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ON GLOBAL IN TIME EXISTENCE OF SOLUTIONS OF STOCHASTIC INCLUSIONS WITH BACKWARD MEAN DERIVATIVES

YURI E. GLIKLIKH AND ALINA D. GRANKINA

Abstract. The paper is devoted to investigation of stochastic differential inclusions with backward mean derivatives. We deal with the property of global in time existence of solutions of "inverse" Cauchy problem for these inclusions. A condition that guarantees the global in time existence of such solutions is obtained.

1. Introduction

The notion of mean derivatives was introduced by Edward Nelson (see. [5–7]) for the needs of stochastic mechanics (a version of quantum mechanics). The equation of motion in this theory (called the Newton-Nelson equation) was the first example of equations with mean derivatives. Later it turned out that the equations with mean derivatives arose also in many other mathematical models in mathematical physics.(see, e.g., [3, 4]).

The equations and inclusions with backward mean derivatives arise in description of some special stochastic processes of mathematical physics. Say, a second order equation with backward mean derivatives of the group of Sobolev diffeomorphisms is derived that describes a process whose expectation is a flow of viscous incompressible fluid (see, e.g. [3, 4]).

This paper is the first attempt to find conditions, under which the "inverse" Cauchy problem for such inclusion exists on every interval $[0, T] \subset \mathbb{R}$. We find such conditions by generalizing the methods elaborated in [1].

Some remarks on notations. In this paper we deal with equations and inclusions in the linear space $\mathbb{R}^n$, for which we use coordinate presentation of vectors and linear operators. Vectors in $\mathbb{R}^n$ are considered as columns. If X is such a vector, the transposed row vector is denoted by X^*. Linear operators from $\mathbb{R}^n$ to $\mathbb{R}^n$ are represented as $n \times n$ matrices, the symbol $*$ means transposition of a matrix (pass to the matrix of conjugate operator). The space of $n \times n$ matrices is denoted by $L(\mathbb{R}^n, \mathbb{R}^n)$.

By $\mathrm{S}(n)$ we denote the linear space of symmetric $n \times n$ matrices that is a subspace in $L(\mathbb{R}^n, \mathbb{R}^n)$. The symbol $\mathrm{S}_+(n)$ denotes the set of positive definite symmetric $n \times n$ matrices that is a convex open set in $\mathrm{S}(n)$. Its closure, i.e., the set of positive semi-definite symmetric $n \times n$ matrices, is denoted by $\bar{\mathrm{S}}_+(n)$.

We use the Einstein convention of summation with respect to repeated upper and lower indices.

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2. Mean derivatives

In this section we briefly describe preliminary facts about mean derivatives. See details in [4–7].

Consider a stochastic process $\xi(t)$ in $\mathbb{R}^n$, $t \in [0,T]$, given on a certain probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and such that $\xi(t)$ is an L_1 random element for all t. It is known that such a process determines 3 families of σ-subalgebras of the σ-algebra $\mathcal{F}$:

(i) "the past" $\mathcal{P}_t^\xi$ generated by preimages of Borel sets from $\mathbb{R}^n$ under all mappings $\xi(s) : \Omega \to \mathbb{R}^n$ for $0 \le s \le t$;

(ii) "the future" $\mathcal{F}_t^\xi$ generated by preimages of Borel sets from $\mathbb{R}^n$ under all mappings $\xi(s) : \Omega \to \mathbb{R}^n$ for $t \le s \le T$;

(iii) "the present" ("now") $\mathcal{N}_t^\xi$ generated by preimages of Borel sets from $\mathbb{R}^n$ under the mapping $\xi(t) : \Omega \to \mathbb{R}^n$.

All the above families we suppose to be complete, i.e., containing all sets of probability zero.

For the sake of convenience we denote by E_t^ξ the conditional expectation $E(\cdot|\mathcal{N}_t^\xi)$ with respect to the "present" $\mathcal{N}_t^\xi$ for $\xi(t)$.

Following [5–7], introduce the following notions of forward and backward mean derivatives.

Definition 2.1. (i) The forward mean derivative $D\xi(t)$ of $\xi(t)$ at the time instant t is an L_1 random element of the form

$$D\xi(t) = \lim_{\triangle t \to +0} E_t^\xi\Big(\frac{\xi(t+\triangle t) - \xi(t)}{\triangle t}\Big), \tag{2.1}$$

where the limit is supposed to exist in $L_1(\Omega, \mathcal{F}, \mathsf{P})$ and $\triangle t \to +0$ means that $\triangle t$ tends to 0 and $\triangle t > 0$.

(ii) The backward mean derivative $D_*\xi(t)$ of $\xi(t)$ at t is the L_1-random element

$$D_*\xi(t) = \lim_{\Delta t \to +0} E_t^\xi\Big(\frac{\xi(t) - \xi(t-\Delta t)}{\Delta t}\Big) \tag{2.2}$$

where (as well as in (i)) the limit is assumed to exist in $L^1(\Omega, \mathcal{F}, \mathsf{P})$ and $\Delta t \to +0$ means that $\Delta t \to 0$ and $\Delta t > 0$.

Remark 2.2. If $\xi(t)$ is a Markov process then evidently E_t^ξ can be replaced by $E(\cdot|\mathcal{P}_t^\xi)$ in (2.1) and by $E(\cdot|\mathcal{F}_t^\xi)$ in (2.2). In initial Nelson's works there were two versions of definition of mean derivatives: as in our Definition 2.1 and with conditional expectations with respect to "past" and "future" as above that coincide for Markov processes. We shall not suppose $\xi(t)$ to be a Markov process and give the definition with conditional expectation with respect to "present" taking into account the physical principle of locality: the derivative should be determined by the present state of the system, not by its past or future.

Following [1] (see also [4]) we introduce the differential operator D_2 that differentiates an L_1 random process $\xi(t)$, $t \in [0,T]$ according to the rule

$$D_2\xi(t) = \lim_{\triangle t \to +0} E_t^\xi\Big(\frac{(\xi(t+\triangle t) - \xi(t))(\xi(t+\triangle t) - \xi(t))^*}{\triangle t}\Big), \tag{2.3}$$

where $(\xi(t+\triangle t)-\xi(t))$ is considered as a column vector (vector in $\mathbb{R}^n$), $(\xi(t+\triangle t)-\xi(t))^*$ is a row vector (transposed, or conjugate vector) and the limit is supposed to exists in $L_1(\Omega,\mathcal{F},\mathsf{P})$. We emphasize that the matrix product of a column on the left and a row on the right is a matrix. It is shown that $D_2\xi(t)$ takes values in $\bar{S}_+(n)$, the set of symmetric semi-positive definite matrices. We call D_2 the quadratic mean derivative.

Remark 2.3. From the properties of conditional expectation (see, e.g., [8]) it follows that there exist Borel mappings $a(t,x)$, $a_*(t,x)$ and $\alpha(t,x)$ from $R\times\mathbb{R}^n$ to $\mathbb{R}^n$ and to $\bar{\mathrm{S}}_+$, respectively, such that $D\xi(t)=a(t,\xi(t))$, $D_*\xi(t)=a_*(t,\xi(t))$ and $D_2\xi(t)=\alpha(t,\xi(t))$. Following [8] we call $a(t,x)$, $a_*(t,x)$ and $\alpha(t,x)$ the regressions.

Let Borel measurable mappings $a(t,x)$ and $\alpha(t,x)$ from $[0,T]\times\mathbb{R}^n$ to $\mathbb{R}^n$ and to $\bar{\mathrm{S}}_+(n)$, respectively, be given. We call the system of the form

$$\begin{cases} D\xi(t)=a(t,\xi(t)), \\ D_2\xi(t)=\alpha(t,\xi(t)), \end{cases} \tag{2.4}$$

a first order differential equation with forward mean derivatives.

Definition 2.4. We say that (2.4) has a solution on $[0,T]$ with initial condition $\xi(0)=x_0$, if there exist a probability space $(\Omega,\mathcal{F},\mathsf{P})$ and a process $\xi(t)$ given on $(\Omega,\mathcal{F},\mathsf{P})$ and taking values in $\mathbb{R}^n$ such that P-a.s. and for almost all t (2.4) is satisfied.

Several existence of solution theorems for (2.4) can be found in [1].

Definition 2.5. The smooth function $\varphi: X\to\mathbb{R}$ sending the topological space X to $\mathbb{R}$ is called proper if the preimage of every relatively compact set in $\mathbb{R}$ is relatively compact in X.

Denote by $\mathcal{L}$ the generator of Markov process generated by equation (2.4).

Theorem 2.6. *Let on $\mathbb{R}^n$ there exist a smooth proper positive function $\varphi:\mathbb{R}^n\to\mathbb{R}$ such that $\mathcal{L}\varphi<C$ for all $t\in[0,+\infty)$ and $x\in\mathbb{R}^n$ where $C>0$ is a certain real constant. Then the flow generated by equation (2.4) is complete, i.e. all solutions of (2.4) with deterministic initial values exist for $t\in[0,+\infty)$.*

Theorem 2.6 is a reformulation of [2, Theorem IX. 6A].

3. Differential inclusions with backward mean derivatives

The system

$$\begin{cases} D_*\xi(t)=a(t,\xi(t)) \\ D_2\xi(t)=\alpha(t,\xi(t)) \end{cases} \tag{3.1}$$

is called a first order differential equation with backward mean derivatives.

Notice that we do not introduce the notion of backward analog of operator D_2 since, applying the properties of Itô integral, one can easily prove that for a diffusion process $\xi(t)$ the result of application of that analog coincides with $D_2\xi(t)$ (for the case of diffusion processes this follows from the results of [6, 7]).

Let $\mathbf{a}(t,x)$ and $\boldsymbol{\alpha}(t,x)$ be set-valued mappings from $[0,T]\times\mathbb{R}^n$ to $\mathbb{R}^n$ and to $\bar{\mathrm{S}}_+(n)$, respectively. The system of the form

$$
\left\{ \begin{array}{l} D_*\xi(t) \in \mathbf{a}(t,\xi(t)), \\ D_2\xi(t) \in \boldsymbol{\alpha}(t,\xi(t)). \end{array} \right. \tag{3.2}
$$

is called a first order differential inclusion with backward mean derivatives.

Definition 3.1. We say that (3.2) has a solution on $[0,T]$ with "inverse" Cauchy condition $\xi(T)=\xi_0$, if there exist a probability space $(\Omega,\mathcal{F},\mathsf{P})$ and a process $\xi(t)$ given on $(\Omega,\mathcal{F},\mathsf{P})$ and taking values in $\mathbb{R}^n$ such that $\xi(T)=\xi_0$ and P-a.s. and for almost all t inclusion (3.2) is satisfied.

For equations with backward mean derivatives and inclusions with forward mean derivatives the definition of solution is quite analogous.

Consider a solution $\eta(t)$, given on $t\in[0,T]$, with initial condition $\eta(0)=\xi_0$ of the following differential inclusion with forward mean derivatives

$$
\left\{ \begin{array}{l} D\eta(t) \in -\mathbf{a}(1-t,\eta(t)), \\ D_2\eta(t) \in \boldsymbol{\alpha}(1-t,\eta(t)). \end{array} \right. \tag{3.3}
$$

Theorem 3.2. *The process $\xi(t)=\xi_0-\eta(T)+\eta(T-t)$ is a solution of (3.2) with condition $\xi(T)=\xi_0$ where $\eta(t)$ is a solution of (3.3) with initial condition $\eta(0)=\xi_0$.*

Indeed, $D_*\xi(t) = -D\eta(T-t) \in \boldsymbol{a}(t,\eta(T-t)) = \boldsymbol{a}(t,\xi(t))$. For $D_2\xi(t)$ the arguments are analogous.

Now we are in position to find conditions, under which solutions of (3.2) exist on every interval $[0,T]$.

Specify $t\in[0,T]$, $x\in\mathbb{R}^n$, a point $a\in\mathbf{a}(t,x)$ with coordinates a^i of this vector and a point $\alpha\in\boldsymbol{\alpha}(t,x)$ with elemets α^{ij} of this matrix. Consider the differential operator $\mathcal{L}(t,x.a,\alpha) = -a^i\frac{\partial}{\partial x^i} + \alpha^{ij}\frac{\partial^2}{\partial x^i\partial x^j}$

Theorem 3.3. *Let $\mathbf{a}$ and $\boldsymbol{\alpha}$ be lower semicontinuous and have closed convex values. If on $\mathbb{R}^n$ there exists a smooth proper positive function $\varphi:\mathbb{R}^n\to\mathbb{R}$ such that for every $t\in[0,T]$, $x\in\mathbb{R}^n$, $a\in\mathbf{a}(t,x)$ and $\alpha\in\boldsymbol{\alpha}(t,x)$ the estimate $\mathcal{L}(t,x.a,\alpha)\varphi<C$ holds for some real $C>0$, on every interval $[0,T]$ there exists a solution of (3.2) with deterministic value of "inverse" Cauchy problem with $\xi(T)=\xi_0$.*

Proof. By Michael's theorem their exist continuous selectors $a(t.x)$ of $\mathbf{a}(\mathbf{t},\mathbf{x})$ and $\alpha(t,x)$ of $\boldsymbol{\alpha}(t.x)$, respectively. So, it is sufficient to prove the statement of theorem for the solution of (3.1) with those $a(t.x)$ and $\alpha(t,x)$. But this solution is a solution of equation with forward mean derivatives

$$
\left\{ \begin{array}{l} D\eta(t) \in -a(1-t,\eta(t)), \\ D_2\eta(t) \in \alpha(1-t,\eta(t)). \end{array} \right. \tag{3.4}
$$

where $\eta(t)$ is a solution of (3.4) with initial condition $\eta(0)=\xi_0$.The generator $\mathcal{L}$ of the flow generated by equation (3.4) is a selector of $\mathcal{L}(t,x.a,\alpha)$. Hence, by the hypothesis of theorem, equation (3.4) satisfies the conditions of Theorem 2.6 and so the solution exists for all $t\in[0,\infty)$. Thus on every interval $[0,T]$ the solution of "inverse" Cauchy problem for (3.2) with $\xi(T)=\xi_0$ exists. □

References

[1] S. V. Azarina and Yu. E. Gliklikh, *Differential inclusions with mean derivatives*, Dynamic Systems and Applications **16** (2007), 49–71.
[2] K. D. Elworthy *Stochastic Differential Equations on Manifolds*, (Lect. Notes of London Math. Soc. vol. 70), Cambridge University Press, Cambridge, 1982.
[3] Yu. E. Gliklikh *Deterministic viscous hydrodynamics via stochastic processes on groups of diffeomorphisms*, in: Probabilistic Metohds in Fluids, I. M. Davis et al., (eds.), World Scientific, Singapore, 2003, pp.179–190.
[4] Yu. E. Gliklikh *Global and Stochastic Analysis with Applications to Mathematical Physics*, Springer-Verlag, London, 2011.
[5] E. Nelson *Derivation of the Schrödinger equation from Newtonian mechanics*, Phys. Reviews **150** (1966), 1079–1085
[6] E. Nelson *Dynamical Theory of Brownian Motion*, Princeton University Press, Princeton, 1967.
[7] E. Nelson *Quantum Fluctuations*, Princeton University Press, Princeton, 1985.
[8] K. R. Parthasarathy *Introduction to Probability and Measure*, Springer-Verlag, New York, 1978.

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Yu. Gliklikh
Voronezh State University, Voronezh, Russian Federation and I.A. Bunin Yelets State University, Yelets, Russian Federation
E-mail address: `yeg@math.vsu.ru`

A. Grankina
Voronezh State University, Voronezh, Russian Federation
E-mail address: `grankina@math.vsu.ru`