Applied Analysis and Optimization
Volume 5, Number 2, 2021, 203–216

Yokohama Publishers
ISSN 2189-1664 Online Journal

A STRING OSCILLATIONS SIMULATION WITH A PERTURBED SWEEPING PROCESS AS A BOUNDARY CONDITION

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Abstract. In the present paper we investigate an initial - boundary value problem, describing an oscillation process for a string with a hysteresis type boundary condition. This condition arises due to the presence of a limiter on the motion of the elastically fixed right end of the string and has the form $-u'(l,t) - \gamma u(l,t) \in N_{C(t)}(u(l,t))$. Here $N_{C(t)}(u(l,t))$ denotes the outward normal cone at the point $u(l,t) \in C(t)$ to the close convex set $C(t)$. The analogue of the d'Alembert formula is obtained. A control problem of the string oscillations is analyzed and an explicit form of the control function is presented.

1. Introduction

Various problems of control and stabilization of solutions for wave equations with different types of boundary conditions are the subject of a large number of papers. We refer the reader to [3, 5, 6, 9–12, 14, 17, 18, 20, 23, 25] and the references therein. In the cycle of works [10–12] a new method of explicit construction of boundary controls for one-dimensional wave equation with classical boundary conditions was suggested. These controls allow to translate the system from an initial state to a final one in a certain time.

In the present paper we investigate the initial - boundary value problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}, \\ u(x,0) = \varphi(x), \\ \dfrac{\partial u}{\partial t}(x,0) = 0, \\ u(0,t) = 0, \\ u(l,t) \in C(t), \\ -u'_x(l,t) - \gamma u(l,t) \in N_{C(t)}(u(l,t)), \end{cases}$$

describing oscillation process with a boundary condition of a hysteresis type. Here $N_{C(t)}(u(l,t))$ denotes the outward normal cone at the point $u(l,t)$ to a close convex set $C(t)$. This kind of problems arises in a simulation of the string oscillations, where the movement of the end of the string is restricted by a limiter $C(t)$, concentrated at one point. Here the function $u(x,t)$ ($0 \le x \le l$, $t \ge 0$) determines the oscillations of the string. We suppose the initial string velocity is zero, and the initial string form is determined by the absolutely continuous on $[0,l]$ function $\varphi(x)$, where $\varphi(0) = 0$, $\varphi(l) \in C(0)$, $-\varphi'(l) - \gamma\varphi(l) \in N_{C(0)}(\varphi(l))$. We assume that at the point $x = 0$ the string is rigidly fixed, that can be expressed by the condition $u(0,t) = 0$. At the

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2020 *Mathematics Subject Classification.* Primary: 35L20, 35Q93; Secondary: 34C55, 35A01.

Key words and phrases. Initial - boundary value problem, wave equation, sweeping process, control problem.

point $x = l$ we have a limiter $C(t)$ on the motion of the elastically fixed right end of the string, so that $u(l,t) \in C(t)$. Notice that we consider the case, when the limiter can move in perpendicular to the axis Ox direction and its movement is given by $C(t) = [-h, h] + \xi(t)$, where the function $\xi(t)$ is absolutely continuous on $[0, +\infty)$. During oscillations, the right end of the string either touches the boundary points of the limiter, or remains the interior point of the set $C(t)$. The last condition can be written in the form $-u'_x(l,t) - \gamma u(l,t) \in N_{C(t)}(u(l,t))$. The analogue of the d'Alembert formula is obtained. A boundary control problem of the string oscillations is analyzed and an explicit form of the control function is presented.

A particular case of this problem was investigated in [14].

2. Preliminaries

In this section we recall some notions and definitions which we will need in the sequel (details can be found in [8,22]).

Let H be a Hilbert space. The inner product in H is denoted by $\langle \cdot, \cdot \rangle$. For a closed convex set $C \subset H$ the set

$$N_C(x) = \{\xi \in H : \langle \xi, c - x \rangle \leq 0 \quad \forall c \in C\}$$

denotes the outward normal cone to C at $x \in C$.

Notice that we always have $0 \in N_C(x)$, $N_{\{x\}}(x) = H$, and $N_C(x) = \{0\}$ for $x \in intC$, where $intC$ is the interior of C. The last relation shows that the outward normal cone is nontrivial only for $x \in \partial C$, where ∂C is the boundary of C.

Recall that the Hausdorff distance $d_H(C_1, C_2)$ between closed sets C_1 and C_2 is given by the formula

$$d_H(C_1, C_2) = \max\{\sup_{x \in C_2} \text{dist}(x, C_1), \sup_{x \in C_1} \text{dist}(x, C_2)\},$$

where $dist(x, C) = inf\{\|x - c\|, c \in C\}$.

There are many papers dedicated to sweeping processes (see [1,2,4,7,8,19,21,22]). Consider so called perturbed sweeping process

$$-x'(t) \in N_{C(t)}(x(t)) + f(t, x(t)), \qquad a.e. \quad t \in [T_0, T] \tag{2.1}$$

$$x(T_0) = x_0, \tag{2.2}$$

where $C(\cdot)$ is a set-valued map from $I = [T_0, T]$ to H, $N_{C(t)}(x(t))$ is the outward normal cone to $C(t)$ at $x(t)$.

A function $x : I \to H$ is a solution of Problem (2.1),(2.2) if
(a) $x(T_0) = x_0$;
(b) $x(t) \in C(t)$ for all $t \in I$;
(c) x is differentiable at almost every point $t \in I$;
(d) $-x'(t) \in N_{C(t)}(x(t)) + f(t, x(t))$ for almost every $t \in I$.

Later we will use the next theorem (Theorem 1 in [8]).

Theorem 2.1. *Assume that for each $t \in I$ the set $C(t) \subset H$ is nonempty, closed and convex. The set $C(t)$ varies in an absolutely continuous way, i.e., there exists an absolutely continuous on I function $v(t)$ such that the Hausdorff distance $d_H(C(t), C(s))$ satisfies the inequality*

$$d_H(C(t), C(s)) \leq |v(t) - v(s)|,$$

for any $s, t \in I$. Let $f : I \times H \to H$ be a separately measurable map on I such that (i) for every $\eta > 0$ there exists a non-negative function $k_\eta(\cdot) \in L^1(I, R)$ such that for all $t \in I$ and for any $(x, y) \in B[0, \eta] \times B[0, \eta]$

$$\|f(t, x) - f(t, y)\| \leq k_\eta(t)\|x - y\|,$$

where $B[0, \eta]$ denotes the closed ball of radius r centered at 0;
(ii) there exists a non-negative function $\beta(\cdot) \in L^1(I, R)$ such that for all $t \in I$ and for all $x \in \bigcup_{s \in I} C(s)$ $\|f(t, x)\| \leq \beta(t)(1 + \|x\|)$.
Then for any $x_0 \in C(T_0)$ the perturbed sweeping process (2.1),(2.2) has one and only one absolutely continuous solution $x(\cdot)$.

In the present paper we will consider a case, when the set $C(t)$ is driven by a point $v(t)$, i.e. $C(t) = [-h, h] + v(t)$, where $v(t)$ is an absolutely continuous on I function. Let us show that $d_H(C(t), C(s)) \leq |v(t) - v(s)|$. To see this, fix $x \in C(s)$. Then $y = (x - v(s)) + v(t) \in [-h, h] + v(t) = C(t)$. Thus, $dist(x, C(t)) \leq |v(t) - v(s)|$. Hence, $\sup\limits_{x \in C(s)} dist(x, C(t)) \leq |v(t) - v(s)|$. Similarly, $\sup\limits_{y \in C(t)} dist(y, C(s)) \leq |v(t) - v(s)|$, and we obtain the claim.

3. A string with a hysteresis type boundary value condition

Suppose a string is located along the segment $[0, l]$. Let $u(x, t)$ be a deviation from the equilibrium position at the time t. Assume that the left end of the string is rigidly fixed, i.e., $u(0, t) = 0$. At the point $x = l$ we have a limiter $C(t)$ on the motion of the elastically fixed right end of the string so, that for all $t \geq 0$ we have $u(l, t) \in C(t)$. Notice that we consider the case, when the limiter can move in perpendicular to the axis Ox direction, and its movement is given by $C(t) = [-h, h] + \xi(t)$, where the function $\xi(t)$ is absolutely continuous on $[0, +\infty)$. While $u(l, t)$ is the interior point of the set $C(t)$, we have the condition $u'_x(l, t) + \gamma u(l, t) = 0$, where $\gamma > 0$. If the string reaches the boundary points of the limiter, then the conditions $u(l, t) = h + \xi(t)$, or $u(l, t) = -h + \xi(t)$ are satisfied at a certain moment, respectively.

Suppose the initial string velocity is zero, and the initial string form is determined by the absolutely continuous on $[0, l]$ function $\varphi(x)$, where $\varphi(0) = 0$, $\varphi(l) \in C(0)$, $-\varphi'(l) - \gamma\varphi(l) \in N_{C(0)}(\varphi(l))$.

The mathematical model of such problem can be described as

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}, \\ u(x, 0) = \varphi(x), \\ \dfrac{\partial u}{\partial t}(x, 0) = 0, \\ u(0, t) = 0, \\ u(l, t) \in C(t), \\ -u'_x(l, t) - \gamma u(l, t) \in N_{C(t)}(u(l, t)), \end{cases} \tag{3.1}$$

where $N_{C(t)}u(l, t)$ denotes the outward normal cone at the point $u(l, t)$ to the set $C(t)$. The condition $-u'_x(l, t) - \gamma u(l, t) \in N_{C(t)}(u(l, t))$ means, that if $u(l, t)$ is the interior point of $C(t)$ then $N_{C(t)}u(l, t) = \{0\}$, and we obtain $u'_x(l, t) + \gamma u(l, t) = 0$.

When the right end of the string contacts with the boundary points of the limiter, the reaction force of the support $f(t)$ (from the side of the limiter) occurs, which blocks this end, and we have $-u'_x(l,t) - \gamma u(l,t) = -f(t) \in N_{C(t)}(u(l,t))$, where $N_{C(t)}(u(l,t)) = (-\infty, 0]$, if $u(l,t) = -h + \xi(t)$, and $N_{C(t)}(u(l,t)) = [0, +\infty)$, if $u(l,t) = h + \xi(t)$.

Let Q_T be the rectangle $Q_T = [0 \leq x \leq l] \times [0 \leq t \leq T]$. By a solution of (3.1) we mean a function $u(x,t)$ such that $u(l,t) \in C(t)$ for all $t \geq 0$, the condition $-u'_x(l,t) - \gamma u(l,t) \in N_{C(t)}(u(l,t))$ holds for almost every $t \geq 0$, $u(x,0) = \varphi(x)$ for all $x \in [0,l]$, $u'_t(x,0) = 0$ for almost every $x \in [0,l]$. We suppose for all $T > 0$
(j) the function $u(x,t)$ is continuous in the closed rectangle Q_T;
(jj) for any fixed $t \in [0,T]$ the function $u(x,t)$ is absolutely continuous on $[0,l]$; for any fixed $x \in [0,l]$ the function $u(x,t)$ is absolutely continuous on $[0,T]$; $u'_x(x,t) \in L^1(Q_T)$, $u'_t(x,t) \in L^1(Q_T)$; $u'_x(x,t) \in L^1([0,T])$, $u'_t(x,t) \in L^1([0,l])$.
(jjj) the integral identity

$$\int_0^l \int_0^T u(x,t)[\Psi''_{tt}(x,t) - \Psi''{}_{xx}(x,t)]dxdt + \int_0^l \Psi'_t(x,0)\varphi(x)dx -$$

$$- \int_0^T \Psi(l,t)u'_x(l,t)dt + \int_0^T \Psi'_x(l,t)u(l,t)dt = 0 \tag{3.2}$$

holds for any function $\Psi(x,t) \in C^2(Q_T)$, which satisfies the conditions $\Psi(0,t) = 0$, $\Psi(x,T) = 0$, $\Psi'_t(x,T) = 0$.

Let $\Phi(x)$ be an odd function, coinciding with $\varphi(x)$ on the segment $[0,l]$. On any interval of the form $[(n+1)l, (n+2)l]$ $(n = 0, 1, 2...,)$ the function $\Phi(x)$ is represented as a finite sum

$$2 \cdot \sum_{k=0}^{\frac{n}{2}} g_{2k}(x - (n+1-2k)l) - \varphi((n+2)l - x),$$

when n is an even number, and

$$2 \cdot \sum_{k=1}^{\frac{n+1}{2}} g_{2k-1}(x - (n+2-2k)l) + \varphi(x - (n+1)l),$$

when n is an odd number. Here the functions $g_0(t)$ and $g_1(t)$ are solutions of the problems

$$\begin{cases} -g'_0(t) \in N_{C(t)}(g_0(t)) + \gamma g_0(t) + \varphi'(l-t), t \in [0,l] \\ g_0(0) = \varphi(l), \end{cases}$$

$$\begin{cases} -g'_1(t) \in N_{C(t)}(g_1(t)) + \gamma g_1(t) + \varphi'(t-l), t \in [l,2l] \\ g_1(l) = g_0(l). \end{cases}$$

The functions $g_i(t)$ $(t \in [il, (i+1)l])$ for even numbers $i \geq 2$ are solutions of the problems

$$\begin{cases} -g'_i(t) \in N_{C(t)}(g_i(t)) + \gamma g_i(t) + 2 \sum\limits_{k=0}^{\frac{i-2}{2}} g'_{2k}(t - il + 2kl) + \varphi'(il + l - t) \\ g_i(il) = g_{i-1}(il), \end{cases}$$

and for odd numbers are solutions of the problems

$$\begin{cases} -g_i'(t) \in N_{C(t)}(g_i(t)) + \gamma g_i(t) + 2\sum\limits_{k=1}^{\frac{i-1}{2}} g_{2k-1}'(t-l-il+2kl) + \varphi'(t-il) \\ g_i(il) = g_{i-1}(il). \end{cases}$$

Theorem 3.1. *Let the function $\xi(t)$ be absolutely continuous on $[0,+\infty)$, the function $\varphi(x)$ be absolutely continuous on $[0,l]$ and $\varphi(0)=0$, $\varphi(l)\in C(0)$, $-\varphi'(l)-\gamma\varphi(l)\in N_{C(0)}(\varphi(l))$. Then the solution of Problem (3.1) can be represented as*

$$u(x,t) = \frac{\Phi(x-t)+\Phi(x+t)}{2}. \tag{3.3}$$

Proof. Suppose formally that a solution of Problem (3.1) has form (3.3). Then $u(x,0)=\Phi(x)=\varphi(x)$, where $x\in[0,l]$, and $\Phi(-x)=-\Phi(x)$. We have equalities

$$u_x'(l,t) = \frac{\Phi'(l-t)+\Phi'(l+t)}{2},$$

$$u_t'(l,t) = \frac{-\Phi'(l-t)+\Phi'(l+t)}{2}.$$

Hence,

$$u_x'(l,t) - u_t'(l,t) = \Phi'(l-t),$$

and

$$-u_t'(l,t) \in N_{C(t)}(u(l,t)) + \gamma u(l,t) + \Phi'(l-t). \tag{3.4}$$

Consider the case, when $t\in[0,l]$. Then $\Phi'(l-t)=\varphi'(l-t)$. Let $g_0(t)=u(l,t)$. We have

$$\begin{cases} -g_0'(t) \in N_{C(t)}(g_0(t)) + \gamma g_0(t) + \varphi'(l-t), t\in[0,l] \\ g_0(0) = \varphi(l). \end{cases} \tag{3.5}$$

Let us show that Problem (3.5) has a unique solution. Denote by

$$x(t) = g_0(t) + \int_0^t \varphi'(l-s)ds,$$

$$D(t) = [-h,h] + \xi(t) + \int_0^t \varphi'(l-s)ds.$$

The function $v(t)=\xi(t)+\int_0^t\varphi'(l-s)ds$ is absolutely continuous on $[0,l]$. Hence,

$$d_H(D(t),D(s)) \le |v(t)-v(s)|.$$

Since $N_{D(t)}(x(t)) = N_{C(t)}(g_0(t))$, we obtain the problem

$$\begin{cases} -x'(t) \in N_{D(t)}(x(t)) + \gamma x(t) - \gamma \int\limits_0^t \varphi'(l-s)ds \\ x(0) = \varphi(l). \end{cases} \tag{3.6}$$

Let us apply Theorem 2.1 to Problem (3.6). Here

$$f(t,x(t)) = \gamma x(t) - \gamma\int_0^t \varphi'(l-s)ds.$$

We have

$$|f(t,x)-f(t,y)|=\gamma|x-y|;$$
$$|f(t,x)|\le\beta(1+|x|),$$

where $\beta=\max\{\gamma,\gamma\int\limits_0^l|\varphi'(l-s)|ds\}$. Thus, Problem (3.6) has a unique absolutely continuous solution $x(t)$. Hence, Problem (3.5) has a unique absolutely continuous solution $g_0(t)$, which is defined on the whole interval $[0,l]$. On the other hand, we have

$$\Phi(l-t)+\Phi(l+t)=2g_0(t).$$

Thus, we obtain

$$\Phi(x)=2g_0(x-l)-\varphi(2l-x),$$

where $x\in[l,2l]$. Notice that $\Phi(x)$ is absolutely continuous function on $[l,2l]$, and $\Phi(l-0)=\Phi(l)=\Phi(l+0)=\varphi(l)$.

Consider now a case, when $t\in[l,2l]$. Let $g_1(t)=u(l,t)$. Consider a problem

$$\begin{cases}-g_1'(t)\in N_{C(t)}(g_1(t))+\gamma g_1(t)+\varphi'(t-l), t\in[l,2l]\\ g_1(l)=g_0(l).\end{cases}$$

Similarly, we obtain that the last problem has a unique solution $g_1(t)$. Then we define $\Phi(x)$, where $x\in[2l,3l]$ as

$$\Phi(x)=2g_1(x-l)+\varphi(x-2l).$$

Notice that $\Phi(2l)=\Phi(2l-0)=\Phi(2l+0)$.

Thus, for $n=0,1$ the representation of $\Phi(x)$ is true. Let it hold for $n\le m$. Let us show, that for $n=m+1$ the representation of $\Phi(x)$ is true. Assume that $m+1$ is an even number. Let us show, that for all $x\in[(m+2)l,(m+3)l]$

$$\Phi(x)=2\cdot\sum_{k=0}^{\frac{m+1}{2}}g_{2k}(x-(m+2-2k)l)-\varphi((m+3)l-x).$$

Let $u(l,t)=g_{m+1}(t)$, where $t\in[(m+1)l,(m+2)l]$. Using (3.4), we have

$$-g_{m+1}'(t)\in N_{C(t)}(g_{m+1}(t))+\gamma g_{m+1}(t)+\Phi'(l-t).$$

Since $\Phi(l-t)=-\Phi(t-l)$, according to the inductive assumption we have

$$\Phi(l-t)=-2\cdot\sum_{k=0}^{\frac{m-1}{2}}g_{2k}(t-l-(m-2k)l)+\varphi((m+1)l-(t-l)),$$

and

$$\Phi'(l-t)=2\cdot\sum_{k=0}^{\frac{m-1}{2}}g_{2k}'(t-l-(m-2k)l)+\varphi'((m+1)l-(t-l)).$$

We obtain

$$\begin{cases}-g_{m+1}'(t)\in N_{C(t)}(g_{m+1}(t))+\gamma g_{m+1}(t)\\ \qquad +2\sum\limits_{k=0}^{\frac{m-1}{2}}g_{2k}'(t-l-(m-2k)l)+\varphi'(ml+2l-t)\\ g_{m+1}((m+1)l)=g_m((m+1)l),\end{cases}$$

where (similar to Problem (3.5)) the solution of last problem exists, is unique and is the absolutely continuous function on $[(m+1)l,(m+2)l]$. Thus,

$$\Phi(l-t)+\Phi(l+t)=2g_{m+1}(t).$$

Hence,

$$\begin{aligned}\Phi(l+t)&=2g_{m+1}(t)-\Phi(l-t)\\&=2g_{m+1}(t)+2\sum_{k=0}^{\frac{m-1}{2}}g_{2k}(t-l-(m-2k)l)-\varphi((m+1)l-(t-l)).\end{aligned}$$

Denote by $l+t=x$. We obtain

$$\Phi(x)=2\cdot\sum_{k=0}^{\frac{m+1}{2}}g_{2k}(x-(m+2-2k)l)-\varphi((m+3)l-x),$$

where $x\in[(m+2)l,(m+3)l]$. The case, when $m+1$ is an odd number, can be considered similarly.

Let us show, that the function $u(x,t)$, defined by (3.3) is a solution of (3.1). Since $\Phi(x)$ is absolutely continuous, we have that the function $u(x,t)$ satisfies the conditions (j)-(jj). The conditions $u(x,0)=\varphi(x)$, $u'_t(x,0)=0$ are obviously satisfied. Let us show, that for all t we have $u(l,t)\in C(t)$. If $t\in[0,l]$ then

$$u(l,t)=\frac{\varphi(l-t)+\Phi(l+t)}{2}=g_0(t)\in C(t).$$

Let t belong to $[(n+1)l,(n+2)l]$. Assume that n is an even number. Then

$$\Phi(l-t)=-2\cdot\sum_{k=1}^{\frac{n}{2}}g_{2k-1}(t-2l-nl+2kl)-\varphi(t-l-nl),$$

$$\Phi(l+t)=2\cdot\sum_{k=1}^{\frac{n+2}{2}}g_{2k-1}(t-2l-nl+2kl)+\varphi(t-l-nl).$$

Thus,

$$u(l,t)=\frac{\Phi(l-t)+\Phi(l+t)}{2}=g_{n+1}(t)\in C(t).$$

The case, when n is an odd number, can be considered similarly.

For the case, when n is an even number, let us show that for almost every $t\in[(n+1)l,(n+2)l]$

$$-u'_x(l,t)-\gamma u(l,t)\in N_{C(t)}(u(l,t)).$$

We have

$$u'_x(l,t)=\frac{\Phi'(l-t)+\Phi'(l+t)}{2},$$

$$\Phi'(l-t)=2\cdot\sum_{k=1}^{\frac{n}{2}}g'_{2k-1}(t-2l-nl+2kl)+\varphi'(t-l-nl),$$

$$\Phi'(l+t) = 2 \cdot \sum_{k=1}^{\frac{n+2}{2}} g'_{2k-1}(t-2l-nl+2kl) + \varphi'(t-l-nl).$$

Thus,

$$\frac{\Phi'(l-t)+\Phi'(l+t)}{2} = \varphi'(t-l-nl) + 2 \cdot \sum_{k=1}^{\frac{n}{2}} g'_{2k-1}(t-2l-nl+2kl) + g'_{n+1}(t).$$

Since $u(l,t) = g_{n+1}(t)$, we obtain the claim. Other cases can be investigated similarly.

Our aim now is to prove the integral equality. We have

$$\begin{aligned}
&\int_0^l \left(\int_0^T u(x,t)\Psi''_{tt}(x,t)dt \right) dx - \int_0^T \left(\int_0^l u(x,t)\Psi''_{xx}(x,t)dx \right) dt \\
&\quad + \int_0^l \Psi'_t(x,0)\varphi(x)dx - \int_0^T \Psi(l,t)u'_x(l,t)dt + \int_0^T \Psi'_x(l,t)u(l,t)dt \\
&= \int_0^l \big(u(x,T)\Psi'_t(x,T) - u(x,0)\Psi'_t(x,0) \big)\, dx - \int_0^l \int_0^T u'_t(x,t)\Psi'_t(x,t)dtdx \\
&\quad - \int_0^T \big(\Psi'_x(l,t)u(l,t) - \Psi'_x(0,t)u(0,t) \big)\, dt \\
&\quad + \int_0^T \int_0^l u'_x(x,t)\Psi'_x(x,t)dxdt + \int_0^l \Psi'_t(x,0)\varphi(x)dx \\
&\quad - \int_0^T \Psi(l,t)u'_x(l,t)dt + \int_0^T \Psi'_x(l,t)u(l,t)dt.
\end{aligned}$$

We need to prove that

$$\int_0^T \int_0^l u'_x(x,t)\Psi'_x(x,t)dxdt - \int_0^T \int_0^l u'_t(x,t)\Psi'_t(x,t)dxdt = \int_0^T \Psi(l,t)u'_x(l,t)dt.$$

According to (3.3), we obtain that

$$\begin{aligned}
&\frac{1}{2}\int_0^T \int_0^l (\Phi'(x-t)+\Phi'(x+t))\Psi'_x(x,t)dxdt - \frac{1}{2}\int_0^l \int_0^T (\Phi'(x+t) \\
&\quad - \Phi'(x-t))\Psi'_t(x,t)dtdx \\
&= \frac{1}{2}\int_0^l \big(\Psi'_x(x,T)(\Phi(x+T)-\Phi(x-T)) - \Psi'_x(x,0)(\Phi(x)-\Phi(x)) \big)\, dx \\
&\quad - \frac{1}{2}\int_0^l \int_0^T (\Phi(x+t)-\Phi(x-t))\Psi''_{xt}(x,t)dtdx \\
&\quad - \frac{1}{2}\int_0^T \big(\Psi'_t(l,t)(\Phi(l+t)-\Phi(l-t)) - \Psi'_t(0,t)(\Phi(t)-\Phi(-t)) \big)\, dt \\
&\quad + \frac{1}{2}\int_0^l \int_0^T (\Phi(x+t)-\Phi(x-t))\Psi''_{xt}(x,t)dtdx \\
&= -\frac{1}{2}\int_0^T \Psi'_t(l,t)(\Phi(l+t)-\Phi(l-t))dt
\end{aligned}$$

On the other hand,

$$\int_0^T \Psi(l,t)u'_x(l,t)dt = \frac{1}{2}\int_0^T \Psi(l,t)(\Phi'(l-t)+\Phi'(l+t))dt$$
$$= \frac{1}{2}(\Psi(l,T)(\Phi(l+T)-\Phi(l-T)) - \Psi(l,0)(\Phi(l)-\Phi(l)))$$
$$-\frac{1}{2}\int_0^T (\Phi(l+t)-\Phi(l-t))\Psi'_t(l,t)dt.$$

This completes the proof of the theorem. □

Remark 3.2. Notice that Problem (3.1) has a unique solution. Assume that $\varphi(l) \in (-h+\xi(0), h+\xi(0))$. Then the oscillation process occurs as for the string with an elastic support for all $t \in [0, t_1]$, and the string form is the solution of the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}, \quad 0<x<l, 0<t<t_1 \\ u(x,0) = \varphi(x), \\ \dfrac{\partial u}{\partial t}(x,0) = 0, \\ u(0,t) = 0, \\ u'_x(l,t) + \gamma u(l,t) = 0. \end{cases}$$

The last problem has a unique solution $u(x,t)$. If the relation $u(l,t_1) = \pm h + \xi(t)$ holds at the moment t_1 then for all $t \in [t_1, t_2]$ a string form is a solution of the problem

$$\begin{cases} \dfrac{\partial^2 v}{\partial x^2} = \dfrac{\partial^2 v}{\partial t^2}, \quad 0<x<l, t_1<t<t_2 \\ v(x,t_1) = u(x,t_1), \\ \dfrac{\partial v}{\partial t}(x,t_1) = u'_t(x,t_1), \\ v(0,t) = 0, \\ v(l,t) = -h + \xi(t) \end{cases}$$

or

$$\begin{cases} \dfrac{\partial^2 v}{\partial x^2} = \dfrac{\partial^2 v}{\partial t^2}, \quad 0<x<l, t_1<t<t_2 \\ v(x,t_1) = u(x,t_1), \\ \dfrac{\partial v}{\partial t}(x,t_1) = u'_t(x,t_1), \\ v(0,t) = 0, \\ v(l,t) = h + \xi(t). \end{cases}$$

Each of the above problems have a unique solution for every $t \in [t_1, t_2]$. By a similar reasoning, we obtain that the original problem has a unique solution.

4. A boundary control problem

Consider the problem

$$\left\{\begin{array}{l} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}, \\ u(x,0) = \varphi(x), \\ \dfrac{\partial u}{\partial t}(x,0) = 0, \\ u(0,t) = \mu(t), \\ u(l,t) \in C(t), \\ -u'_x(l,t) - \gamma u(l,t) \in N_{C(t)}(u(l,t)), \end{array}\right. \tag{4.1}$$

where $C(t) = [-h,h] + \xi(t)$; $\xi(t)$ is absolutely continuous function such that $|\xi(t)| \leq h$ for every $t \in [0,T]$; φ is absolutely continuous on $[0,l]$, $\varphi(0) = 0$, $\varphi(l) \in C(0)$, $-\varphi'(l) - \gamma\varphi(l) \in N_{C(0)}(\varphi(l))$; μ is absolutely continuous on $[0,T]$, $\mu(0) = 0$.

By a solution of this problem we mean a function $u(x,t)$ satisfying (j)-(jj), satisfying the condition $u(l,t) \in C(t)$ for all $t \in [0,T]$; the condition $-u'_x(l,t) \in N_{C(t)}(u(l,t))$ for almost all $t \in [0,T]$; $u(x,0) = \varphi(x)$ for all $x \in [0,l]$; $u(0,t) = \mu(t)$ for all $t \in [0,T]$; $\dfrac{\partial u}{\partial t}(x,0) = 0$ for almost every $x \in [0,l]$ and such, that the integral identity

$$\int_0^l \int_0^T u(x,t)[\Psi''_{tt}(x,t) - \Psi''_{xx}(x,t)]dxdt + \int_0^l \Psi'_t(x,0)\varphi(x)dx -$$

$$- \int_0^T \Psi(l,t)u'_x(l,t)dt + \int_0^T \Psi'_x(l,t)u(l,t)dt - \int_0^T \Psi'_x(0,t)\mu(t)dt = 0 \tag{4.2}$$

holds for every $\Psi(x,t) \in C^2(Q_T)$, satisfying conditions $\Psi(0,t) = 0$, $\Psi(x,T) = 0$, $\Psi_t(x,T) = 0$. The boundary control problem is to find a function $\mu(t)$, such that at the time T the equalities

$$u(x,T) = \varphi^*(x), \quad u'_t(x,T) = \psi^*(x)$$

hold, where φ^*, ψ^* are the given functions, where φ^* is absolutely continuous function on $[0,l]$, $\psi^* \in L_1[0,l]$. Assume that $T < l$. Consider the problem

$$\left\{\begin{array}{l} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}, \\ u(x,0) = 0, \\ \dfrac{\partial u}{\partial t}(x,0) = 0, \\ u(0,t) = \mu(t), \\ u(l,t) \in C(t), \\ -u'_x(l,t) - \gamma u(l,t) \in N_{C(t)}(u(l,t)). \end{array}\right. \tag{4.3}$$

Let us show, that the function $u(x,t) = \underline{\mu}(t-x)$, where

$$\underline{\mu}(t) = \left\{\begin{array}{l} \mu(t), t \geq 0 \\ 0, t < 0 \end{array}\right.$$

is the solution of Problem (4.3).

Notice that since $|\xi(t)| \leq h$, we have $u(l,t) = 0 \in C(t)$. Thus, $u'_x(l,t) - \gamma u(l,t) = 0 \in N_{C(t)}(0)$. Let us verify Equality (4.2), where $\varphi(x) = 0$. We have

$$\begin{aligned}
&\int_0^l \left(\int_0^T u(x,t)\Psi''_{tt}(x,t)dt\right)dx - \int_0^T \left(\int_0^l u(x,t)\Psi''_{xx}(x,t)dx\right)dt \\
&\quad - \int_0^T \Psi(l,t)u'_x(l,t)dt + \int_0^T \Psi'_x(l,t)u(l,t)dt - \int_0^T \Psi'_x(0,t)\mu(t)dt \\
&= - \int_0^l \int_0^T u'_t(x,t)\Psi'_t(x,t)dtdx - \int_0^T \left(\Psi'_x(l,t)u(l,t) - \Psi'_x(0,t)u(0,t)\right)dt \\
&\quad + \int_0^T \int_0^l u'_x(x,t)\Psi'_x(x,t)dxdt - \int_0^T \Psi(l,t)u'_x(l,t)dt \\
&\quad + \int_0^T \Psi'_x(l,t)u(l,t)dt - \int_0^T \Psi'_x(0,t)\mu(t)dt \\
&\quad - \int_0^l \int_0^T \underline{\mu}'(t-x)\Psi'_t(x,t)dtdx - \int_0^T \int_0^l \underline{\mu}'(t-x)\Psi'_x(x,t)dxdt \\
&= - \int_0^l (\int_0^T \underline{\mu}'(t-x)\Psi'_x(x,t)dt)dx - \int_0^T (\int_0^l \underline{\mu}'(t-x)\Psi'_t(x,t)dx)dt \\
&= - \int_0^l (\Psi'_x(x,T)\underline{\mu}(T-x) - \Psi'_x(x,0)\underline{\mu}(-x))dx \\
&\quad + \int_0^l \int_0^T \underline{\mu}(t-x)\Psi''_{xt}(x,t)dtdx + \int_0^T \Psi'_t(l,t)\underline{\mu}(t-l)dt \\
&\quad - \int_0^T \Psi'_t(0,t)\underline{\mu}(t)dt - \int_0^l \int_0^T \underline{\mu}(t-x)\Psi''_{xt}(x,t)dtdx = 0,
\end{aligned}$$

that is required.

Let $v(x,t)$ $(0 \leq x \leq l, 0 \leq t \leq T)$ be the solution of the problem

$$\begin{cases}
\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}, \\
u(x,0) = \varphi(x), \\
\dfrac{\partial u}{\partial t}(x,0) = 0, \\
u(0,t) = 0, \\
u(l,t) \in C(t), \\
-u'_x(l,t) - \gamma u(l,t) \in N_{C(t)}(u(l,t)).
\end{cases}$$

Similarly, we can verify, that the function $u(x,t) = \underline{\mu}(t-x) + v(x,t)$ is the solution of Problem (4.1). Then

$$\underline{\mu}(T-x) = \varphi^*(x) - v(x,T) = \varphi_1(x), \tag{4.4}$$

$$\underline{\mu}'(T-x) = \psi^*(x) - v'_t(x,T) = \psi_1(x). \tag{4.5}$$

Taking the derivative in x in (4.4), we obtain

$$\varphi'_1(x) = -\underline{\mu}'(T-x), \tag{4.6}$$

which implies

$$\psi_1(x) + \varphi_1'(x) = 0. \tag{4.7}$$

Fixing an arbitrary number t_0 from $[0, l]$, denoting by t an arbitrary point of this interval and integrating (4.7) in x in limits from t_0 to t we obtain the equality

$$\widehat{\psi_1}(t) - \widehat{\psi_1}(t_0) + \varphi_1(t) - \varphi_1(t_0) = 0, \tag{4.8}$$

which is true for all $t \in [0, l]$. Here $\widehat{\psi_1}(t)$ is an arbitrary primitive of the function $\psi_1(t)$. We require that this primitive $\widehat{\psi_1}(t)$ satisfies the condition

$$\widehat{\psi_1}(t_0) + \varphi_1(t_0) = 0. \tag{4.9}$$

Thus, we have

$$\widehat{\psi_1}(t) + \varphi_1(t) = 0 \tag{4.10}$$

for all $t \in [0, l]$. Notice that for all $x \in [T, l]$ the equality

$$\psi_1(x) - \varphi_1'(x) = 0 \tag{4.11}$$

holds. Let t_0 be an arbitrary fixed number from $[T, l]$, and t is any number from this interval. Integrating (4.11) in x in limits from t_0 to t, we find that identity

$$\widehat{\psi_1}(t) - \widehat{\psi_1}(t_0) - \varphi_1(t) + \varphi_1(t_0) = 0 \tag{4.12}$$

is true for all $t \in [T, l]$, where $\widehat{\psi_1}(t)$ is an arbitrary primitive of the function $\psi_1(t)$. From equalities (4.7) and (4.11) it follows that $\varphi_1(x) \equiv C = const$ on the segment $[T, l]$, and since $\varphi_1(l) = \underline{\mu}(T - l) = 0$, then $\varphi_1(x) = 0$ for all $x \in [T, l]$. In particular, for any $t_0 \in [T, l]$

$$\varphi_1(t_0) = 0. \tag{4.13}$$

If now we require, that the primitive $\widehat{\psi_1}(t)$ in (4.12) satisfies the relation

$$\widehat{\psi_1}(t_0) - \varphi_1(t_0) = 0 \tag{4.14}$$

for some arbitrary fixed $t_0 \in [T, l]$, then from (4.12) we obtain the equality

$$\widehat{\psi_1}(t) - \varphi_1(t) \equiv 0, \quad t \in [T, l]. \tag{4.15}$$

It remains to note, that according to (4.13), relations (4.9) and (4.14) are equivalent. Therefore, we obtain the validity of (4.10) and (4.15), where the symbol $\widehat{\psi_1}(t)$ denotes the primitive of function $\psi_1(t)$, satisfying

$$\widehat{\psi_1}(t_0) - \varphi_1(t_0) = 0 \tag{4.16}$$

for arbitrary fixed $t_0 \in [T, l]$. Thus, Equalities (4.10), (4.15), (4.16) can be rewritten as

$$\widehat{\psi^*}(t) - \varphi^*(t) + \Phi(t - T) \equiv 0, \quad T \le t \le l,$$
$$\widehat{\psi^*}(t) + \varphi^*(t) - \Phi(t + T) \equiv 0, \quad 0 \le t \le l,$$
$$\widehat{\psi^*}(t_0) - \varphi^*(t_0) + \Phi(t_0 - T) = 0,$$

where $t_0 \in [T, l]$ is fixed. Using Theorem 3.1, we obtain the following connections between the initial and final data

$$\widehat{\psi^*}(t) - \varphi^*(t) + \varphi(t - T) \equiv 0, \quad T \le t \le l,$$

$$\widehat{\psi^*}(t_0) - \varphi^*(t_0) + \varphi(t_0 - T) = 0, t_0 \in [T, l],$$
$$\widehat{\psi^*}(t) + \varphi^*(t) - \varphi(t + T) \equiv 0, \quad 0 \le t \le l - T,$$
$$\widehat{\psi^*}(t) + \varphi^*(t) - 2g_0(T + t - l) + \varphi(2l - t - T) \equiv 0, \quad l - T \le t \le l.$$

At the same time, the required boundary control has the form

$$\mu(t) = \frac{1}{2}(\varphi(t) - \widehat{\psi^*}(T - t) + \varphi^*(T - t)),$$

that solves the problem.

Acknowledgements

This research was supported by the Ministry of Science and Higher Education of the Russian Federation within the framework of the state task in the field of science (topic number FZGF-2020-0009), RFBR and CNRS, project number 20-51-15003.

References

[1] L. Adam and J. Outrata, *On optimal control of a sweeping process coupled with an ordinary differential equation*, Discrete Contin. Dyn. Syst. Ser B. **19** (2014), 2709–2738.

[2] S. Adly and B. K. Le, *Unbounded second-order state-dependent Moreau's sweeping processes in Hilbert spaces*, J. Optim. Theory Appl. **169** (2016), 407–423.

[3] A. V. Borovskikh, *Formulas of boundary control of an inhomogeneous string*, Differ. Equ. **43** (2007), 69–95.

[4] M. Brokate and P. Krejci, *Optimal control of ODE systems involving a rate independent variational inequality*, Discrete Contin. Dyn. Syst. Ser B. **18** (2013), 331–348.

[5] F. Bullo, *Averaging and vibrational control of mechanical systems*, SIAM J. Control Optim. **41** (2002), 542–562.

[6] G. Chen, *Control and stabilization for the wave equations in a bounded domain*, SIAM Control Optim. **17** (1979), 66–81.

[7] C. Castaing and M. Monteiro Marques, *BV periodic solutions of an evolution problem associated with continuous moving convex sets*, Set-Valued Anal. **3** (1995), 381–399.

[8] J. F. Edmond and L. Thibault, *Relaxation of an optimal control problem involving a perturbed sweeping process*, Math. Program. Ser. B. **104** (2005), 347–373.

[9] A. I. Egorov and L. N. Znamenskaya, *On boundary observability of elastic vibrations of connected objects with distributed and lumped parameters*, Autom. Remote Control. **68** (2007), 296–302.

[10] V. A. Il'in, *Two-end point boundary control of vibrations described by a finite energy generalized solution of the wave equation*, Differ. Equ. **36** (2000), 1659–1675.

[11] V. A. Il'in, *Boundary control of oscillations at one endpoint with the other endpoint fixed in terms of a finite-energy generalized solution of the wave equation*, Differ. Equ. **36** (2000), 1832–1849.

[12] V. A. Il'in and E. I. Moiseev, *Optimization of boundary controls of string vibrations*, Russ. Math. Surv. **60** (2005), 1093–1119.

[13] M. Kamenskii, Ch-F. Wen, Zh. Zalukaeva and M. Zvereva, *The influence function properties for a problem with discontinuous solutions*, Appl. Anal. Optim. **1** (2017), 259–281.

[14] M. Kamenskii, Ch.-F. Wen and M. Zvereva, *A string oscillations simulation with boundary conditions of hysteresis type*, Optimization **6** (2018), 1321–1332.

[15] M. Kamenskii, Y.-Ch. Liou, Ch.-F. Wen and M. Zvereva, *On a hyperbolic equation on a geometric graph with hysteresis type boundary conditions*, Optimization **69** (2020), 283–304.

[16] M. Kamenskii, Ch.-F. Wen and M. Zvereva, *On a variational problem for a model of a Stieltjes string with a backlash at the end*, Optimization **69** (2020), 1935–1959.

[17] M. Kamenskii, N. Voskovskaya and M. Zvereva, *On periodic oscillations of some points of a string with a nonlinear boundary condition*, Appl. Set-Valued Anal. Optim. **2** (2020), 35–48.

[18] M. Kamenskii, Ch.-F. Wen and M. Zvereva, *Oscillations of the string with singuliarities*, Journal of nonlinear and convex analysis. **20** (2019), 1525–1545.
[19] M. Kamenskii and O. Makarenkov, *On the response of autonomous sweeping processes to periodic perturbations*, Set-Valued Variational Anal. **24** (2000), 551–563.
[20] V. Komornik and E. Zuazua, *A direct method for boundary stabilisation of the wave equation*, J. Math. Pures et Appl. **69** (1990), 33–54.
[21] P. Krejci and T. Rochem, *Lipschitz continuous data dependence of sweeping processes in BV spaces*, Discrete Contin. Dyn. Syst. Ser B. **15** (2011), 637–650.
[22] M. Kunze and M. Monteiro Marques, *An Introduction to Moreau's Sweeping Process*, LNP 551. Berlin, Heidelberg: Springer-Verlag, 2000.
[23] M. Zvereva, *A string oscillations simulation with nonlinear conditions*, Memoirs on Differential Equations and Mathematical Physics **72** (2017), 141–150.
[24] T. Taniguchi, *Exponential boundary stabilisation for nonlinear wave equations with localized damping and nonlinear boundary condition*, Commun. Pure Appl. Anal. **16** (2017), 1571–1585.
[25] N.A. Tikhonov and A. A. Samarskii, *Equations of MathematicalPphysics*, Courier Corporation, New York, 2003.

Manuscript received February 28 2021
revised March 18 2021

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