superpotentials.

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