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WEAK SOLUTIONS OF COUPLED SYSTEMS OF CAPUTO TYPE MODIFICATION OF THE ERDÉLYI-KOBER FRACTIONAL DIFFERENTIAL INCLUSIONS WITH RETARDED AND ADVANCED ARGUMENTS

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ABSTRACT. In this paper, we study the existence of weak solutions for a system of Caputo type modification of the Erdélyi–Kober fractional differential inclusions with retarded and advanced arguments. The main result of the paper is based on the fixed point theorem of Mönch's type and the technique of measure of weak noncompactness. We illustrate our results by an example in the last section.

1. Introduction

Fractional differential equations and inclusions appear in several areas such as engineering, mathematics, bio-engineering, physics, viscoelasticity, electrochemistry, control, porous media, electromagnetism, etc. For examples and details on the progress of fractional calculus, we refer the reader to the monographs [1–3, 18, 25, 32, 34, 36], and the references therein.

In [4, 6, 11, 12, 16], the authors studied the existence and uniqueness of solutions for boundary value problems of Hadamard-type fractional functional differential equations and inclusions involving both retarded and advanced arguments. In [28] the authors provide some properties of Caputo-type modification of the Erdélyi–Kober fractional derivative. More details on the Erdélyi-Kober fractional integral and fractional derivative are given in [7, 25–27, 35]. In our investigation, we apply the measure of weak noncompactness introduced by De Blasi [15] and the fixed point theorem of Mönch type [29]. For a comprehensive study of the measure of non compactness we refer, for example, to [9, 22]. It was subsequently considered and used in many papers; see, for example, [8,13,20,21,29]. As far as we know, there are very few results devoted to weak solutions for nonlinear fractional differential equations [33]. In [5], Abbas *et al.* studied the coupled system of Hilfer fractional differential inclusions in Banach spaces given by

$$\begin{cases} {}^{H}D_{1}^{\alpha,\beta}u(t) \in F_1(t, u(t), v(t)) \\ {}^{H}D_{1}^{\alpha,\beta}v(t) \in F_2(t, u(t), v(t)) \end{cases} ; \ t \in [1, T],$$

with the initial conditions

$$\begin{cases} ({}^{H}I^{1-\gamma}u_1)(t) \mid_{t=1} = \phi_1 \\ ({}^{H}I^{1-\gamma}u_2)(t) \mid_{t=1} = \phi_2, \end{cases}$$

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where $T > 1$, $\alpha \in (0,1), \beta \in [0,1]$ $F_i : [1,T] \times E^2 \to \mathcal{P}(E)$; $i = 1,2$ are given multivalued maps, $(E, \|\cdot\|_E)$ is a real separable Banach space , ${}^H D_1^{\alpha,\beta}$ is the Hilfer–Hadamard fractional derivative of order α and type β.

In this paper we discuss the existence of weak solutions to the coupled system of Caputo type modification of the Erdélyi–Kober fractional differential inclusions involving both retarded and advanced arguments given by:

$$\begin{cases} ({}^{\rho}_{c}D^{\alpha_1}_{a^+}u)(t) \in F_1(t, u^t, v^t) \\ ({}^{\rho}_{c}D^{\alpha_2}_{a^+}v)(t) \in F_2(t, u^t, v^t) \end{cases} ; \; t \in I := [a, T], \tag{1.1}$$

$$\begin{cases} (u(t), v(t)) = (\phi_1(t), \phi_2(t)), \; t \in [a-r, a], \; r > 0 \\ (u(t), v(t)) = (\psi_1(t), \psi_2(t)), \; t \in [T, T+\beta], \; \beta > 0, \end{cases} \tag{1.2}$$

where $T > a \geq 0$, $\alpha_i \in (1,2]$; $i = 1,2$; $(E, \|\cdot\|)$ is a real separable Banach space, ${}^{\rho}_{c}D^{\alpha_i}_{a^+}$ is the Caputo type modification of the Erdélyi–Kober fractional derivative, $F_i : I \times C([-r,\beta],E) \times C([-r,\beta],E) \to \mathcal{P}(E)$ are given functions, $\phi_i \in C([a-r,a],E)$ with $\phi_i(a) = 0$ and $\psi_i \in C([T,T+\beta],E)$ with $\psi_i(T) = 0$; $i = 1,2$. We denote by y^t the element of $C([-r,\beta])$ defined by:

$$y^t(s) = y(t+s) : s \in [-r, \beta].$$

This paper initiates the study of differential inclusions involving the Erdélyi–Kober fractional derivative, which include the Hadamard fractional derivative as special case.

2. Preliminaries

In this part, we present notations and definitions we will use throughout this paper. By $C([-r,\beta],E)$ we denote the Banach space of all continuous functions from $[-r,\beta]$ into E equipped with the norm

$$\|y\|_{[-r,\beta]} = \sup\{\|y(t)\| : -r \leq t \leq \beta\}.$$

Also, let $E_1 = C([a-r,a],E)$, $E_2 = C([T,T+\beta],E)$. We denote by $AC(I)$ the space of absolutely continuous functions.

$$AC^1(I) := \{w : I \longrightarrow E : w' \in AC(I)\},$$

where

$$w'(t) = t\frac{d}{dt}w(t), \quad t \in I.$$

$$\mathcal{C} = \{y : [a-r, T+\beta] \longmapsto E : y\,|_{[a-r,a]} \in C([a-r,a],E), y\,|_{[a,T]} \in AC^1(I)$$
$$\text{and } y\,|_{[T,T+\beta]} \in C([T,T+\beta],E)\}$$

be the spaces endowed, respectively, with the norms

$$\|y\|_{[a-r,a]} = \sup\{\|y(t)\| : a-r \leq t \leq a\},$$

$$\|y\|_{[T,T+\beta]} = \sup\{\|y(t)\| : T \leq t \leq T+\beta\},$$

$$\|y\|_{\mathcal{C}} = \sup\{\|y(t)\| : a-r \leq t \leq T+\beta\}.$$

Define the weighted product space $\overline{\mathcal{C}} := \mathcal{C} \times \mathcal{C}$ with the norm

$$\|(u, v)\|_{\overline{\mathcal{C}}} := \|u\|_{\mathcal{C}} + \|v\|_{\mathcal{C}}.$$

Let $E_\omega = (E, \sigma(E, E^*))$ be the Banach space E endowed with the weak topology generated by the continuous linear functionals on E, and $C(I, E_\omega)$ the Banach space of weakly continuous functions on I, with the topology of weak uniform convergence. Consider the space $X_c^p(a, b)$, $(c \in \mathbb{R},\ 1 \leq p \leq \infty)$ of those complex-valued Lebesgue measurable functions f on $[a, b]$ for which $\|f\|_{X_c^p} < \infty$, where the norm is defined by:

$$\|f\|_{X_c^p} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t}\right)^{\frac{1}{p}}, \quad (1 \leq p < \infty, c \in \mathbb{R}).$$

In particular, in case when $c = \frac{1}{p}$ the space $X_c^p(a, b)$ coincides with the $L^p(a, b)$ space, i.e., $X_{\frac{1}{p}}^p(a, b) = L^p(a, b)$.

Let $L^1(I, E)$ be the Banach space of Bochner integrable functions $y : I \longrightarrow E$ with norm $\|y\|_{L^1} = \int_a^T |y(t)|dt$.

Definition 2.1 ([24,26,27]): (Erdélyi–Kober fractional integral))**.** Let $\alpha, c \in \mathbb{R}$. The Erdélyi-Kober fractional integral of order α of a function $g \in X_c^p(a, b)$ is defined by :

$$({}^\rho I_{a^+}^\alpha g)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{\rho-1} (t^\rho - s^\rho)^{\alpha-1} g(s)ds, \qquad t > a;\ \rho > 0, \tag{2.1}$$

where Γ is the Euler gamma function defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1}e^{-t}dt,\ \xi > 0.$$

Definition 2.2 ([23])**.** The generalized fractional derivative, corresponding to the fractional integral (2.1) is defined,by:

$$\begin{aligned} {}^\rho D_{a^+}^\alpha g(t) &= \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)} \left(t^{1-\rho}\frac{d}{dt}\right)^n \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-n+\alpha}} g(s)ds \\ &= \delta_\rho^n ({}^\rho I_{a^+}^{n-\alpha} g)(t);\ 0 \leq a < t, \end{aligned} \tag{2.2}$$

where $\delta_\rho^n = \left(t^{1-\rho}\frac{d}{dt}\right)^n$.

Definition 2.3 ([23, 28])**.** The Caputo-type generalized fractional derivative ${}_c^\rho D_{a^+}^\alpha$ is defined via the above generalized fractional derivative (2.2) as follows:

$$({}_c^\rho D_{a^+}^\alpha g)(t) = \left({}^\rho D_{a^+}^\alpha \left[g(t) - \sum_{k=0}^{n-1} \frac{g^{(k)}(a)}{k!} (s-a)^k\right]\right). \tag{2.3}$$

Lemma 2.4 ([23])**.** *Let $\alpha, \rho \in \mathbb{R}^+$, then*

$$({}^\rho I_{a^+}^\alpha {}_c^\rho D_{a^+}^\alpha g)(t) = g(t) - \sum_{k=0}^{n-1} c_k \left(\frac{t^\rho - a^\rho}{\rho}\right)^k, \tag{2.4}$$

for some $c_k \in \mathbb{R}$, $n = [\alpha] + 1$.

We define the following subsets of $\mathcal{P}(E)$:

$P_{cl}(E) = \{Y \in \mathcal{P}(E) : Y \text{ is closed}\}$,
$P_b(E) = \{Y \in \mathcal{P}(E) : Y \text{ is bounded}\}$,
$P_{cp}(E) = \{Y \in \mathcal{P}(E) : Y \text{ is compact}\}$
$P_{cv}(E) = \{Y \in \mathcal{P}(E) : Y \text{ is convex}\}$
$P_{cp,cv}(E) = P_{cp}(E) \cap P_{cv}(E)$.

Definition 2.5. A multivalued map $G : I \to P_{cl}(E)$ is said to be measurable if for every $y \in E$, the function:

$$t \to d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.

Definition 2.6. A Banach space X is called weakly compactly generated (WCG, for short) if it contains a weakly compact set whose linear span is dense in X.

Definition 2.7 ([31])**.** A function $u : I \to E$ is said to be Pettis integrable on I if and only if there is an element $u_J \in E$ corresponding to each $J \subset I$ such that $\phi(u_J) = \int_J \phi(u(s))ds$ for all $\phi \in E^*$, where the integral on the right hand side is assumed to exist in the sense of Lebesgue, (by definition, $u_J = \int_J u(s)ds$).

Let $P(I, E)$ be the space of all $E-$valued Pettis integrable functions on I, and $L^1(I, E)$ be the Banach space of Bochner integrable functions $u : I \to E$. Let $P_1(I, E)$ denote the space $P_1(I, E) = \{u \in P(I, E) : \varphi(u) \in L^1(I, \mathbb{R});\ for\ every\ \varphi \in E^*\}$ normed by

$$\|u\|_{P_1} = \sup_{\varphi \in E^*,\ \|\varphi\| \le 1} \int_a^T |\varphi(u(x))| d\lambda x,$$

where λ stands for a Lebesgue measure on I.

The following result is due to Pettis (see [[31], Theorem 3.4 and Corollary 3.41]).

Proposition 2.8 ([31])**.** *If $u \in P_1(I, E)$ and h is a measurable and essentially bounded real-valued function, then $uh \in P_1(I, E)$.*

Definition 2.9. A function $h : E \to E$ is said to be weakly sequentially continuous if h takes each weakly convergent sequence in E to a weakly convergent sequence in E (i.e., for any (x_n) in E with $x_n \to x$ in E_ω then $h(x_n) \to h(x)$ in E_ω).

Definition 2.10. Let $\mathcal{P}_{cl,cv}(Q) = \{Y \in \mathcal{P}(Q) : Y \text{ is closed and convex}\}$. A function $F : Q \to \mathcal{P}_{cl,cv}(Q)$ has a weakly sequentially closed graph, if for any sequence $(x_n, y_n) \in Q \times Q, y_n \in F(x_n)$ for $n \in \{1, 2, \ldots\}$, with $x_n \to x$ in E_ω, and $y_n \to y$ in E_ω, then $y \in F(x)$.

From the Hahn–Banach theorem, we have the following result

Proposition 2.11. *Let E be a normed space, and $x_0 \in E$ with $x_0 \neq 0$. Then, there exists $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\varphi(x_0) = \|x_0\|$.*

For a given set V of functions $v : I \to E$ let us denote by $V(t) = \{v(t) : v \in V\};\ t \in I$, and $V(I) = \{v(t) : v \in V,\ t \in I\}$.

Recall that the map $\mu : \Omega_E \to [0, \infty)$ defined by

$$\mu(X) = \inf\{\epsilon > 0 : \text{there exists a weakly compact } \Omega \subset E \text{ such that } X \subset \epsilon B_1 + \Omega\}$$

is the De Blasi measure of weak noncompactness, where Ω_E is the bounded subset of the Banach space E and B_1 is the unit ball of E (for details, see [15]). The Blasi measure of weak noncompactness satisfies the following properties.

Lemma 2.12 ([15]). *Let A and B bounded sets.*

(1) $\mu(B) = 0 \Leftrightarrow \overline{B}$ *is compact (B is weakly relatively compact).*
(2) $\mu(cov(B)) = \mu(B)$.
(3) $\mu(B) = \alpha(\overline{B}^{\omega})$, *($\overline{B}^{\omega}$ denote the weak closure of B.)*
(4) $A \subset B \Rightarrow \mu(A) \leq \mu(B)$.
(5) $\mu(A + B) \leq \mu(A) + \mu(B)$, *where* $A + B = \{x + y : x \in A, \quad y \in B\}$.
(6) $\mu(\lambda B) = |\lambda|\mu(B)$; $\lambda \in \mathbb{R}$, *where* $\lambda B = \{\lambda x : x \in B\}$.
(7) $\mu(A \cup B) = max\{\mu(A), \mu(B)\}$.
(8) $\mu(B + x_0) = \mu(B)$ *for any* $x_0 \in E$.

Lemma 2.13 ([14,17]). *Let $H \subset C(I, E_\omega)$ be a bounded and equicontinuous subset. Then the function $t \to \mu(H(t))$ is continuous on I, and*

$$\mu_C(H) = \max_{t \in I} \mu(H(t)),$$

and

$$\mu\left(\left\{\int_I u(s)ds : u \in H\right\}\right) \leq \int_I \mu(H(s))ds,$$

where $H(t) = \{u(t) : u \in H\}$; $t \in I$, and μ_C is the De Blasi measure of weak noncompactness defined on the bounded sets of $C(I, E_\omega)$.

In the sequel, we rely on the following fixed point theorem.

Theorem 2.14 ([30]). *Let E be a Banach space with Q a nonempty, bounded, closed, convex and equicontinuous subset of a metrizable locally convex vector space C such that $0 \in Q$. Suppose $T : Q \to \mathcal{P}_{cl,cv}(Q)$ has weakly sequentially closed graph. If the implication*

$$\overline{V} = \overline{conv}(\{0\} \cup T(V)) \Rightarrow V \quad \text{is relatively weakly compact}, \tag{2.5}$$

holds for every subset $V \subset Q$, then the operator T has a fixed point.

3. Existence of Solutions

We start this section by defining what we mean by weak solution.

Definition 3.1. A function $x : [a - r, T + \beta] \to E$ is called weak solution of the problem (1.1) – (1.2) if $x \in C([a - r, T + \beta], E_\omega)$ satisfies (1.1) and (1.2).

Lemma 3.2. *Let $1 < \alpha \leq 2$, $\phi \in C([a-r, a], E)$ with $\phi(a) = 0$, $\psi \in C([T, T+\beta], E)$ with $\psi(T) = 0$, and $h : I \to E$ be an integrable function. Then the linear problem*

$$ {}^{\rho}_{c}D^{\alpha}_{a^+} y(t) = h(t), \; \textit{for a.e.} \;\; t \in I := [a, T], \; 1 < \alpha \leq 2, \tag{3.1}$$

$$y(t) = \phi(t), \; t \in [a - r, a], \; r > 0, \tag{3.2}$$

$$y(t) = \psi(t), \; t \in [T, T + \beta], \; \beta > 0, \tag{3.3}$$

has a unique solution, which is given by

$$y(t)=\begin{cases}\phi(t), \;\; \textit{if} \quad t\in[a-r,a],\\ \\ -\int_a^T G(t,s)h(s)ds, \;\; \textit{if} \quad t\in I\\ \\ \psi(t), \;\; \textit{if} \quad t\in[T,T+\beta],\end{cases} \tag{3.4}$$

where

$$G(t,s)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\begin{cases}\frac{(t^\rho-a^\rho)(T^\rho-s^\rho)^{\alpha-1}s^{\rho-1}}{(T^\rho-a^\rho)}-s^{\rho-1}(t^\rho-s^\rho)^{\alpha-1}, & a\le s\le t\le T,\\ \\ \frac{(t^\rho-a^\rho)(T^\rho-s^\rho)^{\alpha-1}s^{\rho-1}}{(T^\rho-a^\rho)}, & a\le t\le s\le T.\end{cases} \tag{3.5}$$

Here $G(t,s)$ is called the Green function of the boundary value problem (3.1)-(3.3).
Proof. *From* (2.4)*, we have*

$$y(t)=c_0+c_1\left(\frac{t^\rho-a^\rho}{\rho}\right)+{}^\rho I^\alpha_{a^+}h(s), \quad c_0,c_1\in\mathbb{R}, \tag{3.6}$$

therefore

$$y(a)=c_0=0,$$

$$y(T)=c_1\left(\frac{T^\rho-a^\rho}{\rho}\right)+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^T(T^\rho-s^\rho)^{\alpha-1}s^{\rho-1}h(s)ds,$$

and

$$c_1=-\frac{\rho^{2-\alpha}}{(T^\rho-a^\rho)\Gamma(\alpha)}\int_a^T(T^\rho-s^\rho)^{\alpha-1}s^{\rho-1}h(s)ds.$$

Substitute the value of c_0 and c_1 into equation (3.6), we get equation (3.4).

$$y(t)=\begin{cases}\phi(t), \;\; \textit{if} \quad t\in[a-r,a],\\ \\ -\int_a^T G(t,s)h(s)ds, \;\; \textit{if} \quad t\in I\\ \\ \psi(t), \;\; \textit{if} \quad t\in[T,T+\beta],\end{cases}$$

where G is defined by equation (3.5), the proof is complete.

Lemma 3.3. *Let $F_i : I\times C([-r,\beta],E)\times C([-r,\beta],E)\longrightarrow\mathcal{P}(E), i=1,2$ be such that $S_{F\circ u}\subset\mathcal{C}$ for any $u\in\mathcal{C}$ and $S_{F\circ v}\subset\mathcal{C}$ for any $v\in\mathcal{C}$ Then solving the* *system* $(1.1)-(1.2)$ *is equivalent to the finding the solutions of the system of integral equations*

$$u(t)=\begin{cases}\phi_1(t), \ \textit{if} \quad t\in[a-r,a],\\ \\ -\int_a^T G_{\alpha_1}(t,s)w_1(s)ds, \ \textit{if} \quad t\in I\\ \\ \psi_1(t), \ \textit{if} \quad t\in[T,T+\beta],\end{cases}$$

and

$$v(t)=\begin{cases}\phi_2(t), \ \textit{if} \quad t\in[a-r,a],\\ \\ -\int_a^T G_{\alpha_2}(t,s)w_2(s)ds, \ \textit{if} \quad t\in I\\ \\ \psi_2(t), \ \textit{if} \quad t\in[T,T+\beta],\end{cases}$$

where

$$w_1\in S_{F_1\circ u}, w_2\in S_{F_2\circ v},$$

and

$$\widetilde{G_{\alpha_i}}=\sup\left\{\int_a^T |G_{\alpha_i}(t,s)|ds, t\in I\right\}; \ i=1,2.$$

The following hypotheses will be used in the sequel:

(H_1) $F_1,F_2: I\times C([-r,\beta],E)\times C([-r,\beta],E)\to\mathcal{P}_{cp,cl,cv}(E)$ have weakly sequentially closed graph;

(H_2) For all continuous functions $u,v:[-r,\beta]\to E$, there exist measurable and Pettis integrable functions $w\in S_{F_1\circ u}$, $z\in S_{F_2\circ v}$, a.e. on I;

(H_3) There exist $p_1,p_2\in L^\infty(I,\mathbb{R}_+)$ such that for all $\varphi\in E^*$, we have

$$\|F_1(t,u,v)\|_{\mathcal{P}}\le p_1(t), \text{ for a.e. } t\in I, \text{ and each } u,v\in C([-r,\beta],E),$$

$$\|F_2(t,u,v)\|_{\mathcal{P}}\le p_2(t), \text{ for a.e. } t\in I, \text{ and each } u,v\in C([-r,\beta],E);$$

(H_4) For all bounded sets $B_i\subset C([-r,\beta],E), i=1,2$ and each $t\in I$, we have

$$\mu(F_1(t,B_1,B_2))\le p_1(t)\sup_{s\in[-r,\beta]}\mu(B_1(s)),$$

$$\mu(F_2(t,B_1,B_2))\le p_2(t)\sup_{s\in[-r,\beta]}\mu(B_2(s)),$$

where

$$B_i(t)=\{u(t): u\in B_i\}; \ i=1,2.$$

Set

$$p_i^*=ess\sup_{t\in I}p_i(t); \ i=1,2.$$

Now, we state and prove our existence result for problem (1.1)-(1.2) based on Theorem 2.14.

Theorem 3.4. *Assume that hypotheses* (H_1)*-*(H_4) *hold. If*

$$\widetilde{G_{\alpha_1}}p_1^*+\widetilde{G_{\alpha_2}}p_2^*<1, \tag{3.7}$$

then problem (1.1)-(1.2) has at least one solution.

Proof: Let the operator $N_i : \mathcal{C} \longmapsto \mathcal{P}(\mathcal{C}), i = 1, 2$ defined by

$$(N_1 u)(t) = \left\{ \begin{array}{c} h_1 : [a-r, T+\beta] \longrightarrow \mathcal{C} : \\ \\ h_1(t) = \begin{cases} \phi_1(t), \text{ if } \quad t \in [a-r, a], \\ \\ -\int_a^T G_{\alpha_1}(t,s) w_1(s) ds, \text{ if } \quad t \in I \\ \\ \psi_1(t), \text{ if } \quad t \in [T, T+\beta]. \end{cases} \end{array} \right\}, \tag{3.8}$$

and

$$(N_2 v)(t) = \left\{ \begin{array}{c} h_2 : [a-r, T+\beta] \longrightarrow \mathcal{C} : \\ \\ h_2(t) = \begin{cases} \phi_2(t), \text{ if } \quad t \in [a-r, a], \\ \\ -\int_a^T G_{\alpha_2}(t,s) w_2(s) ds, \text{ if } \quad t \in I \\ \\ \psi_2(t), \text{ if } \quad t \in [T, T+\beta]. \end{cases} \end{array} \right\} \tag{3.9}$$

where

$$w_1 \in S_{F_1 \circ u} = \{u : \Omega \longrightarrow L^1(I, E) : w_1(t) \in F_1(t, u^t, v^t) \text{ a.e. } t \in I\},$$

and

$$w_2 \in S_{F_2 \circ v} = \{v : \Omega \longrightarrow L^1(I, E) : w_2(t) \in F_2(t, u^t, v^t) \text{ a.e. } t \in I\}.$$

Consider the multi-valued map $N : \overline{\mathcal{C}} \to \mathcal{P}(\overline{\mathcal{C}})$ defined by:

$$(N(u,v))(t) = ((N_1 u)(t), (N_2 v)(t)).$$

From Lemma 3.3 it is clear that the fixed points of N are solutions of (1.1)-(1.2) .

Set

$$R \geq \max\left\{L_1 + L_2, \|\phi_1\|_{[a-r,a]} + \|\phi_2\|_{[a-r,a]}, \|\psi_1\|_{[T,T+\beta]} + \|\psi_2\|_{[T,T+\beta]}\right\}, \tag{3.10}$$

and define

$$Q = \left\{ (u,v) \in \mathcal{C} \times \mathcal{C} : \begin{array}{r} \|(u,v)\|_{\overline{\mathcal{C}}} \leq R, \\ \\ \|u(t_2) - u(t_1)\|_E \leq p_1^* \int_a^T |G_{\alpha_1}(t_2,s) - G_{\alpha_1}(t_1,s)| ds, \\ \\ and \ \ \|v(t_2) - v(t_1)\|_E \leq p_2^* \int_a^T |G_{\alpha_2}(t_2,s) - G_{\alpha_2}(t_1,s)| ds; \ t_1, t_2 \in I. \end{array} \right\}$$

It is clear that Q is a bounded, closed and convex subset of $\mathcal{C}$.

Step 1. *$N(u,v)$ is convex for each $(u,v) \in \mathcal{C}$.*

If $(h_1, d_1), (h_2, d_2)$ belong to $N(u,v)$, then there exist $v_1, v_2 \in S_{F \circ u}$ and $z_1, z_2 \in S_{F \circ v}$ such that for each $t \in I$ we have

$$h_i(t) = \int_a^T G_{\alpha_1}(t,s) v_i(s) ds; \ i = 1, 2,$$

and

$$d_i(t) = \int_a^T G_{\alpha_2}(t,s)z_i(s)ds;\ i = 1,2.$$

Let $0 \leq \lambda \leq 1$. For each $t \in I$, we have

$$(\lambda h_1 + (1-\lambda)h_2)(t) = \int_a^T G_{\alpha_1}(t,s)(\lambda v_1(s) + (1-\lambda)v_2(s))ds.$$

Since $S_{F\circ u}$ is convex (because F has convex values), we have $\lambda h_1 + (1-\lambda)h_2 \in N_1(u)$. Also, for each $t \in I$, we have

$$(\lambda d_1 + (1-\lambda)d_2)(t) = \int_a^T G_{\alpha_2}(t,s)(\lambda z_1(s) + (1-\lambda)z_2(s))ds.$$

Since $S_{F\circ v}$ is convex (because F has convex values), we have $\lambda d_1 + (1-\lambda)d_2 \in N_2(v)$. Hence $\lambda(h_1,d_1) + (1-\lambda)(h_2,d_2) \in N(u,v)$.

Step 2. *N maps Q into itself.*

Let $h_i \in N_i(Q), i = 1,2$ then there exists $u, v \in Q$, such that $h_1 \in N_1(u), h_2 \in N_2(v)$ and there exists a Pettis integrable function $w_1 \in F_1 \circ u$ and $w_2 \in F_2 \circ v$, assume that $h_i(t) \neq 0$. Then there exists $\varphi \in E^*$ such that $\|h_i(t)\|_E = |\varphi(h_i(t))|$. Thus, for any $i \in \{1,2\}$ we have

$$\|h_i(t)\|_E = \varphi\left(\int_a^T G_{\alpha_i}(t,s)w_i(s)ds\right).$$

If $t \in [a-r,a]$, then

$$\|h(t)\|_E = \|(h_1(t),h_2(t))\|_E \leq \|\phi_1\|_{[a-r,a]} + \|\phi_2\|_{[a-r,a]} \leq R,$$

and if $t \in [T, T+\beta]$, then

$$\|h(t)\|_E = \|(h_1(t),h_2(t))\|_E \leq \|\psi_1\|_{[T,T+\beta]} + \|\psi_2\|_{[T,T+\beta]} \leq R.$$

For each $t \in I$, we have

$$\|h_i(t)\|_E \leq \int_a^T |G_{\alpha_i}(t,s)||\varphi(w_i(s))|ds, i = 1,2.$$

By (H_3), we get

$$|\varphi(h_i(t))| \leq p_i^*.$$

Therefore

$$\begin{aligned}\|h_i(t)\|_E &\leq p_i^* \int_a^T |G_{\alpha_i}(t,s)|ds \\ &\leq p_i^* \widetilde{G_{\alpha_i}} = L_i,\end{aligned}$$

which implies that $\|h_i(t)\|_E \leq L_i$.
Hence we get

$$\begin{aligned}\|h(t)\|_E &\leq L_1 + L_2 \\ &\leq R.\end{aligned}$$

Now, suppose that $h_1 \in N_1(u), h_2 \in N_2(v)$ and $t_1, t_2 \in I = [a, T]$ with $t_1 < t_2$ so that $(h_i(t_2) - (h_i(t_1) \neq 0, i = 1, 2$ then, there exists $\varphi \in E^*$ such that,

$$\|h_1(t_2) - h_1(t_1)\|_E = \varphi(h_1(t_2) - h_1(t_1)),$$

and $\|\varphi\| = 1$. Then, for any $i \in \{1, 2\}$, we have

$$\begin{array}{rcl} \|h_1(t_2) - h_1(t_1)\|_E & = & \varphi(h_1(t_2) - h_1(t_1)) \\ & \leq & \varphi\left(\int_a^T |G_{\alpha_1}(t_2, s) - G_{\alpha_1}(t_1, s)| w_1(s)\right). \end{array}$$

Thus, we get

$$\begin{array}{rcl} \|h_1(t_2) - h_1(t_1)\|_E & \leq & \int_a^T |G_{\alpha_1}(t_2, s) - G_{\alpha_1}(t_1, s)||\varphi(w_1(s))| ds \\ & \leq & p_1^* \int_a^T |G_{\alpha_1}(t_2, s) - G_{\alpha_1}(t_1, s)| ds. \end{array}$$

Similarly,

$$\|h_2(t_2) - h_2(t_1)\|_E \leq p_2^* \int_a^T |G_{\alpha_2}(t_2, s) - G_{\alpha_2}(t_1, s)| ds.$$

Consequently,

$$N(Q) \subset Q.$$

Step 3. *N has weakly-sequentially closed graph.*
Let $(u_n, w_n), (x_n, y_n)$ be a sequence in $Q \times Q$, with

$$\begin{cases} u_n(t) \to u(t) & \text{in } E_\omega, \\ x_n(t) \to x(t) & \text{in } E_\omega, \end{cases} \quad \text{for each } t \in I,$$

and

$$\begin{cases} w_n \in N_1(u_n), \\ y_n \in N_2(x_n). \end{cases} \quad \text{for } n \in \{1, 2, 3, \ldots\}. \tag{3.11}$$

We shall show that

$$\begin{cases} w \in N_1(u), \\ y \in N_2(x). \end{cases}$$

By (3.11) there exist $f_n \in S_{F_1 \circ u_n}$ and $g_n \in S_{F_2 \circ x_n}$ such that

$$\begin{cases} w_n = \int_a^T G_{\alpha_1}(t, s) f_n(s) ds, \\ y_n = \int_a^T G_{\alpha_2}(t, s) g_n(s) ds. \end{cases}$$

We must show that there exist $f \in S_{F_1 \circ u}$ and $g \in S_{F_2 \circ x}$ such that for each $t \in I$,

$$\begin{cases} w = \int_a^T G_{\alpha_1}(t, s) f(s) ds, \\ y = \int_a^T G_{\alpha_2}(t, s) g(s) ds. \end{cases}$$

Since $F_i, i = 1, 2$ has compact values (so weakly compact), there exist a Pettis integrable subsequence f_{n_m}, g_{n_m} such that

$$f_{n_m}(t) \in F_1(t, u_n^t, x_n^t) \text{ a.e. } t \in I,$$

$$f_{n_m}(\cdot) \to f(\cdot) \text{ in } E_\omega \text{ as } m \to \infty,$$

and

$$g_{n_m}(t) \in F_2(t, u_n^t, x_n^t) \text{ a.e. } t \in I,$$

$$g_{n_m}(\cdot) \to g(\cdot) \text{ in } E_\omega \text{ as } m \to \infty.$$

As $F_i(t,\cdot,\cdot), i = 1,2$ has weakly sequentially closed graph, $f(t) \in F_1(t, u^t, x^t)$ *and* $g(t) \in F_2(t, u^t, x^t)$. Then by the Lebesgue dominated convergence theorem for the Pettis integral, we obtain

$$\varphi(w_n(t)) \to \varphi\left(\int_a^T G_{\alpha_1}(t,s) f_n(s) ds\right),$$

i.e., $w_n(t) \to (N_1 u)(t)$ in E_ω for each $t \in I$, which implies that $w \in N_1(u)$, and

$$\varphi(y_n(t)) \to \varphi\left(\int_a^T G_{\alpha_2}(t,s) g_n(s) ds\right),$$

i.e., $y_n(t) \to (N_2 u)(t)$ in E_ω for each $t \in I$, which implies that $y \in N_2(x)$.

Step 4. Now let $V = V_1 \times V_2$ be a subset of Q such that $V = conv(N(V) \cup \{(0,0)\})$. Obviously

$$V(t) \subset conv(N(V)(t) \cup \{(0,0)\}).$$

Since V is bounded and equicontinuous, the function $t \longmapsto v(t) = \mu(V(t))$ is continuous on $[a-r, T+\beta]$. By $(H_1) - (H_4)$, Lemma 2.13, and the properties of measure μ, for each $t \in I$, we have

$$\begin{aligned}
v(t) &\leq \mu(N(V)(t) \cup \{(0,0)\}) \\
&\leq \mu((NV)(t)) \\
&\leq \mu\left(\{((N_1u)(t), (N_2v)(t)) : (u,v) \in V\}\right) \\
&\leq \mu\Big\{ \int_a^T |G_{\alpha_1}(t,s)|(d(s),0) ds \\
&+ \int_a^T |G_{\alpha_2}(t,s)|(0, z(s)) ds \quad d(t) \in F_1(t,u^t,v^t), z(t) \in F_2(t,u^t,v^t), (u,v) \in V \Big\} \\
&\leq \int_a^T |G_{\alpha_1}(t,s)| \big(p_1(s)\mu(\{(d(s),0), d(t) \in F_1(t,u^t,v^t), (u,v) \in V\} ds\big) \\
&+ \int_a^T |G_{\alpha_2}(t,s)| (p_2(s)\mu(\{(0,z(s)); z(t) \in F_2(t,u^t,v^t), (u,v) \in V\} ds) \\
&\leq \int_a^T |G_{\alpha_1}(t,s)| p_1(s) \mu(V(s)) ds \\
&+ \int_a^T |G_{\alpha_2}(t,s)| p_2(s) \mu(V(s)) ds \\
&\leq \left(\widetilde{G_{\alpha_1}} p_1^* + \widetilde{G_{\alpha_2}} p_2^*\right) \|v\|_c.
\end{aligned}$$

Thus

$$\|v\|_c \leq \left(\widetilde{G_{\alpha_1}} p_1^* + \widetilde{G_{\alpha_2}} p_2^*\right) \|v\|_c.$$

From (3.7)we get $\|v\|_c = 0$, that is $\mu(V(t)) = 0$ for each $t \in I$.
For $t \in [a-r, a]$, we have

$$\begin{aligned} v(t) &= \mu((\phi_1(t), \phi_2(t))) \\ &= 0. \end{aligned}$$

Also for $t \in [T, T+\beta]$ we have

$$\begin{aligned} v(t) &= \mu(\psi_1(t), \psi_2(t)) \\ &= 0. \end{aligned}$$

Then $V(t)$ is weakly relatively compact in E. In view of Ascoli–Arzela theorem, V is weakly relatively compact in $\overline{\mathcal{C}}$. Applying Theorem 2.14, we conclude that N has a fixed point that is a weak solution of problem (1.1) − (1.2).

4. An Example

Let

$$E = l^1 = \left\{ u = (u_1, u_2, \dots, u_n, \dots), \sum_{n=1}^{\infty} |u_n| < \infty \right\}$$

be the Banach space with the norm

$$\|u\|_E = \sum_{n=1}^{\infty} |u_n|.$$

Consider the coupled system Caputo type modification of the Erdélyi–Kober fractional differential inclusions with retarded and advanced arguments

$$\begin{cases} (u(t), v(t)) = (e^t - 1, t^2) \quad t \in [-1, 0], \\ \\ {}^{2}_{c}D^{\frac{3}{2}}_{0^+} u(t) \in F_n(t, u^t, v^t), \quad t \in I = [0, 1] \\ \\ {}^{2}_{c}D^{\frac{4}{3}}_{0^+} v(t) \in G_n(t, u^t, v^t), \quad t \in I = [0, 1] \\ \\ (u(t), v(t)) = (t - 1, e^t - e) \quad t \in [1, 2], \end{cases} \tag{4.1}$$

where

$$F_n(t, u^t, v^t) = \frac{e^{-3}}{1 + \|u\|_{C([-1,1])} + \|v\|_{C([-1,1])}} \left[u_n^t - 1; u_n^t\right] \quad t \in [0, 1], u, v \in C([-r, \beta], E),$$

and

$$G_n(t, u^t, v^t) = \frac{e^{-t-6}}{1 + \|u\|_{C([-1,1])} + \|v\|_{C([-1,1])}} \left[v_n^t; v_n^t + 1\right] \quad t \in [0, 1], u, v \in C([-r, \beta], E).$$

Set

$$u = (u_1, u_2, \dots, u_n, \dots), \quad F = (F_1, F_2, \dots, F_n, \dots)$$

and

$$v = (v_1, v_2, \dots, v_n, \dots), \quad G = (G_1, G_2, \dots, G_n, \dots),$$

with
$$\alpha_1 = \frac{3}{2}, \alpha_2 = \frac{4}{3}, \ \rho = 2, \ r = 1, \ \beta = 1.$$
For each $u, v \in C([-1,1])$, $t \in [2,4]$, we have
$$\|F(t, u^t, v^t)\|_{\mathcal{P}} \leq e^{-3},$$
and
$$\|G(t, u^t, v^t)\|_{\mathcal{P}} \leq e^{-t-6}.$$
Hence, (H_2) is verified with $p_1^* = e^{-3}, p_2^* = e^{-6}$.
For each $t \in I, i = 1,2$ we have
$$\int_a^T |G_{\alpha_i}(t,s)|ds \leq \frac{1}{\Gamma(\alpha_i)} \left(\frac{t^\rho - a^\rho}{T^\rho - a^\rho}\right) \int_a^T \left| \left(\frac{T^\rho - s^\rho}{\rho}\right)^{\alpha_i - 1} s^{\rho-1} \right| ds$$
$$+ \frac{1}{\Gamma(\alpha_i)} \int_a^t \left| \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_i - 1} s^{\rho-1} \right| ds.$$
Then
$$\int_a^T |G_{\alpha_i}(t,s)|ds \leq \frac{2}{\Gamma(\alpha_i + 1)} \left(\frac{T^\rho - a^\rho}{\rho}\right)^{\alpha_i}.$$
Therefore
$$\widetilde{G_{\alpha_i}} \leq \frac{2}{\Gamma(\alpha_i + 1)} \left(\frac{T^\rho - a^\rho}{\rho}\right)^{\alpha_i}, i = 1,2.$$
Condition (3.7) is satisfied. Indeed, we have
$$\begin{aligned}
\widetilde{G_{\alpha_1}} p_1^* + \widetilde{G_{\alpha_2}} p_2^* &\leq \frac{2e^{-3}}{2^{\frac{3}{2}}\Gamma(\frac{3}{2}+1)} + \frac{2e^{-6}}{2^{\frac{4}{3}}\Gamma(\frac{4}{3}+1)} \\
&\approx 0.02813526732 \\
&< 1,
\end{aligned}$$
with $T = 1, a = 0, \ \alpha_1 = \frac{3}{2}$, and $\alpha_2 = \frac{4}{3}$. Since all conditions of Theorem 3.4 are satisfied, problem (4.1) has at least one solution.

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