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CONSTRAINED PROBLEMS VIA SUB-SUPERSOLUTION

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Abstract. The aim of the paper is to study a quasilinear Dirichlet equation driven by a differential operator of (p, q)-Laplacian type and with a reaction term involving convection and convolution for which pointwise constraints on the solution are imposed. Our main contribution is to handle this constrained problem through a sub-supersolution approach applied to the elliptic equation. The idea is to make a suitable choice of the sub-supersolution with respect to the constraints. Under verifiable hypotheses on the constraints and on the nonlinearity in equation we establish the existence of solutions in a weak sense. In particular, the method of sub-supersolution is developed for nonlinear elliptic problems with both convection and convolution, which is done for the first time. The applicability of our result is demonstrated by an example.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and let $a \in L^1(\mathbb{R}^N)$ and $b \in L^1(\mathbb{R}^N)$ be fixed functions with $a(x) < b(x)$ for almost every $x \in \mathbb{R}^N$. We formulate the following quasilinear elliptic problem on Ω with homogeneous Dirichlet boundary condition, convection, convolution and constraints of obstacle type:

$$(1.1)\qquad \begin{cases} -\Delta_p u - \mu\Delta_q u = f(x, \rho * u, \nabla(\rho * u)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ a(x) < u(x) < b(x) & \text{a.e. in } \Omega \end{cases}$$

for $1 < q < p < +\infty$, $\mu \geq 0$, and $\rho \in L^1(\mathbb{R}^N)$, $\rho \geq 0$ almost everywhere. For the rest of the paper we assume that $p < N$. The complementary case $p \geq N$ can be managed analogously.

We emphasize the presence of the constraints with strict inequalities in the statement of (1.1). In order to handle the multivalued character caused by the pointwise constraints $a(x) < u(x) < b(x)$ for a.e. $x \in \Omega$ we argue along the sub-supersolution method applied to the Dirichlet problem incorporated in (1.1):

$$(1.2)\qquad \begin{cases} -\Delta_p u - \mu\Delta_q u = f(x, \rho * u, \nabla(\rho * u)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

This is an original idea that we think to be fruitful for other multivalued constrained problems. Implicitly, we build a sub-supersolution technique for (1.2) involving both convection and convolution, which is done here for the first time.

In order to simplify the notation, for any real number $r > 1$ we set $r' = r/(r-1)$ (the Hölder conjugate of r). In particular, we have $p' = p/(p-1) < q' = q/(q-1)$.

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In the left-hand side of equation (1.1) there are the negative p-Laplacian $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ expressed as

$$\langle -\Delta_p u, v\rangle = \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x)\,dx \quad \text{for all } u, v \in W_0^{1,p}(\Omega)$$

and the negative q-Laplacian $-\Delta_q : W_0^{1,q}(\Omega) \to W^{-1,q'}(\Omega)$,

$$\langle -\Delta_q u, v\rangle = \int_\Omega |\nabla u(x)|^{q-2}\nabla u(x)\cdot\nabla v(x)\,dx \quad \text{for all } u, v \in W_0^{1,q}(\Omega).$$

For $\mu = 0$ the driving operator reduces to the negative p-Laplacian $-\Delta_p$, whereas for $\mu = 1$ we have the negative (p,q)-Laplacian $-\Delta_p - \Delta_q$. Hereafter, the symbols $|\cdot|$ and $\cdot$ stand, respectively, for the Euclidean norm and the standard product in $\mathbb{R}^N$. Since $1 < q < p < +\infty$, the continuous embedding $W_0^{1,p}(\Omega) \hookrightarrow W_0^{1,q}(\Omega)$ holds, which makes the operator $-\Delta_p - \mu\Delta_q$ be well defined on the space $W_0^{1,p}(\Omega)$. Let p^* denote the Sobolev critical exponent $p^* = Np/(N-p)$ (recall that we assume $p < N$).

The right-hand side of equation (1.1) depends on the solution u and its gradient ∇u, which prevents us to use variational methods. Such a nonlinearity is often called convection. We refer to [3] for recent results on problems involving convection terms. In our case, the situation is more complex because the convection is described by a Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ (i.e., $f(\cdot, s, \xi)$ is measurable on Ω for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and $f(x, \cdot, \cdot)$ is continuous for a.e. $x \in \Omega$) composed with the convolution

$$\rho * u(x) = \int_{\mathbb{R}^N} \rho(x-y)u(y)\,dy \quad \text{for a.e. } x \in \mathbb{R}^N$$

of $\rho \in L^1(\mathbb{R}^N)$ and $u \in W_0^{1,p}(\Omega) \subset W^{1,p}(\mathbb{R}^N)$. Notice that the convolution $\rho * u$ is well defined since $u \in W_0^{1,p}(\Omega)$ can be extended on $\mathbb{R}^N$ with zero outside Ω. It is worth mentioning that the convolution is a nonlocal operator. The study of the problems involving the composition of convection and convolution has been initiated in [4] and continued in [5]. We emphasize that the results in [4]– [5] do not address the method of sub-supersolution.

We impose the following growth condition on the nonlinearity $f(x, s, \xi)$ that only concerns the values of s in the interval $a(x) < b(x)$ for $x \in \Omega$ almost everywhere.

$(H1)$ The Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(x,t,\xi)| \le \sigma(x) + c|\xi|^r \text{ for a.e } x \in \Omega, \quad \text{all } t \in [\rho * a(x), \rho * b(x)],\ \xi \in \mathbb{R}^N,$$

with some $\sigma \in L^{\frac{p}{r}}(\Omega)$ and constants $c > 0$, $r \in [1, \frac{p}{(p^*)'})$.

Remark 1.1. (a) The criterion to reach the greatest magnitude of r in assumption (H) is to have a finite integral $\int_\Omega |\nabla u|^r v dx$ whenever $u, v \in W_0^{1,p}(\Omega)$, thus $\nabla u \in L^p(\Omega, \mathbb{R}^N)$ and $v \in L^{p^*}(\Omega)$. From Hölder's inequality

$$\int_\Omega |\nabla u|^r v dx \le \left(\int_\Omega |\nabla u|^p dx\right)^{\frac{r}{p}} \left(\int_\Omega |v|^{\frac{p}{p-r}} dx\right)^{\frac{p-r}{p}}$$

we must require through Rellich-Kondrachov compact embedding theorem that $\frac{p}{p-r} < p^*$ with $r < p$, which gives exactly $r < \frac{p}{(p^*)'}$.

(b) Suppose $a, b \in W^{1,p}(\Omega)$ and note that $u \in W_0^{1,p}(\Omega)$ being solution of (1.1), in particular of (1.2). Then the required inequality $a \leq u$ a.e. in Ω, in conjunction with $u = 0$ on $\partial\Omega$, forces $a \leq 0$ on $\partial\Omega$. Likewise, upon the formulation of (1.1), we must necessarily have $\overline{u} \geq 0$ on $\partial\Omega$. These facts suggest a powerful link of the functions a and b given in (1.1) with a sub-supersolution of (1.2) (see below).

(c) The assumptions $a, b \in L^1(\mathbb{R}^N)$ with $a(x) < b(x)$ for almost every $x \in \mathbb{R}^N$ and $\rho \in L^1(\mathbb{R}^N)$ with $\rho \geq 0$ almost everywhere ensure that $\rho * a \leq \rho * b$ almost everywhere, thus the ordered interval $[\rho * a(x), \rho * b(x)]$ in hypothesis ($H1$) makes sense.

By a (weak) solution to problem (1.1) we mean a function $u \in W_0^{1,p}(\Omega)$ such that $f(\cdot, \rho * u(\cdot), \nabla(\rho * u)(\cdot)) \in L^{(p^*)'}(\Omega)$, $a(x) < u(x) < b(x)$ for a.e. $x \in \Omega$, and

$$\begin{aligned}(1.3)\qquad &\int_\Omega |\nabla u|^{p-2}\nabla u(x) \cdot \nabla v dx + \mu \int_\Omega |\nabla u|^{q-2}\nabla u(x) \cdot \nabla v dx \\ &= \int_\Omega f(x, \rho * u, \nabla(\rho * u)) v dx, \ \forall v \in W_0^{1,p}(\Omega).\end{aligned}$$

In this context, a function $u \in W_0^{1,p}(\Omega)$ is (weak) solution to problem (1.2) if $f(\cdot, \rho * u(\cdot), \nabla(\rho * u)(\cdot)) \in L^{(p^*)'}(\Omega)$ and (1.3) holds true.

We recall the notion of sub-supersolution to the Dirichlet problem (1.2). A function $\underline{u} \in W^{1,p}(\Omega)$ is a subsolution (or lower solution) for problem (1.2) if $\underline{u} \leq 0$ on $\partial\Omega$ (in the sense of traces), $f(\cdot, \underline{u}(\cdot), \nabla\underline{u}(\cdot)) \in L^{(p^*)'}(\Omega)$ and

$$\begin{aligned}(1.4)\qquad &\int_\Omega |\nabla \underline{u}|^{p-2}\nabla \underline{u}(x) \cdot \nabla v dx + \mu \int_\Omega |\nabla \underline{u}|^{q-2}\nabla \underline{u}(x) \cdot \nabla v dx \\ &\leq \int_\Omega f(x, \rho * \underline{u}, \nabla(\rho * \underline{u})) v dx, \ \forall v \in W_0^{1,p}(\Omega), \ v \geq 0 \text{ a.e. in } \Omega.\end{aligned}$$

A function $\overline{u} \in W^{1,p}(\Omega)$ is a supersolution (or upper solution) for problem (1.2) if $\overline{u} \geq 0$ on $\partial\Omega$ (in the sense of traces), $f(\cdot, \overline{u}(\cdot), \nabla\overline{u}(\cdot)) \in L^{(p^*)'}(\Omega)$ and

$$\begin{aligned}(1.5)\qquad &\int_\Omega |\nabla \overline{u}|^{p-2}\nabla \overline{u}(x) \cdot \nabla v dx + \mu \int_\Omega |\nabla \overline{u}|^{q-2}\nabla \overline{u}(x) \cdot \nabla v dx \\ &\geq \int_\Omega f(x, \rho * \overline{u}, \nabla(\rho * \overline{u})) v dx, \ \forall v \in W_0^{1,p}(\Omega), \ v \geq 0 \text{ a.e. in } \Omega.\end{aligned}$$

The second hypothesis that we assume refers to the functions $a(x)$ and $b(x)$ in the formulation of problem (1.1).

($H2$) There exist a subsolution $\underline{u} \in W^{1,p}(\Omega)$ and a supersolution $\overline{u} \in W^{1,p}(\Omega)$ to problem (1.2) such that

$$a(x) < \underline{u}(x) \leq \overline{u}(x) < b(x) \quad \text{for a.e } x \in \Omega.$$

Our main abstract result provides the existence of a (weak) solution to problem (1.1) under assumptions ($H1$)-($H2$). Section 3 is devoted to this result with its proof relying on a few preliminary tools discussed in Section 2. An effective application is presented in Section 4 to obtain positive solutions, thus offering a clear example of the interest to have strict inequalities in the statement of problem (1.1).

2. Preliminary tools

Throughout the rest of the paper, the space $W_0^{1,p}(\Omega)$ is endowed with the norm $\|\nabla(\cdot)\|_{L^p(\Omega,\mathbb{R}^N)}$.

The operators $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ and $-\Delta_q : W_0^{1,q}(\Omega) \to W^{-1,q'}(\Omega)$ are continuous, strictly monotone and bounded (in the sense that they map bounded sets to bounded sets). Therefore, recalling that $1 < q < p < +\infty$ and $\mu \geq 0$, the operator $-\Delta_p - \mu\Delta_q : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ has the same properties.

Assumption $(H2)$ provides a subsolution $\underline{u} \in W^{1,p}(\Omega)$ and a supersolution $\overline{u} \in W^{1,p}(\Omega)$ for equation (1.2) with $\underline{u}(x) \leq \overline{u}(x)$ for a.e $x \in \Omega$. We associate to the ordered pair $\underline{u} \leq \overline{u}$ the truncation operator $T : W_0^{1,p}(\Omega) \to W^{1,p}(\Omega)$ given by

$$(Tu)(x) = \begin{cases} \underline{u}(x) & \text{if} \quad u(x) < \underline{u}(x) \\ u(x) & \text{if} \quad \underline{u}(x) \leq u(x) \leq \overline{u}(x) \\ \overline{u}(x) & \text{if} \quad u(x) > \overline{u}(x) \end{cases} \tag{2.1}$$

for all $u \in W_0^{1,p}(\Omega)$ and a.e. $x \in \Omega$. The operator T is continuous and bounded.

We also need a cut-off function $\pi : \Omega \times \mathbb{R} \to \mathbb{R}$ associated with the ordered pair $\underline{u} \leq \overline{u}$ defined as

$$\pi(x,s) = \begin{cases} -(\underline{u}(x) - s)^{\frac{r}{p-r}} & \text{if} \quad s < \underline{u}(x) \\ 0 & \text{if} \quad \underline{u}(x) \leq s \leq \overline{u}(x) \\ (s - \overline{u}(x))^{\frac{r}{p-r}} & \text{if} \quad s > \overline{u}(x) \end{cases} \tag{2.2}$$

for all $(x,s) \in \Omega \times \mathbb{R}$, with r given by hypothesis $(H1)$. Since $\underline{u}, \overline{u} \in L^{p^*}(\Omega)$, the definition in (2.2) yields the estimate

$$|\pi(x,s)| \leq c_0|s|^{\frac{r}{p-r}} + \eta(x) \text{ for a.e. } x \in \Omega, \text{ all } s \in \mathbb{R}, \tag{2.3}$$

with a constant $c_0 > 0$ and a function $\eta \in L^{\frac{p^*(p-r)}{r}}(\Omega)$. Consequently, by (2.3) and the compact embedding $W_0^{1,p}(\Omega) \subset L^{\frac{p^*(p-r)}{p^*(p-r)-r}}(\Omega)$ (note that

$$\frac{p^*(p-r)}{p^*(p-r)-r} < p^*$$

because $r \in [1, \frac{p}{(p^*)'})$ in $(H1)$), the mapping $\Pi : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^*$ defined by

$$\langle \Pi(u), v \rangle = \int_\Omega \pi(x,u)v dx, \ \forall u, v \in W_0^{1,p}(\Omega)$$

is completely continuous. Another useful estimate derived from (2.2) is

$$\int_\Omega \pi(x,u(x))u(x)\, dx \geq b_1\|u\|^{\frac{p}{p-r}}_{L^{\frac{p}{p-r}}(\Omega)} - b_2 \ \text{ for all } u \in W_0^{1,p}(\Omega), \tag{2.4}$$

with constants $b_1 > 0$ and $b_2 \geq 0$.

Using again hypotheses $(H1)$-$(H2)$ we observe that it is well defined the mapping $N_f : [\rho * \underline{u}, \rho * \overline{u}] \to W^{-1,p'}(\Omega)$ given by

$$\langle N_f(u), v \rangle = \int_\Omega f(x,u(x),\nabla u(x))v(x)\, dx$$

for all $u \in [\rho * \underline{u}, \rho * \overline{u}]$ and $v \in W_0^{1,p}(\Omega)$, where

$$[\rho * \underline{u}, \rho * \overline{u}] := \{w \in W^{1,p}(\Omega) : \rho * \underline{u} \leq w \leq \rho * \overline{u} \text{ a.e. in } \Omega\}.$$

The mapping $N_f : [\rho * \underline{u}, \rho * \overline{u}] \to W^{-1,p'}(\Omega)$ is completely continuous in view of the fact that the embedding $W^{1,p}(\Omega) \subset L^{\frac{p}{p-r}}(\Omega)$ is compact by the Rellich-Kondrachov theorem (note that $\frac{p}{p-r} < p^*$ due to the assumption $r \in [1, \frac{p}{(p^*)'})$ in $(H1)$).

As mentioned before, an element $u \in W_0^{1,p}(\Omega)$ is viewed as belonging to $W^{1,p}(\mathbb{R}^N)$ by identifying it to its extension with zero outside Ω. Therefore the convolution $\rho * u$ of $\rho \in L^1(\mathbb{R}^N)$ and $u \in W_0^{1,p}(\Omega)$ can be done. We have that $\rho * u \in W^{1,p}(\mathbb{R}^{\mathbb{N}})$ with the weak partial derivatives

$$\frac{\partial}{\partial x_i}(\rho * u) = \rho * \frac{\partial u}{\partial x_i} \in L^p(\mathbb{R}^N), \quad \forall i = 1, \ldots, N,$$

for which the following estimates are available

$$\Big\|\rho * \frac{\partial u}{\partial x_i}\Big\|_{L^p(\mathbb{R}^N)} \leq \|\rho\|_{L^1(\mathbb{R}^N)} \Big\|\frac{\partial u}{\partial x_i}\Big\|_{L^p(\Omega)}, \quad \forall i = 1, \ldots, N$$

(see [1, §9.1]). Consequently, the corresponding gradient estimate can be carried out

$$\|\nabla(\rho * u)\|_{L^p(\mathbb{R}^N,\mathbb{R}^N)} \leq N\|\rho\|_{L^1(\mathbb{R}^N)}\|\nabla u\|_{L^p(\Omega,\mathbb{R}^N)}. \tag{2.5}$$

3. Main abstract result

Our main abstract result on problem (1.1) reads as follows.

Theorem 3.1. *Assume that conditions* $(H1)$ *and* $(H2)$ *hold. Then there exists a (weak) solution* $u \in W_0^{1,p}(\Omega)$ *to problem* (1.1) *Moreover, it is a (weak) solution to problem* (1.2) *satisfying the enclosure property* $\underline{u}(x) \leq u(x) \leq \overline{u}(x)$ *for a.e.* $x \in \Omega$, *where* $\underline{u} \leq \overline{u}$ *is the sub-supersolution to equation* (1.2) *guaranteed by hypothesis* $(H2)$.

Proof. For each $\lambda > 0$ we introduce the operator $A_\lambda : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ defined by

$$A_\lambda = -\Delta_p - \mu\Delta_q + \lambda\Pi - N_f(\rho * T(\cdot)). \tag{3.1}$$

We note that the operator $A_\lambda : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ in (3.1) is well defined and bounded.

Next we show that the operator $A_\lambda : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is pseudomonotone. Let $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and

$$\limsup_{n\to\infty}\langle A_\lambda(u_n), u_n - u\rangle \leq 0. \tag{3.2}$$

The compact embedding $W_0^{1,p}(\Omega) \subset L^{\frac{p^*(p-r)}{p^*(p-r)-r}}(\Omega)$ ensures that along a relabeled subsequence, the strong convergence $u_n \to u$ in $L^{\frac{p^*(p-r)}{p^*(p-r)-r}}(\Omega)$ holds. Since the sequence $\{\Pi(u_n)\}$ is bounded in $L^{\frac{p^*(p-r)}{r}}(\Omega)$, we deduce that

$$\lim_{n\to\infty}\langle \Pi(u_n), u_n - u\rangle = 0. \tag{3.3}$$

By assumption $(H1)$, the sequence $\{N_f(\rho * T(u_n))\}$ is bounded in $L^{\frac{p}{r}}(\Omega)$. The compact embedding $W_0^{1,p}(\Omega) \subset L^{\frac{p}{p-r}}(\Omega)$ combined with $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ implies $u_n \to u$ in $L^{\frac{p}{p-r}}(\Omega)$, so we obtain

$$\lim_{n\to\infty} \langle N_f(\rho * T(u_n)), u_n - u\rangle = 0. \tag{3.4}$$

Taking into account (3.1), (3.3), and (3.4), we see that (3.2) becomes

$$\limsup_{n\to\infty} \langle -\Delta_p u_n - \mu\Delta_q u_n, u_n - u\rangle \leq 0. \tag{3.5}$$

At this point we invoke the S_+ property that is satisfied by the operator $-\Delta_p - \mu\Delta_q$ rendering the strong convergence $u_n \to u$ in $W_0^{1,p}(\Omega)$ (refer to [2, Theorem 2.109]). Now the continuity of the operator $A_\lambda : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ allows us to infer that $A_\lambda(u_n) \to A_\lambda(u)$, so $A_\lambda(u_n) \rightharpoonup A_\lambda(u)$ in $W^{-1,p'}(\Omega)$, and that $\langle A_\lambda u_n, u_n\rangle \to \langle A_\lambda u, u\rangle$, thus obtaining that the operator $A_\lambda : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is pseudomonotone (see, e.g., [2, Definition 2.97]).

We pass to check that the operator $A_\lambda : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is coercive provided $\lambda > 0$ is sufficiently large, which means that

$$\lim_{\|\nabla u\|_{L^p(\Omega,\mathbb{R}^N)}\to+\infty} \frac{\langle A_\lambda(u), u\rangle}{\|\nabla u\|_{L^p(\Omega,\mathbb{R}^N)}} = +\infty. \tag{3.6}$$

In order to prove (3.6) we have to estimate

$$\begin{aligned} \langle A_\lambda(u), u\rangle &= \int_\Omega |\nabla u|^p dx + \mu \int_\Omega |\nabla u|^q dx \\ &+\lambda \int_\Omega \pi(x,u)u dx - \int_\Omega f(x, \rho * (Tu), \nabla(\rho * (Tu)))u dx, \ \forall u \in W_0^{1,p}(\Omega). \end{aligned} \tag{3.7}$$

By (3.7), (2.4) and hypothesis $(H1)$, which can be used thanks to $\rho * Tu \in [\rho * \underline{u}, \rho * \overline{u}]$ as known from (2.1) and hypothesis $(H2)$, it turns out

$$\begin{aligned} \langle A_\lambda(u), u\rangle &\geq \|\nabla u\|^p_{L^p(\Omega,\mathbb{R}^N)} + \lambda(b_1 \|u\|^{\frac{p}{p-r}}_{L^{\frac{p}{p-r}}(\Omega)} - b_2) \\ &-\|\sigma\|_{L^{\frac{p}{r}}(\Omega)} \|u\|_{L^{\frac{p}{p-r}}(\Omega)} - c\int_\Omega |\nabla(\rho * (Tu))|^r |u| dx, \ \forall u \in W_0^{1,p}(\Omega). \end{aligned} \tag{3.8}$$

The integral term in (3.8) can be estimated on the basis of (2.1) and the decomposition

$$\begin{aligned} \int_\Omega |\nabla(\rho * (Tu))|^r |u| dx &= \int_{\{\underline{u}\leq u\leq \overline{u}\}} |\nabla(\rho * u)|^r |u| dx \\ &+ \int_{\{u<\underline{u}\}} |\nabla(\rho * \underline{u})|^r |u| dx + \int_{\{u>\overline{u}\}} |\nabla(\rho * \overline{u})|^r |u| dx. \end{aligned}$$

Then Hölder's inequality and Sobolev embedding theorem imply

$$\int_\Omega |\nabla(\rho * (Tu))|^r |u| dx \leq \int_\Omega |\nabla(\rho * u)|^r |u| dx + c_1 \|\nabla u\|_{L^p(\Omega,\mathbb{R}^N)} \tag{3.9}$$

for all $u \in W_0^{1,p}(\Omega)$, with a constant $c_1 > 0$. Through Hölder's inequality, Young's inequality with an $\varepsilon > 0$ and (2.5) we are able to estimate the first integral term in the right-hand side of (3.9) as follows

$$\int_\Omega |\nabla(\rho * u)|^r |u| dx \leq \|\nabla(\rho * u)\|^r_{L^p(\Omega,\mathbb{R}^N)} \|u\|_{L^{\frac{p}{p-r}}(\Omega)}$$
$$\leq \varepsilon N^p \|\rho\|^p_{L^1(\mathbb{R}^N)} \|\nabla u\|^p_{L^p(\Omega,\mathbb{R}^N)} + c(\varepsilon)\|u\|^{\frac{p}{p-r}}_{L^{\frac{p}{p-r}}(\Omega)},$$

with a constant $c(\varepsilon) > 0$ depending on ε. Returning to (3.8) we get from (3.9) that

$$\langle A_\lambda(u), u\rangle \geq \|\nabla u\|^p_{L^p(\Omega,\mathbb{R}^N)} + \lambda\left(b_1\|u\|^{\frac{p}{p-r}}_{L^{\frac{p}{p-r}}(\Omega)} - b_2\right)$$
$$-c_2\|\nabla u\|_{L^p(\Omega,\mathbb{R}^N)} - c\left(\varepsilon N^p\|\rho\|^p_{L^1(\mathbb{R}^N)}\|\nabla u\|^p_{L^p(\Omega,\mathbb{R}^N)} + c(\varepsilon)\|u\|^{\frac{p}{p-r}}_{L^{\frac{p}{p-r}}(\Omega)}\right) \tag{3.10}$$

for all $u \in W_0^{1,p}(\Omega)$, with a constant $c_2 > 0$. If we fix $\varepsilon > 0$ sufficiently small and then choose $\lambda > 0$ sufficiently large, (3.10) entails (3.6) (note that $p > 1$ and $b_1 > 0$).

We have shown that the operator $A_\lambda : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is bounded, pseudomonotone and coercive provided $\lambda > 0$ is large enough. We are thus enabled to apply the main theorem for pseudomonotone operators (see, e.g., [2, Theorem 2.99])) ensuring the existence of $u \in W_0^{1,p}(\Omega)$ solving the equation $A_\lambda(u) = 0$, which in view of (3.1) reads as

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v dx + \mu\int_\Omega |\nabla u|^{q-2}\nabla u \cdot \nabla v dx$$
$$+\lambda\int_\Omega \pi(x,u)v dx$$
$$= \int_\Omega f(x, \rho * (Tu), \nabla(\rho * (Tu)))v dx,\ \forall v \in W_0^{1,p}(\Omega). \tag{3.11}$$

Our next goal is to show that $\underline{u} \leq u \leq \overline{u}$ a.e. in Ω thus achieving comparison with the subsolution $\underline{u}$ and supersolution $\overline{u}$ whose existence is assumed in hypothesis $(H2)$. First, we prove that $u \leq \overline{u}$ a.e. in Ω. To this end, we use as test function $(u-\overline{u})^+ = \max\{u-\overline{u}, 0\}$. Due to the condition $\overline{u} \geq 0$ on $\partial\Omega$ in the sense of traces, we have indeed that $(u-\overline{u})^+ \in W_0^{1,p}(\Omega)$, so $v = (u-\overline{u})^+$ can be inserted in (3.11) and (1.5) obtaining

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla(u-\overline{u})^+ dx + \mu\int_\Omega |\nabla u|^{q-2}\nabla u \cdot \nabla(u-\overline{u})^+ dx$$
$$+\lambda\int_\Omega \pi(x,u(x))(u-\overline{u})^+(x)dx$$
$$= \int_\Omega f(x,\rho * (Tu), \nabla(\rho * (Tu)))(u-\overline{u})^+(x)dx \tag{3.12}$$

and

$$\int_\Omega |\nabla \overline{u}|^{p-2}\nabla \overline{u} \cdot \nabla(u-\overline{u})^+ dx + \mu\int_\Omega |\nabla \overline{u}|^{q-2}\nabla \overline{u} \cdot \nabla(u-\overline{u})^+ dx$$
$$\geq \int_\Omega f(x,\rho * \overline{u}, \nabla(\rho * \overline{u}))(u-\overline{u})^+ dx. \tag{3.13}$$

From (3.12) and (3.13) we derive

$$\begin{aligned}\int_{\{u>\overline{u}\}} & \left(|\nabla u|^{p-2}\nabla u - |\nabla\overline{u}|^{p-2}\nabla\overline{u}\right)\cdot\nabla(u-\overline{u})dx \\ & + \mu\int_{\{u>\overline{u}\}} \left(|\nabla u|^{p-2}\nabla u - |\nabla\overline{u}|^{p-2}\nabla\overline{u}\right)\cdot\nabla(u-\overline{u})dx \\ & + \lambda\int_{\{u>\overline{u}\}} \pi(x,u(x))(u-\overline{u})dx \\ & \leq \int_{\{u>\overline{u}\}} (f(x,\rho*(Tu),\nabla(\rho*(Tu))) - f(x,\rho*\overline{u},\nabla(\rho*\overline{u})))(u-\overline{u})dx.\end{aligned}$$

Replacing $(Tu)(x)$ and $\pi(x,u(x))$ with their expressions according to (2.1) and (2.2), we find that

$$\begin{aligned}& \int_{\{u>\overline{u}\}} \left(|\nabla u|^{p-2}\nabla u - |\nabla\overline{u}|^{p-2}\nabla\overline{u}\right)\cdot\nabla(u-\overline{u})dx \\ & + \mu\int_{\{u>\overline{u}\}} \left(|\nabla u|^{p-2}\nabla u - |\nabla\overline{u}|^{p-2}\nabla\overline{u}\right)\cdot\nabla(u-\overline{u})dx \\ & + \lambda\int_{\{u>\overline{u}\}} (u(x)-\overline{u}(x))^{\frac{p}{p-r}}dx \\ & \leq \int_{\{u>\overline{u}\}} \big(f(x,\rho*\overline{u},\nabla(\rho*\overline{u})) - f(x,\rho*\overline{u},\nabla(\rho*\overline{u}))\big)(u-\overline{u})dx = 0.\end{aligned}$$

Since the mappings $\xi\mapsto|\xi|^{p-2}\xi$ and $\xi\mapsto|\xi|^{q-2}\xi$ on $\mathbb{R}^N$ are monotone, we infer

$$\int_{\{u>\overline{u}\}} (u(x)-\overline{u}(x))^{\frac{p}{p-r}}dx \leq 0,$$

which yields $u\leq\overline{u}$ a.e in Ω.

In order to prove that $\underline{u}\leq u$ a.e in Ω we argue by taking as test function $(\underline{u}-u)^+ = \max\{\underline{u}-u,0\}$ which is an element of $W_0^{1,p}(\Omega)$ because $\underline{u}\leq 0$ on $\partial\Omega$ in the sense of traces. If we plug $v=(\underline{u}-u)^+$ in (3.11) and (1.4) we note that

$$\begin{aligned}\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla(\underline{u}-u)^+dx + \mu\int_\Omega |\nabla u|^{q-2}\nabla u\cdot\nabla(\underline{u}-u)^+dx \\ +\lambda\int_\Omega \pi(x,u(x))(\underline{u}-u)^+(x)dx \\ = \int_\Omega f(x,\rho*(Tu),\nabla(\rho*(Tu)))(\underline{u}-u)^+(x)dx\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}\int_\Omega |\nabla\underline{u}|^{p-2}\nabla\underline{u}\cdot\nabla(\underline{u}-u)^+dx + \mu\int_\Omega |\nabla\underline{u}|^{q-2}\nabla\underline{u}\cdot\nabla(\underline{u}-u)^+dx \\ \leq \int_\Omega f(x,\rho*\underline{u},\nabla(\rho*\underline{u}))(\underline{u}-u)^+dx.\end{aligned} \tag{3.15}$$

From (3.14) and (3.15) we derive

$$\begin{aligned}\int_{\{\underline{u}>u\}} & \big(|\nabla \underline{u}|^{p-2}\nabla \underline{u} - |\nabla u|^{p-2}\nabla u\big)\cdot\nabla(\underline{u}-u)dx \\ & + \mu\int_{\{\underline{u}>u\}} \big(|\nabla \underline{u}|^{p-2}\nabla \underline{u} - |\nabla u|^{p-2}\nabla u\big)\cdot\nabla(\underline{u}-u)dx \\ & + \lambda\int_{\{\underline{u}>u\}} \pi(x,u(x))(\underline{u}-u)dx \\ & \leq \int_{\{\underline{u}>u\}} (f(x,\rho*\underline{u},\nabla(\rho*\underline{u})) - f(x,\rho*(Tu),\nabla(\rho*(Tu)))(\underline{u}-u)dx.\end{aligned}$$

Arguing as before along (2.1), (2.2) and the monotonicity of the mappings $\xi \mapsto |\xi|^{p-2}\xi$ and $\xi\mapsto|\xi|^{q-2}\xi$ on $\mathbb{R}^N$, we get

$$\begin{aligned}\lambda\int_{\{\underline{u}>u\}} (\underline{u}(x)-u(x))^{\frac{p}{p-r}}dx \leq & \int_{\{\underline{u}>u\}} (f(x,\rho*\underline{u},\nabla(\rho*\underline{u})) \\ & - f(x,\rho*\underline{u},\nabla(\rho*\underline{u})))(\underline{u}-u)dx \\ = & \ 0,\end{aligned}$$

which amounts to saying that $\underline{u}\leq u$ a.e. in Ω.

Therefore we have established the enclosure property $\underline{u}\leq u\leq\overline{u}$ a.e. in Ω. Then, in view of (2.1) and (2.2), we see that (3.11) reduces to (1.3). Moreover, again on the basis of the location $\underline{u}\leq u\leq\overline{u}$ a.e. in Ω, it follows from assumption $(H2)$ that $a(x)<u(x)<b(x)$ for a.e. $x\in\Omega$, thereby $u\in W_0^{1,p}(\Omega)$ is a solution of problem (1.1). The proof is thus complete.

□

4. An example

In this section we present an example of how Theorem 3.1 can be applied to concrete problems of type (1.1). The main difficulty is the effective constructions of sub-supersolution $\underline{u}\leq\overline{u}$ required in assumption $(H2)$.

In order to simplify the discussion, we suppose that $\Omega\subset\mathbb{R}^N$ is a bounded domain with a C^∞ boundary $\partial\Omega$, $q=2$, $p\in(2,+\infty)$, and $\mu=1$, i.e., the case of $(p,2)$-Laplacian is taken. Let $r\in[1,\frac{p}{(p^*)'})$, $\alpha>0$ and $\beta>0$ be given constants and let $\rho\in L^1(\mathbb{R}^N)$ with compact support, $\rho\geq0$ a.e., $\rho\not\equiv0$, and $\gamma\in L^\infty(\Omega)$, $\gamma>0$ a.e. in Ω. Seeking positive and bounded solutions to problem (1.1), we are led to assume that $a(x)\equiv0$ and $b(x)\equiv\beta$.

With the mentioned data, we state the problem

$$\begin{cases} -\Delta_p u - \Delta u = f(x,\rho*u,\nabla(\rho*u)) & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega \\ 0<u(x)<\beta & \text{a.e. in } \Omega, \end{cases} \tag{4.1}$$

where $f:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is given by

$$f(x,s,\xi) = \begin{cases} \alpha & \text{if } s<0 \\ \alpha+s(\gamma(x)+|\xi|^r) & \text{if } 0\leq s\leq 1 \\ \alpha+(2-s)(\gamma(x)+|\xi|^r) & \text{if } s>1 \end{cases} \tag{4.2}$$

whenever $(x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$.

It is straightforward to check that f is a Carathéodory function for which condition $(H1)$ is fulfilled. In order to comply with condition $(H2)$ we further assume

$$\operatorname*{ess\,sup}_{x\in\Omega} \frac{\alpha + 2\gamma(x)}{\gamma(x)} < \beta \|\rho\|_{L^1(\mathbb{R}^N)}. \tag{4.3}$$

We construct the subsolution $\underline{u}$ required in $(H2)$ by means of the first eigenvalue of the negative Laplacian $-\Delta$ with homogeneous Dirichlet boundary condition which is

$$\lambda_1 = \min\left\{ \frac{\|\nabla u\|^2_{L^2(\Omega,\mathbb{R}^N)}}{\|u\|^2_{L^2(\Omega)}} : u \in W_0^{1,2}(\Omega) \setminus \{0\} \right\}. \tag{4.4}$$

It is well-known that there exists a corresponding eigenfunction $u_1 \in C^\infty(\overline{\Omega})$ with $u_1 > 0$ in Ω (see, e.g., [1, pages 311-312]).

Fix $\varepsilon > 0$ so small to have $\varepsilon(\rho * u_1)(x) \le 1$ and

$$\varepsilon^{p-1}(-\Delta_p u_1)(x) + \varepsilon\lambda_1 u_1(x) \le \alpha$$

for all $x \in \Omega$. If we choose $\underline{u} = \varepsilon u_1$, then through (4.2), (4.4) and the choice of ε it holds pointwise

$$\begin{aligned} -\Delta_p \underline{u} - \Delta \underline{u} &= -\varepsilon^{p-1}\Delta_p u_1 + \varepsilon\lambda_1 u_1 \le \alpha \\ &\le \alpha + \rho * (\varepsilon u_1)(x)(\gamma(x) + |\nabla(\rho * (\varepsilon u_1))(x)|^r) \\ &= f(x, \rho * (\varepsilon u_1)(x), \nabla(\rho * (\varepsilon u_1))(x)). \end{aligned}$$

Hence (1.4) is true, that is, $\underline{u}$ is a subsolution of problem (4.1) taking into account that $\underline{u} = 0$ on $\partial\Omega$.

Concerning the construction of a supersolution for problem (4.1), we note from (4.3) that we can choose a constant $C > 0$ such that

$$\|\rho\|^{-1}_{L^1(\mathbb{R}^N)} \operatorname*{ess\,sup}_{x\in\Omega} \frac{\alpha + 2\gamma(x)}{\gamma(x)} < C < \beta \tag{4.5}$$

and $\varepsilon u_1(x) \le C$ for all $x \in \Omega$, with a possibly smaller $\varepsilon > 0$. We find by (4.5) that $\rho * C(x) = C\|\rho\|_{L^1(\mathbb{R}^N)} > 2$ for all $x \in \Omega$. Setting $\overline{u} = C$, the first inequality in (4.5) and (4.2) lead to the pointwise inequality

$$\begin{aligned} -\Delta_p \overline{u} - \Delta \overline{u} &= -\Delta_p C - \Delta C \\ &= 0 \\ &\ge \alpha + (2 - C\|\rho\|_{L^1(\mathbb{R}^N)})\gamma(x) \\ &= f(x, \rho * C(x), \nabla(\rho * C)(x)) \\ &= f(x, \rho * \overline{u}(x), \nabla(\rho * \overline{u})(x)). \end{aligned}$$

This shows the validity of (1.5), which establishes that $\overline{u}$ is a supersolution to problem (4.1). In view of $0 < \underline{u} = \varepsilon u_1 \le C = \overline{u} < \beta$, Theorem 3.1 can be applied to problem (4.1) ensuring the existence of a weak solution.

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