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REMARKS ON THE PERIODIC PROBLEM FOR SEMILINEAR FUNCTIONAL DIFFERENTIAL INCLUSIONS WITH DELAY IN BANACH SPACES

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Abstract. In this paper we give a more complete approach for the study of the periodic problem for semilinear functional differential inclusions with finite delay. Our approach allows to have a clear idea on the structure of the set of all T-periodic mild solutions and, it is more adapted to applications. In particular, we explain why the approaches which were developed through the use of the translation operator are too restrictive. Finally, to validate our approach an existence result is established.

1. Introduction

In this paper we give a more complete approach for the study of the periodic problem for a semilinear functional differential inclusion with finite delay in a Banach space, of the form

$$x'(t) \in Ax(t) + F(t, x_t), \ t \geq 0,$$

where A is a linear not necessarily bounded operator which generates a C_0-semigroup, F is a multifunction with convex compact values which is T-periodic on the first argument and, satisfies some a χ-regularity and a boundedness conditions and x_t is a function which represents the history of the state from the time $t - r$ up to the time t.

The periodic problem described above was studied by developing several methods, the method of the translation operator along the trajectories of solutions was the most used (whether for single and multivalued cases), see e.g., [3, 7, 8, 11, 12]. More details on the topic can be found in e.g, [13] and [15].

It should be noted that in previous work, the study of the periodic problem for semilinear differential equations with delay, described above, is equivalent to the study of the periodic boundary value problem:

$$\begin{cases} x'(t) \in A\,x(t) + F(t, x_t), & t \in [0, T], \\ x_0 = x_T. \end{cases}$$

Our aim in this paper is to develop a more complete approach for the study of the periodic problem for semilinear differential equations with delay. More precisely:

(1) We prove that the periodic boundary value problem described above is too restrictive for the study of the periodic problem.
(2) We give a more complete formulation which allows to give a more complete definition of a periodic mild solutions.

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(3) We explain why our approach remains valid for semilinear functional differential equations where the initial memories are not necessarily observed in the space of continuous functions, that is, the space where the initial memories are observed can be of a very different nature from where the system evolves, in particular, we explain why our approach remains valid in the case of infinite delay (without giving results).
(4) We give two illustrations that explain, in general, why our approach is more complete and effective compared to those that were used before from the point of view of applications.
(5) To validate our approach, we establish an existence result.

The above mentioned points (1) − (4) will be considered in Section 3. In Section 4 we consider the point (5).

2. Preliminaries

Multivalued maps and measures of noncompactness. Let X, Y be two topological vector spaces. We denote by $\mathcal{P}(Y)$ the family of all nonempty subsets of Y and by $K(Y)$ (resp. $\mathcal{K}v(Y)$) we denote the collection of all nonempty compact (resp. nonempty compact convex) subsets of X.

A multivalued map $F : X \to \mathcal{P}(Y)$ is said to be: (i) upper semicontinuous (u.s.c) if $F^{-1}(O) = \{x \in X : F(x) \subset O\}$ is an open subset of X for every open $O \subset Y$;

(ii) closed if its graph $\Gamma_F = \{(x, y) \in X \times Y : y \in F(x)\}$ is a closed subset of $X \times Y$;

(iii) compact if $\overline{F(X)}$ is compact in Y;

Let $\mathcal{Z}$ be a Banach space and $(\mathcal{O}, \leq)$ a partially ordered set.

A multifunction $\digamma : [0, T] \to K(\mathcal{Z})$ is said to be strongly measurable if there exists a sequence $\{\digamma_n\}_{n=1}^{\infty}$ of step multifunctions such that

$Haus\,(\digamma(t), \digamma_n(t)) \to 0$ as $n \to \infty$ for $\mu - a.e.\ t \in [0, T]$ where μ denotes a Lebesgue measure on $[0, T]$ and $Haus$ is the Hausdorff metric on $K(\mathcal{Z})$.
Every strongly measurable multivalued map $\digamma$ admits a strongly measurable selection f i.e., $f : [0, T] \to \mathcal{Z}$ measurable such that $f(t) \in \digamma(t)$ for a.e. $t \in [0, T]$.

A function $\Psi : \mathcal{P}(\mathcal{Z}) \to \mathcal{O}$ is called a measure of noncompactness in E if

$$\Psi(\Omega) = \Psi(\bar{co}\,\Omega)$$

for every $\Omega \subset \mathcal{P}(E)$, where $\bar{co}\,\Omega$ denotes the closed convex hull of Ω.

The measure Ψ is called :

(i) nonsingular if for every $a \in \mathcal{Z}$, $\Omega \in \mathcal{P}(\mathcal{Z})$, $\Psi(\{a\} \cup \Omega) = \Psi(\Omega)$;

(ii) monotone, if Ω_0, $\Omega_1 \in \mathcal{P}(\mathcal{Z})$ and $\Omega_0 \subseteq \Omega_1$ imply $\Psi(\Omega_0) \leq \Psi(\Omega_1)$;

(iii) If $\mathcal{O}$ is a cone in a Banach space we will say that Ψ is regular if $\Psi(\Omega) = 0$ is equivalent to the relative compactness of the set Ω.

One of most important example of a measure of noncompactness possessing all these properties is the Hausdorff measure of noncompactness defined by:

$$\chi(\Omega) = \inf\{\varepsilon > 0; \Omega \text{ has a finite } \varepsilon\text{-net in } \mathcal{Z}\}.$$

Let $\Psi : \mathcal{P}(\mathcal{Z}) \to (Y, \leq)$ be a measure of noncompactness in $\mathcal{Z}$. Let $Z \subset \mathcal{Z}$ be a closed subset. A multifunction $G : Z \to K(\mathcal{Z})$ is called Ψ-condensing, if for every bounded set $\Omega \subset Z$, the relation $\Psi(G(\Omega)) \geq \Psi(\Omega)$ implies the relative compactness of Ω.

In the proof of existence result will need the following results [9, Corollary 3.3.1 and Proposition 3.5.1].

Theorem 2.1. *Let M be a convex closed subset of $\mathcal{Z}$. If $\mathcal{G} : M \to \mathcal{K}v(M)$ is a closed β-condensing multimap, where β is a nonsingular measure of noncompactness defined on subsets of $\mathcal{Z}$, then the fixed points set $Fix\,\mathcal{G} = \{x : x \in \mathcal{G}(x)\}$ is nonempty.*

Proposition 2.2. *Let $\mathcal{C} \subset \mathcal{Z}$ be a closed subset and $\mathcal{F} : \mathcal{C} \to \mathcal{K}(\mathcal{Z})$ a closed β-condensing multifunction, where β is a monotone measure of noncompactness. If the fixed point $Fix\,\mathcal{F}$ is bounded then it is compact.*

The Cauchy operator and its properties. By the symbol $L^1([0,T];\mathcal{Z})$ we denote the space of all Bochner summable functions equipped with the usual norm.

Definition 2.3. A sequence $\{f_n\}_{n=1}^{\infty} \subset L^1([0,T];\mathcal{Z})$ is semicompact if:
(i) it is integrably bounded: $\|f_n(t)\| \leq p(t)$ for a.e. $t \in [0,T]$ and for every $n \geq 1$ where $p(.) \in L^1([0,T],\mathbb{R}^+)$;
(ii) the set $\{f_n(t)\}_{n=1}^{\infty}$ is relatively compact for almost every $t \in [0,T]$.

Lemma 2.4 ([2]). *Any semicompact sequence in $L^1([0,T];\mathcal{Z})$ is weakly compact in $L^1([0,T];E)$.*

Let $A : \mathcal{D}(A) \subset \mathcal{Z} \to \mathcal{Z}$ be a linear operator generating a C_0-semigroup, $(\mathcal{T}(t))_{t\geq 0}$. Then, the Cauchy operator $S : \mathcal{Z} \times L^1([0,T];\mathcal{Z}) \longrightarrow C([0,T];\mathcal{E})$, such that $S(u,g)$ stands for the unique mild solution to the Cauchy problem

$$\begin{cases} x'(t) = Ax(t) + g(t), \ t \in [0,T], \\ x(0) = u, \end{cases}$$

is well defined. Moreover, by means of the variation of constants formula, $S(\cdot,\cdot)$ can be expressed explicitly by

$$S(u,g)(t) = \mathcal{T}(t)u + \int_0^t \mathcal{T}(t-s)\,g(s)\,ds. \tag{2.1}$$

Remark 2.5. Since $(\mathcal{T}(t))_{t\geq 0}$ is a strongly continuous semigroup, then there exists $D > 0$ such that,

$$\|\mathcal{T}(t)\|_{\mathcal{Z}} \leq D \quad \text{for all } t \in [0,T].$$

(see e.g. [1, Theorem 1.3.1]).

The basic properties of the operator $S(\cdot,\cdot)$ can be summarized in the following two lemmas. For the first lemma see [9, Lemma 4.2.1], for the second see [9, Lemma 4.2.1, Theorem 4.2.2].

Lemma 2.6. *Assume that $(\mathcal{T}(t))_{t\geq 0}$ is a C_0- semigroup. Then, for every semicomapct sequence $\{f_n\}_{n=1}^{\infty} \subset L^1([0,T];\mathcal{Z})$, the set $\{S(u,f_n)\}_{n=1}^{\infty}$ is relatively compact in $C([0,T];\mathcal{Z})$, for every u in $\mathcal{Z}$.*

Lemma 2.7. *Assume that* $(\mathcal{T}(t))_{t\geq 0}$ *is a* C_0*-semigroup. Let* $\{f_n\}_{n=1}^{\infty}$ *be an integrably bounded sequence in* $L^1([0,T],\mathcal{Z})$ *such that* $\chi(\{f_n(t)\}_{n=1}^{\infty}) \leq \zeta(t)$ *a.e.* $t \in [0,T]$, *where* $\zeta \in L^1_+([0,T]$. *Then, for every* $u \in \mathcal{Z}$,

$$\chi_{\mathcal{E}}(S(u,\{f_n(t)\}_{n=1}^{\infty})) \leq 2\,D\int_0^t \zeta(s)ds; \tag{2.2}$$

where D *has been defined in Remark 2.5.*

Remark 2.8. If $\mathcal{E}$ is separable, then (2.2) has the form

$$\chi_{\mathcal{Z}}(S(u,\{f_n(t)\}_{n=1}^{\infty})) \leq \ D\int_0^t \zeta(s)ds.$$

Using Lemma 2.6 and following the same lines as the proof of [5, Lemma 4.5] with almost obvious modifications, we obtain,

Lemma 2.9. *Assume that* $(\mathcal{T}(t))_{t\geq 0}$ *is a* C_0*- semigroup. Then, for every relatively compact subset* K *of* $\mathcal{Z}$ *and every semicomapct sequence* $\{f_n\}_{n=1}^{\infty} \subset L^1([0,T];\mathcal{Z})$, *the set* $\{S(K,f_n)\}_{n=1}^{\infty}$ *is relatively compact in* $C([0,T];\mathcal{Z})$. *Moreover, if* $f_n \underset{w}{\rightarrow} f_0$ *in* $L^1([0,T];\mathcal{Z})$ *and* $u_n \to u^0$ *in* E, *then* $S(u_n,f_n) \to S(u^0,f_0)$ *in* $C([0,T];\mathcal{Z})$.

Using Lemma 2.7 and following the same lines as the proof of [5, Lemma 4.4] with almost obvious modifications, we get,

Lemma 2.10. *Assume that* $(\mathcal{T}(t))_{t\geq 0}$ *is a* C_0*- semigroup such that, for every* $t \in [0,T]$, $\|\mathcal{T}(t)\| \leq q\,e^{-p\,t}$, *for some constants* $q \geq 1$ *and* $p > 0$. *Let* $\{f_n\}_{n=1}^{\infty}$ *be an integrably bounded sequence in* $L^1([0,T],\mathcal{Z})$. *Suppose that* $\chi(\{f_n(t)\}) \leq \zeta(t)$, *for a.e.* $t \in [0,T]$, *where* $\zeta(.) \in L^1_+[0,T]$. *Then for every bounded subset* $\Theta \subset E$ *and for all* $t \in [0,T]$:

$$\chi_{\mathcal{Z}}\left\{S(\Theta,\{f_n\}_{n=1}^{\infty})(t)\right\} \leq 2\,q\int_0^t \zeta(s)ds + q\,e^{-p\,t}\chi_{\mathcal{Z}}(\Theta), \tag{2.3}$$

where

$$\{S(\Theta,\{f_n\}_{n=1}^{\infty})(t)\} = \bigcup_{\substack{u\in\Theta\\ n\geq 1}} S(u,f_n)(t).$$

Remark 2.11. If $\mathcal{Z}$ is separable, then (2.3) has the form

$$\chi_{\mathcal{Z}}\left\{S(\Theta,\{f_n\}_{n=1}^{\infty})(t)\right\} \leq q\int_0^t \zeta(s)ds + q\,e^{-p\,t}\chi_E(\Theta). \tag{2.4}$$

3. More complete formulation of the periodic problem

We aim in this section to give a complete definition of a periodic solution to semilinear functional differential equations with finite delay described by

$$x'(t) = Ax(t) + \mathcal{H}(t,x_t), \quad t \geq 0. \tag{3.1}$$

It is natural to treat the associated periodic boundary value problem first. The associated periodic boundary value problem that was considred is written in the form

$$\begin{cases} x'(t) = Ax(t) + \mathcal{H}(t, x_t), & t \in [0,T], \\ x_0 = x_T, \end{cases} \tag{3.2}$$

where $A : D(A) \subset E \to E$, is a linear operator generating a C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ and $\mathcal{H} : [0,T] \times C([-r,0];E) \to E$, is a given map satisfying some conditions. For any continuous function $x : [-r,T] \to E$ and any $t \geq 0$, the function x_t denotes the element of $C([-r,0];E)$, defined by $x_t(\theta) = x(t+\theta),\ \theta \in [-r,0]$.

Such a consideration of the problem (3.2) is due to the fact that the later is a well-defined, in the sense that, we know how to study the associated Cauchy problem. It is clear that there is a direct influence of the approach used in the study of the periodic problem where the delay is absent.

Recall that the Cauchy operator $S : E \times L^1([0,T];E) \longrightarrow C([0,T];E)$, is defined by (2.1).

Definition 3.1. A function $x \in C([-r,T];E)$ is a mild solution to the problem (3.2) if $x_0 = x_T$ and for every $t \in [0,T]$, $x(t) = S(x(0), f)$, where $f(t) = \mathcal{H}(t, x_t)$ a.e. $t \in [0,T]$.

Let us show that the formulation given by Problem (3.2) is too restrictive for the study of the periodic problem (3.1) even from theoretical point of view. For the sake of simplicity, we set,

Definition 3.2. A function z in $C([a,b];E)$ $(b-a \geq T)$, is said to be T-periodic if z is a restriction on $[a,b]$ of some T-periodic function defined from $\mathbb{R}$ with values in E. In such a context, we denote by $C_T([a,b];E)$ the space of continuous T-periodic functions defined on $[a,b]$ and with values in E.

The expression $x_0 = x_T$ implies that every mild solution of the problem (3.2) is necessarily an element of $C_T([-r,T];E)$. Then, the problem (3.2) can be written equivalently as:

$$\begin{cases} y'(t) = Ay(t) + \mathcal{H}(t, \hat{y}_t), & t \in [0,T], \\ y(0) = y(T), \end{cases} \tag{3.3}$$

where $\hat{y}$ is a T-periodic extension of $y \in C([0,T];E)$ on all $[-r,T]$. It follows that the problem (3.2) can be seen as the one without delay. The multivalued version of the problem (3.3) (equivalently of the problem (3.2)) was studied in [10]. So, for the study of the problem (3.2), we simply have to deal with the problem (3.3) where $\hat{y}$ is a T-periodic extension of $y \in C([0,T];E)$ on all $[-r,T]$ and use, for example, the same approach given in [10]. Note that, if in place of $C([-r,0];E)$ we choose another space, we must be sure that the latter contains T-periodic extension of the elements of $C([0,T];E)$ on $]-r,0]$, if not, then the study of the problem (3.3) (as a consequence the problem (3.2)) will be absurd as will be shown in the next remark. To explain in a clearer way why the formulation given by Problem (3.2) is too restrictive even from theoretical point of view, consider the problem,

$$\begin{cases} x'(t) = Ax(t) + \mathcal{H}(t, x_t), & t \in [0,T], \\ x(0) = x(T). \end{cases} \tag{3.4}$$

Definition 3.3. A function $x \in C([-r,T];E)$ is a mild solution to the problem (3.4) if $x(0) = x(T)$ and for every $t \in [0,T]$, $x(t) = S(x(0), f)$, where $f(t) = H(t, x_t)$ a.e. $t \in [0,T]$.

Remark 3.4. It is clear that if x is a mild solution to Problem (3.2) then x is a mild solution to Problem (3.4).

Because of the existence of the delay (initial memories), one must initially specify, by words, the meaning of a mild solution to the problem (3.4). Let us agree on this general principle: finding a mild solution to the problem (3.4) is equivalent to finding an initial condition (a memory) that gives rise to periodic evolution ($x(0) = x(T)$). It is the evolution of the solution (i.e., $x\mid_{[0,T]}$) which must be generated by an initial memory and not the reverse. Of course, the formulation given by problem (3.2) does not respect this general principle because for a given solution $\hat{y}$ to Problem (3.3) (equivalently to Problem (3.2)), the initial memory $\hat{y}_0$ is entirely defined by the evolution of $\hat{y}$ ($\hat{y}_0$ is necessarily a T-periodic extension of y on $[-r,0]$). So, finding a solution through the formulation given by problem (3.2) (equivalently by Problem (3.3)) is like finding the evolution of the solution on $[0,T]$ by imposing, a priori, that its memory is nothing but its T -periodic extension on $[-r,0]$. This explains that the formulation given by Problem (3.2) is too restrictive and moreover uninteresting.

Remark 3.5. It is interesting to note that the reasoning developed above remains true even in the case of an infinite delay, i.e., $r = \infty$. If x is a mild solution of the problem (3.2), then $x\mid_{[0,T]}$ is continuous, the expression $x_0 = x_T$ implies that x is necessarily an element of $C_T(]-\infty,T];E)$ if the phase space is $C_T(]-\infty,0];E)$ or it can be represented by an element of $C_T(]-\infty,T];E)$, as being an element belonging to an equivalence class of measurable functions almost everywhere equal, if the phase space is the one introduced by Hale and Kato [6] . As a consequence, for the study of the problem (3.2), we simply have to deal with the problem (3.3) where $\hat{y}$ is a T-periodic extension of $y \in C([0,T];E)$ on all $]-\infty,T]$ and use, for example, the same approach given in [10]. Now, take as phase space, the space C^0 (see [6]), defined by $C^0 = \{\varphi \in BC : \lim_{\theta\to\infty} \varphi(\theta) = 0\}$, where BC is the space of all continuous bounded function from $]-\infty,0]$ into E endowed with the sup-norm. It is clear that for any $T > 0$, the unique T-periodic element of C^0 is the null function. As a consequence, the null function is the unique candidate as a solution to the problem (3.2) (we will deal with the infinite delay case in detail in another work).

So, to study the periodic Problem (3.1), it is more natural to study Problem (3.4). This will allow us thereafter to make a complete study of the Problem (3.1). Of course, the study of Problem (3.4) cannot be done directly because the latter is poorly defined, in the sense that, the memories are observed only at the time $t = 0$ and, the associated Cauchy problem is not well defined. To face this constraint, we will show that the set of mild solutions of Problem (3.4) can be observed as an infinite union of mild solutions sets of well-defined problems (i.e., problems that we know how to study the associated Cauchy problems) and, to justify our approach, we will show that the manner with which we will treat Problem (3.4) is also interesting from the point of view of applications.

Let us start by studying Problem (3.4). It is formally assumed, for the moment, that the solutions sets to the problems that we are going to consider are nonempty. Conditions which ensure that such sets are nonempty will be given in Section 4.

Let us denote by Σ, the mild solution set of the problem (3.4) i.e.,

$$\Sigma = \{x \in C([-r,T];E) :\, x \,\text{is a mild solution to Problem (3.4)}\}. \tag{3.5}$$

For any $z \in C([0,T];E)$ and any $\psi \in C([-r,0];E)$ such that $z(0) = \psi(0)$, denote by $z[\psi]$ the element of $C([-r,T];E)$, given by

$$z[\psi](t) = \begin{cases} z(t), & t \in [0,T], \\ \psi(t), & t \in [-r,0]. \end{cases}$$

One can easily check that

$$C([-r,T];E) = \{y[\psi] :\, y \in C([0,T];E),\ \psi \in C([-r,0];E);\ z(0) = \psi(0)\}.$$

Moreover, each $x \in C([-r,T];E)$ can be written in a unique way as $x = y[\psi]$, for some $y \in C([0,T];E)$ and, $\psi \in C([-r,0];E)$ such that, $\psi(0) = y(0)$.

Let us consider the Banach subspace $\mathcal{B}_0$ of $C([-r,0];E)$ defined as follow,

$$\mathcal{B}_0 = \{\varphi \in C([-r,0];E) :\ \varphi(0) = 0\}. \tag{3.6}$$

For $\varphi \in \mathcal{B}_0$, let $\Lambda_\varphi : C([0,T];E) \to C([-r,0];E)$, be an operator such that, for every $y \in C([0,T];E)$,

$$\Lambda_\varphi(y) = y(0) + \varphi. \tag{3.7}$$

It is clear the operator Λ_φ is well defined, continuous and maps bounded sets onto bounded sets. Let us consider the periodic boundary value problem

$$(P_\varphi) \qquad \begin{cases} y'(t) = Ay(t) + F(t, y[\Lambda_\varphi(y)]_t), & t \in [0,T], \\ y(0) = y(T). \end{cases}$$

Denote by $\Sigma^\varphi[0,T]$, the mild solution set of the problem (P_φ), i.e.,

$$\Sigma^\varphi[0,T] = \{y \in C([0,T];E) :\, y \,\text{is a mild solution to Problem}\,(P_\varphi)\}. \tag{3.8}$$

It is clear that if $y \in \Sigma^\varphi[0,T]$ then,

$$y[\Lambda_\varphi(y)](t) = \begin{cases} y(t), & t \in [0,T], \\ \Lambda_\varphi(y)(t), & t \in [-r,0]. \end{cases} = \begin{cases} y(t), & t \in [0,T], \\ y(0) + \varphi(t), & t \in [-r,0], \end{cases} \tag{3.9}$$

is a mild solution of the problem (3.4). For every $\varphi \in \mathcal{B}_0$, set

$$\Sigma^\varphi[-r,T] = \{y[\Lambda_\varphi(y)] :\ y \in \Sigma^\varphi[0,T]\} = \{y[y(0) + \varphi] :\ y \in \Sigma_\varphi\}. \tag{3.10}$$

So, for any $\varphi \in \mathcal{B}_0$, we have $\Sigma^\varphi[-r,T] \subset \Sigma$. Let us show that $\Sigma = \underset{\varphi \in \mathcal{B}_0}{\cup} \Sigma^\varphi[-r,T]$. Let $x \in \Sigma$. Set $\psi = x_0$ and, $y(t) = x(t)$, $t \in [0,T]$. It is clear that for every $t \in [-r,T]$, we have $x(t) = y[\Lambda_\varphi(y)]$, where $\varphi = \psi - \psi(0)$ an element of $\mathcal{B}_0$. Since $y \in \Sigma^\varphi[0,T]$, we deduce that $x \in \Sigma^\varphi[-r,T]$. Thus,

$$\Sigma = \underset{\varphi \in \mathcal{B}_0}{\cup} \Sigma^\varphi[-r,T]. \tag{3.11}$$

Remark 3.6. Our approach allows a deep study of the periodic problem (3.4) in the meaning that it allows to observe the initial segments of the solutions in other spaces that may be of very different nature than $C([-r,0];E)$. Indeed, for $\tau \in [0,r[$, let us consider the Banach space $L^1_\tau = L^1_\tau([-r,0];E)$ (inspired from [6]), given by

$$L^1_\tau = \left\{ \psi : \psi \mid_{[-\tau,0]} \text{ is continuous and, } \psi \mid_{[-r,-\tau]} \in L^1([-\tau,0];E) \right\}.$$

Suppose that $\mathcal{H} : [0,T] \times L^1_\tau([-r,0];E) \to E$ and for every $\psi \in L^1_\tau$, the Cauchy problem

$$\begin{cases} x'(t) = Ax(t) + \mathcal{H}(t,x_t), \quad t \in [0,T], \\ x_0 = \psi, \end{cases} \tag{3.12}$$

has at least a mild solution. In this case for the study of the problem (3.4) (with $L^1_\tau([-r,0];E)$ in place of $C([-r,0];E)$), we have to study the problems (P_φ) where $\varphi \in L^1_\tau([-r,0];E)$ such that $\varphi(0) = 0$ and $\Lambda_\varphi : C([0,T];E) \to L^1_\tau([-r,0];E)$, is given by, $\Lambda_\varphi(y)(\cdot) = y(0) + \varphi(\cdot)$. A solution $x = y[\Lambda_\varphi(y)]$ does not need to be represented by an element of $C_T([-r,T];E)$.

Mild periodic solution. Now, we have to explain how a solution to the problem (3.4) can generate a periodic solution to the problem (3.1). Suppose that the map $\mathcal{H} : \mathbb{R}_+ \times C([-r,0];E) \to E$, is T-periodic on the first argument, i.e.,

$(\mathcal{H}_T)$ For every $v \in C([-r,0];E)$, $\mathcal{H}(t+T,v) = \mathcal{H}(t,u)$ a.e. $t \in \mathbb{R}_+$.

First of all, note that under Hypothesis $(\mathcal{H}_T)$, if x is a mild solution to the problem (3.2)(recall that in this case $x \in C_T([-r,T];E)$, see Definition 3.2) then its T-periodic extension on all $[-r,+\infty]$ is a T-periodic solution of the problem (3.1). The study of the periodic solutions to the problem (3.1) through Problem (3.2) is a simple consequence. Now, let $x \in C([-r,T];E)$ be a mild solution of the problem (3.4) (i.e., $x \in \Sigma$). Set $y = x \mid_{[0,T]}$. It is clear that, $x = y[x_0]$ and y is a mild solution to the problem

$$\begin{cases} y'(t) &= Ay(t) + \mathcal{H}(t, y[x_0]_t), \quad t \in [0,T], \\ y(0) &= y(T). \end{cases} \tag{3.13}$$

For $n \geq 0$, denote by x_0^n, the translation of x_0 to the interval $[nT-r, nT]$ (we want that the evolution in each time interval $[nT,(n+1)T]$, $n \geq 0$, occurs while keeping the initial memory x_0, we will explain in Remark 3.7 why such a consideration is more natural) and by y^n the translation of y to the interval $[nT,(n+1)T]$ (we want to have a periodic evolution starting from $t \geq 0$), i.e., for every $n \geq 0$, we set,

$$\begin{cases} x_0^n(\theta) = x_0(\theta - nT),\ \theta \in [nT-r, nT], \\ y^n(t) = y(t-nT),\ t \in [nT,(n+1)T]. \end{cases}$$

Note that $x_0^0 = x_0$ and $y^0 = y$. For every $t \in [0,T]$, we have, $y^n[x_0^n]_{t+nT} = y[x_0]_t$ (geometrically this fact is obvious), which is equivalent to write,

$$y^n[x_0^n]_t = y[x_0]_{t-nT}, \text{ for every } t \in [nT,(n+1)T]. \tag{3.14}$$

For every $n \geq 0$, the problem (3.13) can be written equivalently as

$$\begin{cases} y'(t-nT) = Ay(t-nT) + \mathcal{H}(t-nT, y[x_0]_{t-nT}), \quad t \in [nT,(n+1)T], \\ y(0) = y(T). \end{cases}$$

By using (3.14) and the fact that $\mathcal{H}$ is T-periodic on the first argument, we deduce, that every $n \geq 0$, y^n is a mild solution to the problem

$$\begin{cases} (y^n)'(t) &= Ay^n(t) + \mathcal{H}(t, y^n[x_0^n]_t), \quad t \in [nT,(n+1)T], \\ y^n(nT) &= y^n((n+1)T). \end{cases} \tag{3.15}$$

It results that if $\mathcal{H}$ satisfies Hypothesis $(\mathcal{H}_T)$ and $x = y[x_0]$ is a mild solution of the problem (3.4), then $\hat{x} = \hat{y}[x_0]$, where $\hat{y}$ is a T-periodic extension of y on all $[0,+\infty[$, is a T-periodic mild solution to (3.1) that conserves the memory when the equation (3.1) evolves on each time interval $[nT,(n+1)T]$, $n \geq 0$.

Remark 3.7. If the evolution of the solution in each time interval $[nT,(n+1)T]$, $n \geq 0$, occurs without keeping the initial memory x_0, i.e., it occurs in each time interval $[nT,(n+1)T]$, $n \geq 0$, by taking as initial memory x_{nT}, under these conditions we are in the case of the problem (3.2). A periodic solution of the problem (3.1) belong necessarily to $C_T([-r,+\infty[,E)$ and, is nothing but a T-periodic extension of a solution to the problem (3.2) (which is necessarily an element of $C_T([-r,T],E)$), on all the interval $[-r,+\infty]$, in other words, if the initial memory is not preserved, one will be in the case of the formulation given by Problem (3.2), that we now know that it is not very interesting (see also Remark 3.4). Thus, to define a periodic solution to Problem (3.1), it is natural to assume that the evolution of the solution in each time interval $[nT,(n+1)T]$, $n \geq 0$, occurs with keeping the initial memory.

Recall that $C_T = C_T([0,+\infty[;E)$ (see Definition 3.2) denotes the space of continuous T-periodic functions defined on $[0,+\infty[$ with values in E. Let us denote by $C^T = C^T([-r,+\infty[;E)$ the space defined as follows,

$$C^T = \left\{ x : \, x \,\text{is continuous}, \, x_0 \in C([-r,0];E) \,\text{and}, \, x\,|_{[0,+\infty[} \in C_T \right\}.$$

We are now in position to give a more complete definition of a mild periodic solution to the problem (3.1) under the condition that $\mathcal{H}$ satisfies Hypothesis $(\mathcal{H}_T)$.

Definition 3.8. A function $x \in C^T$ is a T-periodic mild solution to the problem (3.1) if and only if, $x\,|_{[-r,T]} \in \Sigma^{\varphi}[-r,T]$ for some $\varphi \in \mathcal{B}_0$.

Let us denote by $\Sigma\Sigma$ the set of all T-periodic mild solutions to the problem (3.1) and, by $\Sigma_T \subset C^T$ the set of all T-periodic extension of the elements of $\Sigma\,|_{[0,T]}$ on all $[0,+\infty[$ (see (3.5)). Then,

$$\Sigma\Sigma = \Sigma_T.$$

Now, if for each $\varphi \in \mathcal{B}_0$, we denote by $\Sigma_T^{\varphi}[-r,T] \subset C^T$, the set of all T-periodic extension of the elements of $\Sigma^{\varphi}[-r,T]\,|_{[0,T]}$ on all $[0,+\infty[$, from (3.11), we get

$$\Sigma\Sigma = \Sigma_T = \bigcup_{\varphi \in \mathcal{B}_0} \Sigma_T^{\varphi}[-r,T].$$

Recall (see (3.9)) that , for each $\varphi \in \mathcal{B}_0$,

$$x \in \Sigma^{\varphi}[-r,T] \Leftrightarrow x = y[\Lambda_{\varphi}] \text{ with } y \in \Sigma^{\varphi}[0,T]. \tag{3.16}$$

So, Definition 3.8 can be written equivalently:

Definition 3.9. A function $y \in C_T([0,+\infty[;E)$ is a T-periodic mild solution to the problem (3.1), if and only if, $y\mid_{[0,T]} \in \Sigma^{\varphi}[0,T]$ for some $\varphi \in \mathcal{B}_0$.

Now, if, for each $\varphi \in \mathcal{B}_0$, we denote by $\Sigma^{\varphi}_T[0,T]$ the set of all T-periodic extension of the elements of $\Sigma^{\varphi}[0,T]$ on all $[0,+\infty[$, from (3.16), we get

$$\underset{\varphi\in\mathcal{B}_0}{\cup} \Sigma^{\varphi}_T[-r,T] = \underset{\varphi\in\mathcal{B}_0}{\cup} \left\{ y[\Lambda_{\varphi}] : \, y \in \Sigma^{\varphi}_T[0,T] \right\}.$$

Thus,

$$\Sigma\Sigma = \underset{\varphi\in\mathcal{B}_0}{\cup} \left\{ y[\Lambda_{\varphi}] : \, y \in \Sigma^{\varphi}_T[0,T] \right\}. \tag{3.17}$$

Remark 3.10. In applications the choice of φ can be dictated by the problem to be studied.

Intuitive Illustration 1. Suppose we want to send a flying object so that its motion law is governed by a finite-delay semilinear differential equation and that after a certain time, say after $t = 0$ (we take $t = -r, r > 0$, the time of the beginning of the flight), its movement is periodic. It is clear that the movement of this object must be programmed following a trajectory of a periodic solution. Suppose that between the instants, $t = -r$ and $t = 0$, (where r is assumed to be large enough compared to T), this object must pass through a danger zone. It is clear that the movement of this object should not be too predictable between the instants, $t = -r$ and $t = 0$. So, programming the movement of this object following a solution found through the translation operator or equivalently following a solution found through the formulation given by the problem (3.2) would be really a bad choice. We can program its motion following a solution belonging to $\Sigma^{\varphi}_T[-r,T]$, where $\varphi \in \mathcal{B}_0$, is a difficult function to predict.

Intuitive Illustration 2. A system that evolves while still keeping its initial memory can be observed in the study of the exhaling operation of a person who inhales air in a uniform way (initial memory). A respirologist who wants to analyze the lungs of a patient, asks the latter to inhale the air in a repeated and uniform manner. So, the graph which defines the inhaling operation is known. The respirologist is interested in the graph which defines the exhaling operation (which is a consequence of the inhaling operation). Now, if for each inhaling-exhaling operation, the respirologist only looks at the graph which defines the exhaling operation in order to subsequently deduce, by a periodic extension, the graph of the inhaling operation, this will be absurd, since he cannot deduce anything about the state of health of the patient's lungs. The formulation given by (3.2) puts us in such a context.

4. Hypotheses and existence result

To validate our approach which was developed in the previous section, we must give at least one existence result which ensures that the sets with which we have developed our approach, i.e., $\Sigma^{\varphi}[0,T]$, $\Sigma^{\varphi}[-r,T]$ and Σ, are nonempty. According to the relations (3.10) and (3.11), it enough to show that, for each $\varphi \in \mathcal{B}_0$, under some conditions, , the set $\Sigma^{\varphi}[0,T]$ is nonempty. Of course, the fact that we will consider the multivalued case (differential inclusions) does not change anything on our approach, what changes is the fact that the solutions are written through their selections.

Let φ be an element of $\mathcal{B}_0$ (see Definition (3)). Consider the problem

$$(\hat{P}_{\varphi}) \qquad \begin{cases} y'(t) \in Ay(t) + F(t, y[\Lambda_{\varphi}(y)]_t), & t \in [0,T], \\ y(0) = y(T). \end{cases}$$

where $A : D(A) \subset E \to E$, is a linear operator not necessarily bounded, $F : [0,T] \times C([-r,0];E) \to \mathcal{K}v(E)$ a multifunction and the map $\Lambda_{\varphi}(\cdot)$ has been defined in (3.7). Suppose that:

(A_1) The operator A generates a $\mathcal{C}_0$ semigroup $(\mathcal{T}(t))_{t\geq 0}$ on E.

(A_2) The semigroup $(\mathcal{T}(t))_{t\geq 0}$ is decreasing, i.e., there exist constants $C \geq 1$ and $\omega > 0$ such that

$$\|\mathcal{T}(t)\| \leq C\, e^{-\omega\, t}, \qquad t \in [0,T].$$

(F_1) For all $u \in C([-r,0];E)$, the mapping $t \to F(t,\, u)$ is measurable.

(F_2) For a.e. $t \in [0,T]$, the map $F(t,\cdot)$ is u.s.c.

(F_3) There exists $\alpha(.) \in L^1_+([0,T])$ such that for all $u \in C([-r,0];E)$,

$$\|F(t,u)\| \leq \alpha(t), \text{ a.e.}\, t \in [0,T].$$

(F_4) There exists a function $\kappa(\cdot) \in L^1_+([0,T])$ such that, for every bounded subset $D \subset C([-r,0];E)$,

$$\chi\left(F(t,D)\right) \leq \kappa(t) \sup_{-r\leq\theta\leq 0} \chi(D(\theta)),$$

for a.e. $t \in [0,T]$, where $D(\theta) = \{x(\theta),\ x \in D\}$ and χ denotes the Hausdorff measure of noncompactness in E.

Remark 4.1. Recall that, under conditions $(F_1) - (F_3)$, for every continuous map $\omega : [0,T] \to C([-r,0];E)$, there exists a summable selection $f : [0,T] \to E$ of the multivalued map $F(\cdot,\omega(\cdot))$. (see Theorem 1.3.5 in [9]). Now consider the map $\Pi : [0,T] \times C([0,T];E) \to C([-r,0];E)$, defined by $\Pi(t,y) = y[\Lambda_{\varphi}(y)]_t$. It is easy to check that, the map $\Pi(\cdot,y)$ is continuous for each $y \in C([0,T];E)$ and, $\Pi(t,\cdot)$ is Lipschitz continuous uniformly with respect to $t \in [0,T]$. The continuity of $\Pi(\cdot,y)$ and the fact that $F(t,\Pi(t,y)) = F(t,y[\Lambda_{\varphi}(y)]_t)$, $a.e.\, t \in [0,T]$, ensure that the superposition operator $\mathrm{sel}_F^{\Lambda_{\varphi}}$,

$$\mathrm{sel}_F^{\Lambda_{\varphi}} : C([0,T];E) \to L^1([0,T];E)$$
$$\mathrm{sel}_F^{\Lambda_{\varphi}}(y) = \left\{ f \in L^1([0,T];E) :\ f(t) \in F(t,y[\Lambda_{\varphi}(y)]_t),\ a.e.\, t \in [0,T] \right\},$$

is well defined. The Lipschitz continuity of $\Pi(t,\cdot)$ uniformly with respect to $t \in [0,T]$ and Lemma 5.1.1 in [9] (see also [4, Lemma 4]), sel_F^{Λ} is weakly closed. More precisely:

Lemma 4.2. *Let* $\{y^n\}_{n=1}^{\infty} \subset C([0,T];E)$ *and* $\{f_n\}_{n=1}^{\infty} \subset L^1([0,T];E)$ *with,* $f_n \in \mathrm{sel}_F^{\Lambda_{\varphi}}(y^n)$, $n \geq 1$. *If* $y^n \to y^0$ *and,* $f_n \xrightarrow{w} f_0$, *then* $f_0 \in \mathrm{sel}_F^{\Lambda_{\varphi}}(y^0)$.

Definition 4.3. A function $y \in C([0,T];E)$ is a mild solution to the problem $(\hat{P}_\varphi)$ if there exists $f \in \mathrm{sel}_F^{\Lambda_\varphi}(y)$ such that, $x(\cdot) = S(x(0), f)(\cdot)$ and, $x(0) = x(T)$.

We denote by $\hat{\Sigma}^\varphi[0,T]$ the mild solution set of the problem $\hat{P}_\varphi$. We have

Theorem 4.4. *Assume Hypotheses* $(A_1) - (A_2)$ *and* (F_1)-(F_4). *If*

$$2\,C\Big(1 + 2\,C\Big)\,\|\kappa(\cdot)\|_{L^1} + C^2\,e^{-\omega T} < 1, \tag{4.1}$$

then the mild solutions set $\hat{\Sigma}_\varphi[0,T]$ *of the problem* $(\hat{P}_\varphi)$ *is nonempty and compact.*

Remark 4.5. If the map $F(t,\cdot)$ is compact for a.e. $t \in [0,T]$, then $\kappa(\cdot) \equiv 0$ and the condition (4.1) is reduced to $C^2\,e^{-\omega T} < 1$, which is always satisfied if $C = 1$.

Before giving the proof of Theorem (4.4), we need to establish some auxiliary results. Let us consider the multivalued operator Γ_φ,

$$\begin{cases} \Gamma_\varphi : & C([0,T];E) \longrightarrow \mathcal{P}(C([0,T];E)) \\ & \Gamma_\varphi(y) = \{S(S(y(0),f)(T),f) : f \in \mathrm{sel}_F^{\Lambda_\varphi}(y)\}, \end{cases} \tag{4.2}$$

where, $S(\cdot,\cdot)$ is the Cauchy operator defined in (2.1).

Lemma 4.6. *We have:*

$$\hat{\Sigma}^\varphi[0,T] = Fix\,\Gamma_\varphi.$$

Proof. Let $y \in \Gamma_\varphi(y)$. Then, there exists $f \in sel_F^{\Lambda}(y)$, such that

$$y = S(S(y(0),f)(T),f). \tag{4.3}$$

Since

$$y(0) = S(S(y(0),f)(T),f)(0) = S(y(0),f)(T),$$

we get,

$$y = S(y(0),f) \text{ with } y(T) = S(y(0),f)(T) = y(0).$$

That is $x \in \hat{\Sigma}^\varphi[0,T]$. Now, let y be a solution to the problem $(\hat{P}_\varphi)$. Then, there exists $f \in sel_F^{\Lambda}(y)$ such that, $y = S(y(0),f)$ and $y(0) = y(T)$. It results that $y = S(y(0),f) = S(y(T),f)$. But $y(T) = S(y(0),f)(T)$. Hence, $y = S(S(y(0),f)(T),f)$, which means that y is a fixed point of Γ_φ. □

Remark 4.7. The idea of the construction of such operator is due to Ioan I. Vrabie [14] (in the study of fully nonlinear differential equation where the delay is absent).

Lemma 4.8. *The operator* Γ_φ *is with convex values.*

Proof. Since the mulimap F is with convex values, we have immediately that, for all convex subsets V of E and, $\mathcal{V}$ of $L^1([0,T];E)$, the set $S(V,\mathcal{V}) = \{S(u,g) : u \in E,\, g \in \mathcal{V}\}$ is a convex subset of $C([0,T];E)$. The result follows from the fact that, for each $y \in C([0,T];E)$, the sets $\mathrm{sel}_F^{\Lambda_\varphi}(y)$ and $\{S(y(0),f)(T) : f \in \mathrm{sel}_F^{\Lambda_\varphi}(y)\}$ are convex subsets of $L^1([0,T];E)$ and E respectively. □

Lemma 4.9. *Let* Ω *be a bounded subset of* $C([0,T];E)$. *Then for for a.e.* $t \in [0,T]$, *we have*

$$\chi\,\{F(t, y[\Lambda_\varphi(y)]_t) : y \in \Omega\} \le 2\,\kappa(t) \sup_{\theta\in[0,t]} \chi(\Omega(\theta)).$$

Proof. By Hypothesis (F_4), for a.e. $t \in [0,T]$ we have,

$$\begin{aligned}
\chi\,\{F(t, y[\Lambda_\varphi(y)]_t) : y \in \Omega\} &\leq \kappa(t) \sup_{\theta\in[-r,0]} \chi\Big(\{y[\Lambda_\varphi(y)]_t(\theta) : y \in \Omega\} \Big) \\
&\leq \kappa(t) \sup_{\theta\in[-r,0]} \chi\Big(\{y[\Lambda_\varphi(y)](t+\theta) : y \in \Omega\} \Big) \\
&\leq \kappa(t) \sup_{\theta\in[-r,-t]} \chi\Big(\{y[\Lambda_\varphi(y)](t+\theta) : y \in \Omega\} \Big) \\
&+ \kappa(t) \sup_{\theta\in[-t,0]} \chi\Big(\{y[\Lambda_\varphi(y)](t+\theta) : y \in \Omega\} \Big) \\
&\leq \kappa(t) \sup_{\theta\in[-r,-t]} \chi\Big(\{\varphi(t+\theta) + y(0) : y \in \Omega\} \Big) \\
&+ \kappa(t) \sup_{\theta\in[-t,0]} \chi\Big(\{y(t+\theta) : y \in \Omega\} \Big) \\
&= 2\,\kappa(t) \sup_{\theta\in[0,t]} \chi(\Omega(\theta)).
\end{aligned}$$

□

Lemma 4.10. *The multimap Γ_φ is closed with compact values.*

Proof. Let $\{y_n\}_n$, $\{z_n\}_n \subset C([0,T];E)$, such that $y_n \longrightarrow y^0$, $z_n \in \Gamma_\varphi(y_n)$, $n \geq 1$, and $z_n \longrightarrow z^0$. Take any sequence $\{f_n\}_n$ from $L^1([0,T],E)$ such that $f_n \in \mathrm{sel}_F^{\Lambda_\varphi}(y_n)$ and $z_n = S(S(y_n(0), f_n)(T), f_n)$, $n \geq 1$. By Hypothesis (F_3) the sequence $\{f_n\}_n$ is integrably bounded. By applying Lemma 4.9, we have

$$\chi(\{f_n(t)\}_{n=1}^{+\infty}) \leq 2\,\kappa(t)\chi(\{y_n(t)\}_{n=1}^{+\infty}) = 0 \quad \text{a.e. } t \in [0,T].$$

It results that the sequence $\{f_n\}_n$ is semicompact. Without loss of generality, one can suppose that $f_n \overset{w}{\to} f_0$. Since $(\{y_n(0)\}_{n=1}^{\infty})$ is a relatively compact subset of E, invoking Lemma 2.9 we get that the sequence $\{S(y_n(0), f_n)\}_n$ is relatively compact in $C([0,T];E)$ and $S(y_n(0), f_n) \to S(y_0(0), f_0)$. In particular we have $S(y_n(0), f_n)(T) \to S(y_0(0), f_0)(T)$ in E. Invoking again Lemma 2.9, we obtain

$$z_n = S(S(y_n(0), f_n)(T), f_n) \to S(S(y_0(0), f_0)(T), f_0).$$

By applying Lemma 4.2 we have $f_0 \in \mathrm{sel}_F^{\Lambda_\varphi}(y_0)$. Therefore, $z_0 \in \Gamma_\varphi(y_0)$, which yields the closedness of Γ_φ. Let $y(\cdot) \in C([0,T];E)$. Lemma 4.9 and Hypothesis (F_3) ensure that every sequence $\{f_n\}_n$, $f_n \in \mathrm{sel}_F^{\Lambda_\varphi}(y)$, is semicompact. Following the same reasoning as above, one can easily show that $S\{(S(y(0), f_n)(T), f_n)\}_{n=1}^{\infty}$ is relatively compact in $C([0,T];E)$. The compactness of $\Gamma_\varphi(y)$ follows from its closeness. □

Let Ψ be a function defined on bounded subsets of $C([0,T];E)$, in the following way

$$\Psi(\Omega) = \max_{\mathcal{D}\in\Delta(\Omega)} \Big(\vartheta(\mathcal{D}),\, \mathrm{mod}_c(\mathcal{D})\Big), \tag{4.4}$$

where

$$\vartheta(\mathcal{D}) = \sup_{t\in[0,T]} \chi_(\mathcal{D}(t)), \tag{4.5}$$

$$\mathrm{mod}_c(\mathcal{D}) = \lim_{\delta\to 0} \sup_{x\in\mathcal{D}} \max_{|t_1-t_2|\leq\delta} \|x(t_1) - x(t_2)\|,$$

and $\Delta(\Omega)$ is the collection of all denumerable subsets of Ω. The range for the function Ψ is a cone $\mathbb{R}^2_+$, max is taken in the sense of the ordering induced by this cone. The function Ψ is well defined, monotone, nonsingular and regular measure of noncompactness in $C([0,T];\,E)$ (see [9]).

Lemma 4.11. *The operator Γ_φ is Ψ-condensing.*

Proof. let $\Omega \subset C([0,T];E)$ be bounded subset such that,

$$\Psi(\Gamma_\varphi(\Omega)) \geq \Psi(\Omega). \tag{4.6}$$

We have to show that (4.6) implies that Ω is relatively compact. Let the maximum on the left-hand side of the inequality (4.6) be achieved for a countable set $D' = \{z^n\}_{n=1}^\infty \subset \Gamma_\varphi(\Omega)$, i.e.,

$$\Psi(\Gamma_\varphi(\Omega)) = (\vartheta(\{z^n\}_{n=1}),\, \mathrm{mod}_c(\{z^n\}_{n=1})).$$

By the construction of the operator Γ_φ, there exists $\{y^n\}_{n=1} \subset \Omega$, such that

$$\begin{cases} z^n = S(S(y^n(0), f_n)(T), f_n),\ n \geq, 1, \\ f_n \in \mathrm{sel}_F^{\Lambda_\varphi}(y^n). \end{cases}$$

Inequality (4.6) implies that

$$\Psi\left(\{y^n\}_{n=1}^\infty\right) \leq \Psi\left(\{z^n\}_{n=1}^\infty\right). \tag{4.7}$$

Therefore,

$$\vartheta\left(\{y^n\}_{n=1}^\infty\right) \leq \vartheta\left(\{z^n\}_{n=1}^\infty\right) = \vartheta\left(\{S(S(y^n(0), f_n))(T), f_n\}_{n=1}^\infty\right). \tag{4.8}$$

By applying Lemma 4.9, we have immediately,

$$\chi(F(t, \{y^n[\Lambda_\varphi(y^n)]_t\}_{n=1}^\infty)) \leq 2\,k(t) \sup_{\theta\in[0,t]} \chi(\{y^n(\theta)\}_{n=1}^\infty) \leq 2\,k(t)\vartheta\left(\{y^n\}_{n=1}^\infty\right).$$

The last inequality and Lemma 2.10 give

$$\begin{aligned} \chi\left(\{z^n(0)\}_{n=1}^\infty\right) &= \chi(\{S(y^n(0), f_n)(T)\}_n) \\ &\leq 4\,C\,\|\kappa\|_{L^1}\,\vartheta\big(\{y^n\}_{n=1}^\infty\big) + C\,e^{-\omega T}\chi\big(\{y^n(0)\}_{n=1}^\infty\big) \\ &\leq \Big(4\,C\,\|\kappa\|_{L^1} + C\,e^{-\omega T}\Big)\vartheta\big(\{y^n\}_{n=1}^\infty\big). \end{aligned}$$

Using the last estimate and again Lemma 2.10, for every $t \in [0,T]$, we get

$$
\begin{aligned}
\chi\left(\{z^n(t)\}_{n=1}^{\infty}\right) &= \chi\left(\{S\Big(S(y^n(0), f_n)(T), f_n\Big)(t)\}_{n=1}^{\infty}\right) \\
&\le 2\,C \int_0^t \kappa(s)\vartheta\big(\{y^n\}_{n=1}^{\infty}\big)ds + C\,e^{-\omega t}\chi_E\left(\{S(y^n(0), f_n)(T)\}_{n=1}^{\infty}\right) \\
&\le 2\,C\,\|\kappa(\cdot)\|_{L^1}\,\vartheta\big(\{y^n\}_{n=1}^{\infty}\big) + C\,\chi_E\left(\{S(y^n(0), f_n)(T)\}_{n=1}^{\infty}\right) \\
&\le \Big[2\,C\,\|\kappa(\cdot)\|_{L^1} + C\Big(4\,C\,\|\kappa(\cdot)\|_{L^1} + C\,e^{-\omega T}\Big)\Big]\vartheta\big(\{y^n\}_{n=1}^{\infty}\big) \\
&\le \Big[2\,C\,\Big(1 + 2\,C\Big)\,\|\kappa(\cdot)\|_{L^1} + C^2\,e^{-\omega T}\Big]\vartheta\big(\{y^n\}_{n=1}^{\infty}\big).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\vartheta\left(\{z^n\}_{n=1}^{\infty}\right) &= \sup_{t\in[0,T]} \chi_E\left(\{z^n(t)\}_{n=1}^{\infty}\right) \\
&\le \Big[2\,C\,\Big(1 + 2\,C\Big)\,\|\kappa(\cdot)\|_{L^1} + C^2\,e^{-\omega T}\Big]\vartheta\big(\{y^n\}_{n=1}^{\infty}\big). \qquad (4.9)
\end{aligned}
$$

The last inequality together with (4.8) give

$$
\vartheta\big(\{y^n\}_{n=1}^{\infty}\big) \le \Big[2\,C\,\Big(1 + 2\,C\Big)\,\|\kappa(\cdot)\|_{L^1} + C^2\,e^{-\omega T}\Big]\vartheta\big(\{y^n\}_{n=1}^{\infty}\big). \qquad (4.10)
$$

By hypothesis, $2\,C\,\Big(1 + 2\,C\Big)\,\|\kappa(\cdot)\|_{L^1} + C^2\,e^{-\omega T} < 1$. Therefore,

$$
\vartheta\big(\{y^n\}_{n=1}^{\infty}\big) = 0. \qquad (4.11)
$$

By using Lemma 4.9 and Hypothesis (F_3) we deduce that the sequence $\{f_n\}_{n=1}^{\infty}$ is semicompact in $L^1([0,T];E)$. It results by Lemma 2.9 that the sequence $\{S(y^n(0), f_n)(T)\}_{n=1}^{\infty} = \{z^n(0)\}_{n=1}^{\infty}$ is relatively compact in E. By applying once again Lemma 2.9, we deduce that the sequence $\{z^n\}_{n=1}^{\infty} = \{S(S(y^n(0), f_n)(T), f_n)\}_{n=1}^{\infty}$ is relatively compact in $C([0,T];E)$. Consequently, $\mathrm{mod}_c\left(\{z^n\}_{n=1}^{\infty}\right) = 0$. From Inequality (4.7), $\mathrm{mod}_c\left(\{y^n\}_{n=1}^{\infty}\right) = 0$. The last equality together with (4.11) imply that $\Psi(\Omega) = (0,0)$. Since Ψ is regular, we deduce that the Ω is relatively compact. □

Proof of Theorem (4.4). From the previous lemmas, we know that the operator Γ_φ is closed with compact convex values and Ψ-condensing, where Ψ is nonsingular, regular and monotone measure of noncompactness. According to Theorem 2.1 and Proposition 2.2, for the proof of Theorem (4.4), it remains only to show that Γ_φ maps a ball of $C([0,T];E)$ into it self and that $\Sigma^{\varphi}[0,T]$ is bounded.

Recall that the constants C and ω are defined in Hypothesis (A_2) and the function $\alpha(\cdot)$ is defined in Hypothesis (F_3). In the space $C([0,T];E)$, let us define an equivalent norm as follows,

$$
\|y\|_L = e^{-L}\,\|y\|_{C([0,T];E)}\,, \qquad (4.12)
$$

where L is a constant chosen big a enough such that,

$$
j = (C^2 + C)\,e^{-L}\,\|\alpha(\cdot)\|_{L^1} < 1. \qquad (4.13)
$$

Let us denote by $\mathbf{B}(0,\,\rho)$ the closed ball in $\Big(C([0,T];E), \|\cdot\|_L\Big)$, centered at 0, with radius ρ:

$$\mathbf{B}(0,\,\rho) = \{y \in C([0,T];E) :\ \|y\|_L \le \rho\}\,.$$

Let ρ^* be a positive real number, such that

$$\rho^* \ge \frac{j}{1 - C^2\, e^{-\omega T}}. \tag{4.14}$$

where j has been defined in (4.13). Note that Inequality (4.1) implies that $C^2\, e^{-\omega T} < 1$. Let us prove that the operator Γ_φ maps $\mathbf{B}(0,\,\rho^*)$ into itself. Let $y \in \mathbf{B}(0,\,\rho^*)$ and $z = S\Big(\Big(S(y(0), f)\Big)(T),\, f\Big)$, $f \in \mathrm{sel}_F^{\Lambda_\varphi}(y)$, an element of $\Gamma_\varphi(y)$. For every $t \in [0,T]$, we have,

$$\begin{aligned}
\|z(t)\| &= \Big\|S\Big(\Big(S(y(0), f)\Big)(T),\, f\Big)(t)\Big\| \\
&\le C\, e^{-\omega\, t}\, \Big\|S\Big(y(0), f\Big)(T)\Big\| + C\int_0^t e^{-\omega(t-\tau)}\, \|f(\tau)\|\, d\tau \\
&\le C\, e^{-\omega\, t}\left[C\, e^{-\omega T}\, \|y(0)\| + C\int_0^T e^{-\omega(T-\tau)}\, \|f(\tau)\|\, d\tau\right] \\
&\quad + C\int_0^t e^{-\omega(t-\tau)}\, \|f(\tau)\|\, d\tau. \\
&\le C^2\, e^{-\omega T} \sup_{t\in[0,T]} \|y(t)\| + (C^2 + C)\ \|\alpha(\cdot)\|_{L^1}\,.
\end{aligned}$$

By using Inequalities (4.13) and (4.14), we obtain

$$\begin{aligned}
\|z(\cdot)\|_L &\le C^2\, e^{-\omega T} \sup_{t\in[0,T]} e^{-L}\, \|y(t)\| + (C^2 + C)\, e^{-L}\, \|\alpha(\cdot)\|_{L^1} \\
&\le C^2\, e^{-\omega T}\, \rho^* + j \le \rho^*.
\end{aligned}$$

That is, Γ_φ maps $\mathbf{B}(0,\,\rho^*)$ into itself. It remains to show that $\Sigma^\varphi[0,T]$ is bounded. This fact results from the global boundedness condition (F_3). Indeed, let $y \in \Sigma_\varphi$, following the same lines as above (by taking $y(\cdot)$ in place of $z(\cdot)$), for every $t \in [0,T]$, we obtain,

$$\|y(t)\| \le C^2\, e^{-\omega T} \sup_{t\in[0,T]} \|y(t)\| + (C^2 + C)\ \|\alpha(\cdot)\|_{L^1}\,.$$

Thus,

$$\sup_{t\in[0,T]} \|y(t)\| \le \frac{(C^2 + C)\ \|\alpha(\cdot)\|_{L^1}}{1 - C^2\, e^{-\omega T}}.$$

The proof is complete. □

Remark 4.12. If E is separable then, Inequality (4.1) in Theorems 4.4 can be reduced to $C\Big(1 + 2C\Big)\, \|\kappa(\cdot)\|_{L^1} + C^2\, e^{-\omega T} < 1$ (see Remark 2.11).

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