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CONTINUOUSLY PARAMETRIZED SOLUTIONS OF A FRACTIONAL INTEGRO-DIFFERENTIAL INCLUSION

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Abstract. Existence of solutions continuously depending on a parameter of a fractional integro-differential inclusion defined by a Caputo type fractional derivative is obtained. By using this result we deduce the existence of a continuous selection of the solution set of the problem considered.

1. Introduction

In the last years one may see a strong development of the theory of differential equations and inclusions of fractional order ([5, 15, 20, 21] etc.). The main reason is that fractional differential equations are very useful tools in order to model many physical phenomena. In the fractional calculus there are several fractional derivatives. From them, the fractional derivative introduced by Caputo in [7] allows to use Cauchy conditions which have physical meanings.

A Caputo type fractional derivative of a function with respect to another function ([20]) that extends and unifies several fractional derivatives existing in the literature like Caputo, Caputo-Hadamard, Caputo-Katugampola was intensively studied in recent years [1–3] etc.. In particular, existence results and qualitative properties of the solutions for fractional differential equations defined by this fractional derivative are obtained in [2, 3, 14].

The present paper is concerned with the following problem

$$D_C^{\alpha,\psi}x(t) \in F(t, x(t), V(x)(t)) \quad a.e. \text{ in } [0, T], \quad x(0) = x_0, \tag{1.1}$$

where $\alpha \in (0, 1]$, $D_C^{\alpha,\psi}$ is the fractional derivative mentioned above, $x_0 \in \mathbf{R}$ and $F : [0, T] \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map. $V : C([0, T], \mathbf{R}) \to C([0, T], \mathbf{R})$ is a nonlinear Volterra integral operator defined by $V(x)(t) = \int_0^t k(t, s, x(s))ds$ with $k(., ., .) : [0, T] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ a given function.

Our goal is to obtain the existence of solutions continuously depending on a parameter for problem (1.1). A such kind of result may be interpreted as a continuous variant of Filippov's theorem ([16]) for problem (1.1). Moreover, as usual at a Filippov type existence theorem, our result provides an estimate between the family of starting "quasi" solutions and the family of solutions of the fractional integro-differential inclusion. By a "quasi" solution or an "almost" solution we mean a function for which we are able to estimate the distance between its derivative and the values of the set-valued map computed in this function. Obviously, if this distance is zero one has a solution of the differential inclusion.

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Also, this result allows us to provide a continuous selection of the solution set of problem (1.1). Our approach is essentially based on a well known theorem of Bressan and Colombo ([6]) concerning the existence of continuous selections of lower semicontinuous multifunctions with decomposable values. We treat, also, the case when the family of "quasi" solutions reduces to a single element and we establish a corresponding theorem.

The results in the present paper extend and unify similar results obtained for fractional differential inclusions defined by Riemann-Liouville fractional derivative ([8,18]), by Caputo fractional derivative ([10]), by Hadamard fractional derivative ([9,11]) and by Caputo-Katugampola fractional derivative ([12,13]).

The paper is organized as follows: in Section 2 we recall some preliminary results that we need in the sequel and Section 3 is devoted to our results.

2. Preliminaries

Let $T > 0$, $I := [0,T]$ and denote by $\mathcal{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I. Let X be a real separable Banach space with the norm $|.|$. Denote by $\mathcal{P}(X)$ the family of all nonempty subsets of X and by $\mathcal{B}(X)$ the family of all Borel subsets of X. If $A \subset I$ then $\chi_A(.) : I \to \{0,1\}$ denotes the characteristic function of A. For any subset $A \subset X$ we denote by $cl(A)$ the closure of A.

The distance between a point $x \in X$ and a subset $A \subset X$ is defined as usual by $d(x,A) = \inf\{|x-a|; a \in A\}$. We recall that Pompeiu-Hausdorff distance between the closed subsets $A, B \subset X$ is defined by $d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\}$, $d^*(A,B) = \sup\{d(a,B);\, a \in A\}$.

As usual, we denote by $C(I,X)$ the Banach space of all continuous functions $x(.) : I \to X$ endowed with the norm $|x(.)|_C = \sup_{t\in I} |x(t)|$ and by $L^1(I,X)$ the Banach space of all (Bochner) integrable functions $x(.) : I \to X$ endowed with the norm $|x(.)|_1 = \int_0^T |x(t)|dt$.

We recall first several preliminary results we shall use in the sequel.

Lemma 2.1 ([22])**.** *Let $u : I \to X$ be measurable and let $G : I \to \mathcal{P}(X)$ be a measurable closed-valued multifunction.*

Then, for every measurable function $r : I \to (0,\infty)$, there exists a measurable selection $g : I \to X$ of $G(\cdot)$ such that

$$|u(t) - g(t)| < d(u(t), G(t)) + r(t) \quad a.e. \ in \ I.$$

Definition 2.2. A subset $D \subset L^1(I,X)$ is said to be *decomposable* if for any $u(\cdot), v(\cdot) \in D$ and any subset $A \in \mathcal{L}(I)$ one has $u\chi_A + v\chi_B \in D$, where $B = I\backslash A$.

We denote by $\mathcal{D}(I,X)$ the family of all decomposable closed subsets of $L^1(I,X)$.

Next (S, d) is a separable metric space; we recall that a multifunction $G(\cdot) : S \to \mathcal{P}(X)$ is said to be lower semicontinuous (l.s.c.) if for any closed subset $C \subset X$, the subset $\{s \in S; G(s) \subset C\}$ is closed.

Lemma 2.3 ([6])**.** *Let $F^*(.,.) : I \times S \to \mathcal{P}(X)$ be a closed-valued $\mathcal{L}(I) \otimes \mathcal{B}(S)$-measurable multifunction such that $F^*(t,.)$ is l.s.c. for any $t \in I$.*

Then the multifunction $G(.) : S \to \mathcal{D}(I,X)$ defined by

$$G(s) = \{v \in L^1(I,X); \quad v(t) \in F^*(t,s) \quad a.e. \ in \ I\}$$

is l.s.c. with nonempty closed values if and only if there exists a continuous mapping $p(.) : S \to L^1(I, X)$ *such that*

$$d(0, F^*(t, s)) \le p(s)(t) \quad a.e. \ in \ I, \ \forall s \in S.$$

Lemma 2.4 ([6])**.** *Let* $G(.) : S \to \mathcal{D}(I, X)$ *be a l.s.c. multifunction with closed decomposable values and let* $\phi(.) : S \to L^1(I, X)$, $\psi(.) : S \to L^1(I, \mathbf{R})$ *be continuous such that the multifunction* $H(.) : S \to \mathcal{D}(I, X)$ *defined by*

$$H(s) = cl\{v(.) \in G(s); \quad |v(t) - \phi(s)(t)| < \psi(s)(t) \quad a.e. \ in \ I\}$$

has nonempty values.

Then $H(.)$ *has a continuous selection.*

Consider $\beta > 0$, $f(.) \in L^1(I, \mathbf{R})$ and $\psi(.) \in C^n(I, \mathbf{R})$ such that $\psi'(t) > 0 \ \forall \, t \in I$.

Definition 2.5 ([20])**.** a) The ψ - Riemann-Liouville fractional integral of f of order β is defined by

$$I^{\beta,\psi} f(t) = \frac{1}{\Gamma(\beta)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\beta-1} f(s) ds,$$

where Γ is the (Euler's) Gamma function defined by $\Gamma(\beta) = \int_0^\infty t^{\beta-1} e^{-t} dt$.

b) The ψ - Riemann-Liouville fractional derivative of f of order β is defined by

$$D^{\beta,\psi} f(t) = \frac{1}{\Gamma(n-\beta)} (\frac{1}{\psi'(t)} \frac{d}{dt})^n \int_0^t \psi'(s)(\psi(t) - \psi(s))^{n-\beta-1} f(s) ds,$$

where $n = [\beta] + 1$.

c) The ψ - Caputo fractional derivative of f of order β is defined by

$$D_C^{\beta,\psi} f(t) = D^{\beta,\psi}[f(t) - \sum_{k=0}^{n-1} \frac{f_\psi^{[k]}(0)}{k!} (\psi(t) - \psi(0))^k],$$

where $f_\psi^{[k]}(t) = (\frac{1}{\psi'(t)} \frac{d}{dt})^k x(t)$, $n = \beta$ if $\alpha \in \mathbf{N}$ and $n = [\beta] + 1$, otherwise .

We note that if $\beta = m \in \mathbf{N}$ then $D_C^{\beta,\psi} f(t) = f_\psi^{[m]}(t)$ and if $n = [\beta] + 1$ then $D_C^{\beta,\psi} f(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{n-\alpha-1} f_\psi^{[n]}(s) ds$. Also, if $\psi(t) \equiv t$ one obtains Caputo's fractional derivative, if $\psi(t) \equiv \ln(t)$ one obtains Caputo-Hadamard's fractional derivative and, finally, if $\psi(t) \equiv t^\sigma$ one obtains Caputo-Katugampola's fractional derivative.

In what follows we need the following technical lemma proved in [2] (namely, Theorem 2 in [2]).

Lemma 2.6. *Let* $\alpha \in [0, 1)$ *and* $\psi(.) \in C^1(I, \mathbf{R})$ *with* $\psi'(t) > 0 \ \forall \, t \in I$. *For a given integrable function* $h(.) : I \to \mathbf{R}$, *the unique solution of the initial value problem*

$$D_C^{\alpha,\psi} x(t) = h(t) \quad a.e. \ in \ I, \quad x(0) = x_0,$$

is given by

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} h(s) ds.$$

Definition 2.7. By a solution of the problem (1.1) we mean a function $x \in C(I, \mathbf{R})$ for which there exists a function $h \in L^1(I, \mathbf{R})$ satisfying $h(t) \in F(t, x(t), V(x)(t))$ *a.e.* in I, $D_C^{\alpha,\psi} x(t) = h(t)$ *a.e.* in I and $x(0) = x_0$.

Also, the set of all solutions of (1.1) will be denoted with $\mathcal{S}(x_0)$.

3. Main results

In what follows $\alpha \in [0, 1)$ and $\psi(.) \in C^1(I, \mathbf{R})$ with $\psi'(t) > 0 \; \forall \, t \in I$.

Hypothesis 3.1. i) $F(.,.) : I \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R} \times \mathbf{R})$ measurable with nonempty closed values.

ii) There exists $l(.) \in L^1(I, (0, \infty))$ such that, for almost all $t \in I$

$$d_H(F(t, u_1, v_1), F(t, u_2, v_2)) \le l(t)(|u_1 - u_2| + |v_1 - v_2|) \quad \forall \, u_1, u_2, v_1, v_2 \in \mathbf{R}.$$

iii) The mapping $k(., ., .) : I \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ verifies: $\forall y \in \mathbf{R}$, $(s, t) \to k(s, t, y)$ is measurable.

iv) $|k(s, t, y) - k(s, t, x)| \le l(t)|y - x| \quad a.e. \; (s, t) \in I \times I, \quad \forall \, y, x \in \mathbf{R}$.

Hypothesis 3.2. (i) S is a separable metric space, the mappings $a(.) : S \to \mathbf{R}$ and $\varepsilon(.) : S \to (0, \infty)$ are continuous.

(ii) There exists $g(.), q(.) : S \to L^1(I, \mathbf{R})$, $y(.) : S \to C(I, \mathbf{R})$ continuous that satisfy

$$(Dy(s))_C^{\alpha,\psi}(t) = g(s)(t) \quad \text{for } a.e. \; t \in I, \quad \forall s \in S,$$
$$d(g(s)(t), F(t, y(s)(t), V(y(s)(.))(t)) \le q(s)(t) \quad \text{for } a.e. \; t \in I, \; \forall \, s \in S.$$

Next, we use the following notation

$$L(t) := l(t)(1 + \int_0^t l(u)du), \quad t \in J,$$

$$\xi(s) = \frac{1}{1 - I^{\alpha,\psi} L} (|a(s) - y(s)(0)| + \varepsilon(s) + I^{\alpha,\psi} q(s)), \; s \in S,$$

where $I^{\alpha,\psi} L := \sup_{t \in I} |I^{\alpha,\psi} L(t)|$ and $I^{\alpha,\psi} q(s) := \sup_{t \in I} |I^{\alpha,\psi} q(s)(t)|$.

Theorem 3.3. *Assume that Hypotheses 3.1 and 3.2 are verified.*

If $I^{\alpha,\psi} L < 1$, then there exists $x(.) : S \to C(I, \mathbf{R})$ continuous such that $x(s)(.)$ is a solution of

$$D_C^{\alpha,\psi} z(t) \in F(t, z(t), V(z)(t)), \quad z(0) = a(s)$$

and

$$|x(s)(t) - y(s)(t)| \le \xi(s) \quad \forall \, (t, s) \in I \times S.$$

Proof. We consider the following notations $q_n(s) := (I^{\alpha,\psi} L)^{n-1} (|a(s) - y(s)(0)| + I^{\alpha,\psi} q(s) + \frac{n}{n+1} \varepsilon(s))$, $n \ge 1$, $x_0(s)(t) = y(s)(t)$, $\forall s \in S, t \in I$. Define the set-valued maps

$$A_0(s) = \{f \in L^1(I, \mathbf{R}); \quad f(t) \in F(t, y(s)(t), V(y(s)(.))(t)) \quad a.e. \text{ in } I\},$$

$$B_0(s) = \text{cl}\{f \in A_0(s); \quad |f(t) - g(s)(t)| < q(s) + \frac{\Gamma(\alpha + 1)}{2(\psi(T) - \psi(0))^\alpha} \varepsilon(s)\}.$$

By hypothesis, $d(g(s)(t), F(t, y(s)(t), V(y(s)(.))(t)) \le q(s)(t) < q(s)(t) + \frac{\Gamma(\alpha+1)}{2(\psi(T) - \psi(0))^\alpha} \varepsilon(s)$; hence with Lemma 2.1, $B_0(s)$ is not empty.

Define $G_0(t,s) = F(t, y(s)(t), V(y(s)(.))(t))$ and one has

$$d(0, G_0(t,s)) \leq |g(s)(t)| + q(s)(t) = q^*(s)(t)$$

with $q^*(.) : S \to L^1(I, \mathbf{R})$ continuous.

Taking into account Lemmas 2.3 and 2.4 we deduce that exists h_0 a selection of B_0 that is continuous, i.e.

$$h_0(s)(t) \in F(t, y(s)(t), V(y(s)(.))(t)) \quad a.e. \text{ in } I, \ \forall s \in S,$$

$$|h_0(s)(t) - g(s)(t)| \leq q(s)(t) + \frac{\Gamma(\alpha+1)}{2(\psi(T)-\psi(0))^\alpha}\varepsilon(s) \quad \forall t \in I, \ s \in S.$$

Define $x_1(s)(t) = a(s) + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(u)(\psi(t)-\psi(u))^{\alpha-1}h_0(s)(u)du$ and we may write

$$|x_1(s)(t) - x_0(s)(t)| \leq |a(s) - y(s)(0)| + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(u)(\psi(t)-\psi(u))^{\alpha-1}.$$
$$|h_0(s)(u) - g(s)(u)|du \leq |a(s) - y(s)(0)| + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(u)(\psi(t)-\psi(u))^{\alpha-1}.$$
$$(q(s)(u) + \frac{\Gamma(\alpha+1)}{2(\psi(T)-\psi(0))^\alpha}\varepsilon(s))du \leq |a(s) - y(s)(0)| + I^{\alpha,\psi}q(s) + \frac{1}{2}\varepsilon(s) = q_1(s).$$

Next, we define the sequences $h_n(.) : S \to L^1(I, \mathbf{R})$, $x_n(.) : S \to C(I, \mathbf{R})$ such that

a) $x_n(.) : S \to C(I, \mathbf{R})$, $h_n(.) : S \to L^1(I, \mathbf{R})$ are continuous.
b) $h_n(s)(t) \in F(t, x_n(s)(t), V(x_n(s)(.))(t))$, $s \in S$, for a.e. $t \in I$.
c) $|h_n(s)(t) - h_{n-1}(s)(t)| \leq L(t)q_n(s)$, $q_n(s) := (I^{\alpha,\psi}L)^{n-1}(|a(s) - y(s)(0)| + I^{\alpha,\psi}q(s) + \frac{n}{n+1}\varepsilon(s))$, $s \in S$, for a.e. $t \in I$.
d) $x_{n+1}(s)(t) = a(s) + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(u)(\psi(t)-\psi(u))^{\alpha-1}h_n(s)(u)du$.

If we assume that $h_i(.), x_i(.)$ are already constructed with properties a)-c) and define $x_{n+1}(.)$ as in d). It follows from c) and d) that

$$\begin{aligned}
&|x_{n+1}(s)(t) - x_n(s)(t)| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(u)(\psi(t)-\psi(u))^{\alpha-1}|h_n(s)(u) - h_{n-1}(s)(u)|du \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(u)(\psi(t)-\psi(u))^{\alpha-1}L(u)q_n(s)du \qquad (3.1)\\
&\leq I^{\alpha,\psi}L \cdot q_n(s) = q_{n+1}(s) - (I^{\alpha,\psi}L)^n\frac{\varepsilon(s)}{(n+1)(n+2)} \\
&< q_{n+1}(s).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&d(h_n(s)(t), F(t, x_{n+1}(s)(t), V(x_{n+1}(s)(.))(t))\\
&\le l(t)(|x_{n+1}(s)(t) - x_n(s)(t)| + \int_0^t l(u)|x_{n+1}(s)(u) - x_n(s)(u)|du)\\
&\le L(t)(q_{n+1}(s) - (I^{\alpha,\psi}L)^n \frac{\varepsilon(s)}{(n+1)(n+2)})\\
&< L(t)(q_{n+1}(s) - (I^{\alpha,\psi}L)^n \frac{\varepsilon(s)}{2(n+1)(n+2)}).
\end{aligned}$$

For $s \in S$ we define

$$A_{n+1}(s) = \{f \in L^1(I, \mathbf{R}); f(t) \in F(t, x_{n+1}(s)(t), V(x_{n+1}(s)(.))(t))\, a.e. \text{ in}\, I\},$$

$$B_{n+1}(s) = \text{cl}\{f \in A_{n+1}(s); \quad |f(t) - h_n(s)(t)| < L(t)q_{n+1}(s) \; a.e. \text{ in } I\}.$$

We are able now to apply Lemma 2.1 and find $w(.) \in L^1(I, \mathbf{R})$ such that $w(t) \in F(t, x_{n+1}(s)(t), V(x_{n+1}(s)(.))(t))$ a.e. (I) and

$$\begin{aligned}
|w(t) - h_n(s)(t)| &< d(h_n(s)(t), F(t, x_{n+1}(s)(t), V(x_{n+1}(s)(.))(t))\\
&\quad + L(t)(I^{\alpha,\psi}L)^n \frac{\varepsilon(s)}{2(n+1)(n+2)})\\
&< L(t)q_{n+1}(s) - L(t)(I^{\alpha,\psi}L)^n \frac{\varepsilon(s)}{2(n+1)(n+2)}\\
&\quad + L(t)(I^{\alpha,\psi}L)^n \frac{\varepsilon(s)}{2(n+1)(n+2)} = L(t)q_{n+1}(s).
\end{aligned}$$

i.e., $B_{n+1}(s)$ is nonempty.

We introduce $G_{n+1}(t, s) = F(t, x_{n+1}(s)(t), V(x_{n+1}(s)(.))(t))$. One may estimate

$$\begin{aligned}
&d(0, G_{n+1}(t, s)) \le |h_n(s)(t)| + L(t)|x_{n+1}(s)(t) - x_n(s)(t)| \le\\
&|h_n(s)(t)| + L(t)q_{n+1}(s) = q^*_{n+1}(s)(t) \quad a.e. \text{ in } I,
\end{aligned}$$

where $q^*_{n+1}(.) : S \to L^1(I, \mathbf{R})$ is continuous.

As above one may find $h_{n+1}(.) : S \to L^1(I, \mathbf{R})$ continuous and such that

$$h_{n+1}(s)(t) \in F(t, x_{n+1}(s)(t), V(x_{n+1}(s)(.))(t)) \quad \forall s \in S, \; a.e. \text{ in } I,$$

$$|h_{n+1}(s)(t) - h_n(s)(t)| \le L(t)q_{n+1}(s) \quad \forall s \in S, \; a.e. \text{ in } I.$$

Using conditions c), d) and (3.1) one has

$$\begin{aligned}
|x_{n+1}(s)(.) - x_n(s)(.)|_C &\le q_{n+1}(s)\\
&\le (I^{\alpha,\psi}L)^n(|a(s) - y(s)(0)| + I^{\alpha,\psi}q(s) + \varepsilon(s))
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
|h_{n+1}(s)(.) - h_n(s)(.)|_1 &\le |L(.)|_1 q_n(s)\\
&\le |L(.)|_1 (I^{\alpha,\psi}L)^n(|a(s) - y(s)(0)|\\
&\quad + I^{\alpha,\psi}q(s) + \varepsilon(s)).
\end{aligned} \tag{3.3}$$

Since by hypothesis $I^{\alpha,\psi}L < 1$ from (3.2) and (3.3) it follows that the sequences $h_n(s)(.)$, $x_n(s)(.)$ are Cauchy in spaces $L^1(I, \mathbf{R})$ and $C(I, \mathbf{R})$, respectively. We denote by $h(.) : S \to L^1(I, \mathbf{R})$ and $x(.) : S \to C(I, \mathbf{R})$ their limits. The function

$s \to |a(s) - y(s)(0)| + |I^{\alpha,\psi}q(s)| + \varepsilon(s)$ is continuous, therefore locally is bounded. Therefore, from (3.3) we obtain the continuity of $s \to h(s)(.)$ from S into $L^1(I, \mathbf{R})$.

As above, from (3.2), we obtain that the Cauchy condition is satisfied for the sequence $x_n(s)(.)$ locally uniformly with respect to s. Hence, the function $s \to x(s)(.)$ is continuous. Since the convergence of $x_n(s)(.)$ to $x(s)(.)$ is uniform and

$$d(h_n(s)(t), F(t, x(s)(t), V(x(s)(.))(t)) \le L(t)|x_n(s)(t) - x(s)(t)| \quad a.e. \text{ in } I,$$

$\forall s \in S$ we may pass to the limit in order to deduce that

$$h(s)(t) \in F(t, x(s)(t), V(x(s)(.))(t)) \quad \forall s \in S, \; a.e. \text{ in } I.$$

We note that we have the following estimate

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\Big|\int_0^t \psi'(u)(\psi(t)-\psi(u))^{\alpha-1}h_n(s)(u)du \\
&\qquad\qquad - \int_0^t \psi'(u)(\psi(t)-\psi(u))^{\alpha-1}\cdot h(s)(u)du\Big| \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(u)(\psi(t)-\psi(u))^{\alpha-1}|h_n(s)(u)-h(s)(u)|du \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(u)(\psi(t)-\psi(u))^{\alpha-1}L(u).|x_{n+1}(s)(.)-x_n(s)(.)|_C du \\
&\le I^{\alpha,\psi}L.|x_{n+1}(s)(.)-x_n(s)(.)|_C.
\end{aligned}$$

It remains to pass to the limit in d) in order to find

$$x(s)(t) = a(s) + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(u)(\psi(t)-\psi(u))^{\alpha-1}h(s)(u)du.$$

For all $n \ge 1$ we add inequalities (3.1) and we deduce

$$\begin{aligned}
|x_{n+1}(s)(t) - y(s)(t)| &\le \sum_{l=1}^n q_l(s) \\
&= \sum_{l=1}^n (I^{\alpha,\psi}L)^{l-1}(b(s) + I^{\alpha,\psi}q(s) + \varepsilon(s)) \\
&= \sum_{l=1}^n (I^{\alpha,\psi}L)^{l-1}(|a(s)-y(s)(0)| + \varepsilon(s) + I^{\alpha,\psi}q(s)) \\
&\le \frac{1}{1 - I^{\alpha,\psi}L}\cdot(|a(s)-y(s)(0)| + \varepsilon(s) + I^{\alpha,\psi}q(s)) \\
&= \xi(s).
\end{aligned}$$

Finally, passing to the limit in the last inequality we conclude the proof. □

Remark 3.4. If in Theorem 3.3 $\psi(t) \equiv t$ we obtain Theorem 3.1 in [10]; if in Theorem 3.3 $\psi(t) \equiv \ln(t)$ and $V(x)(t) \equiv I^{\beta,\psi}x(t)$, $\beta > 0$ we get Theorem 3.6 in [9] and if in Theorem 3.3 $\psi(t) \equiv t^\sigma$ we cover Theorem 3.3 in [12].

By using Theorem 3.3 we may find a continuous selection of the solution set of problem (1.1).

Hypothesis 3.5. Hypothesis 3.1 is satisfied, $I^{\alpha,\psi}L < 1$, $q_0(.) \in L^1(I, \mathbf{R}_+)$ exists with $d(0, F(t, 0, V(0)(t)) \leq q_0(t)$ a.e. (I).

Corollary 3.6. *Hypothesis 3.5 is verified. Then there exists a function $s(.,.) : I \times \mathbf{R} \to \mathbf{R}$ with the following properties*

a) $s(., x) \in \mathcal{S}(x)$, $\forall x \in \mathbf{R}$.
b) $x \to s(., x)$ *is continuous from* $\mathbf{R}$ *into* $C(I, \mathbf{R})$.

Proof. It is enough to take in Theorem 3.3 $S = \mathbf{R}$, $a(x) = x$, $\forall x \in \mathbf{R}$, $\varepsilon(.) : \mathbf{R} \to (0, \infty)$ an arbitrary fixed continuous mapping, $g(.) = 0$, $y(.) = 0$, $q(x)(t) = q_0(t)$ $\forall x \in \mathbf{R}$, $t \in I$. □

In the particular case when S reduces to a single element, Theorem 3.3 reduces to a Filippov type existence result for problem (1.1).

Theorem 3.7. *Assume that Hypothesis 3.1 is satisfied, $I^{\alpha,\psi}L(T) < 1$ and let $y \in C(I, \mathbf{R})$ be such that there exists $q(.) \in L^1(I, \mathbf{R})$ with $I^{\alpha,\psi}q(T) < +\infty$ and $d(D_C^{\alpha,\psi}y(t), F(t, y(t), V(y)(t))) \leq q(t)$ a.e. (I).*

Then there exists $x(.)$ a solution of problem (1.1) satisfying for all $t \in I$

$$|x(t) - y(t)| \leq \frac{1}{1 - I^{\alpha,\psi}L(T)}(|x_0 - y(0)| + I^{\alpha,\psi}q(T)).$$

Sketch of proof. Since the proof is similar with the proof of Theorem 3.3 in [14] we point out only the main steps of the proof. □

The set-valued map $t \to F(t, y(t), V(y)(t))$ is measurable with closed values and

$$F(t, y(t), V(y)(t)) \cap \{D_C^{\alpha,\psi}y(t) + q(t)[-1, 1]\} \neq \emptyset \quad a.e. \text{ in } I.$$

It follows from celebrated Kuratowski and Ryll-Nardzewski selection theorem (e.g., [4]) that there exists a measurable map $f_1(t) \in F(t, y(t), V(y)(t))$ *a.e.* in I such that

$$|f_1(t) - D_C^{\alpha,\psi}y(t)| \leq q(t) \quad a.e. \text{ in } I.$$

Define $x_1(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(u)(\psi(t) - \psi(u))^{\alpha-1}f_1(u)du$ and one has

$$|x_1(t) - y(t)| \leq |x_0 - y(0)| + I^{\alpha,\psi}q(T).$$

Then, by induction we construct two sequences $x_n(.) \in C(I, \mathbf{R})$, $f_n(.) \in L^1(I, \mathbf{R})$, $n \geq 1$ with the following properties

$$x_n(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(u)(\psi(t) - \psi(u))^{\alpha-1}f_n(s)ds, \quad t \in I, \tag{3.4}$$

$$f_n(t) \in F(t, x_{n-1}(t), V(x_{n-1})(t)) \quad a.e. \text{ in } I, \tag{3.5}$$

$$|f_{n+1}(t) - f_n(t)| \leq L(t)(|x_n(t) - x_{n-1}(t)| + \int_0^t L(s)|x_n(s) - x_{n-1}(s)|ds) \; a.e. \, I \tag{3.6}$$

If this construction is realized then, by a similar computation as in the proof of Theorem 3.3, for almost all $t \in I$ we have

$$|x_{n+1}(t) - x_n(t)| \leq (I^{\alpha,\psi}L(T))^n(|x_0 - y(0)| + I^{\alpha,\psi}q(T)) \quad \forall n \in \mathbf{N}.$$

Therefore $\{x_n(.)\}$ is a Cauchy sequence in the Banach space $C(I,\mathbf{R})$, hence converging uniformly to some $x(.) \in C(I,\mathbf{R})$. In particular, it follows from (3.6) that for almost all $t \in I$, the sequence $\{f_n(t)\}$ is Cauchy in $\mathbf{R}$. Let $f(.)$ be the pointwise limit of $f_n(.)$.

Moreover, one has

$$\begin{aligned}
|x_n(t) - y(t)| &\leq |x_1(t) - y(t)| + \sum_{i=1}^{n-1} |x_{i+1}(t) - x_i(t)| \\
&\leq |x_0 - y(0)| + I^{\alpha,\psi} q(T) + \sum_{i=1}^{n-1} (I^{\alpha,\psi} L(T))^i (|x_0 - y(0)| + I^{\alpha,\psi} q(T)) \\
&= \frac{|x_0 - y(0)| + I^{\alpha,\psi} q(T)}{1 - I^{\alpha,\psi} L(T)}.
\end{aligned}$$

On the other hand, from (3.6) and (3.7) we obtain for almost all $t \in I$

$$\begin{aligned}
|f_n(t) - D_C^{\alpha,\psi} y(t)| &\leq \sum_{i=1}^{n-1} |f_{i+1}(t) - f_i(t)| + |f_1(t) - D_C^{\alpha,\psi} y(t)| \\
&\leq L(t) \frac{|x_0 - y(0)| + I^{\alpha,\psi} q(T)}{1 - I^{\alpha,\psi} L(T)} + q(t).
\end{aligned}$$

Thus the sequence $f_n(.)$ is integrably bounded and therefore $f(.) \in L^1(I,\mathbf{R})$.

Using Lebesgue's dominated convergence theorem and passing to the limit we deduce that $x(.)$ is a solution of (1.1) satisfying the desired estimate.

Finally, the construction of the sequences $x_n(.), f_n(.)$ with the properties in (3.4)-(3.6) is done by induction.

Since the first step is already realized, assume that for some $N \geq 1$ we already constructed $x_n(.) \in C(I,\mathbf{R})$ and $f_n(.) \in L^1(I,\mathbf{R})$, $n = 1, 2, ...N$ satisfying (3.4)-(3.6). The set-valued map $t \to F(t, x_N(t), V(x_N)(t))$ is measurable. Moreover, the map $t \to L(t)(|x_N(t) - x_{N-1}(t)| + \int_0^t L(s)|x_N(s) - x_{N-1}(s)|ds)$ is measurable. By the lipschitzianity of $F(t,.)$ we have that for almost all $t \in I$

$$\begin{aligned}
F(t, x_N(t)) \cap \{f_N(t) + L(t)(|x_N(t) - x_{N-1}(t)| \\
+ \int_0^t L(s)|x_N(s) - x_{N-1}(s)|ds)[-1,1]\} \neq \emptyset.
\end{aligned}$$

Kuratowski and Ryll-Nardzewski selection theorem yields that there exist a measurable selection $f_{N+1}(.)$ of $F(., x_N(.), V(x_N)(.))$ such that for almost all $t \in I$

$$|f_{N+1}(t) - f_N(t)| \leq L(t)(|x_N(t) - x_{N-1}(t)| + \int_0^t L(s)|x_N(s) - x_{N-1}(s)|ds).$$

We define $x_{N+1}(.)$ as in (3.4) with $n = N + 1$. Thus $f_{N+1}(.)$ satisfies (3.5) and (3.6).

Remark 3.8. If in Theorem 3.7 $\psi(t) \equiv t$ we find Theorem 3.3 in [10]; if in Theorem 3.7 $\psi(t) \equiv \ln(t)$ and $V(x)(t) \equiv I^{\beta,\psi}x(t)$, $\beta > 0$ a similar result may be found in [11] and if in Theorem 3.7 $\psi(t) \equiv t^\sigma$ we obtain Theorem 3.2 in [13].

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