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A CLASS OF NONLINEAR DIFFERENTIAL OPTIMIZATION PROBLEMS IN FINITE DIMENSIONAL SPACES

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ABSTRACT. In this paper, we introduce a new class of nonlinear optimization problems controlled by differential equations ((DOP), for short) in Euclidean spaces. Under suitable conditions, the existence of weak Carathéodory solutions for (DOP) is proved via applying Weierstrass existence theorem and Filippov implicit function lemma. Then, a numerical approximation approach corresponding to (DOP) is introduced, and further a convergence theorem is established. Finally, we provide an numerical example to verify the validity of the proposed algorithm.

1. Introductionand preliminaries

Let K be a subset of $\mathbb{R}^m$. Also let $g\colon [0,T]\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}$, $f\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ and $B : [0,T]\times\mathbb{R}^n \to \mathbb{R}^{n\times m}$ be given functions, which will be specified in Section 2. In this paper, we consider the following differential optimization problem

$$(1.1)\qquad \begin{cases} \begin{cases} \text{Minimize } g(t,x(t),u(t)),\ t\in[0,T] \\ \text{subject to } u(t)\in K,\ t\in[0,T] \end{cases} & (1.1)_1 \\ \\ \begin{cases} \dot{x}(t) = f(t,x(t)) + B(t,x(t))u(t),\ t\in[0,T] \\ x(0)=x_0. \end{cases} & (1.1)_2 \end{cases}$$

It is well known that in our life there are a lot of problems from economics, transportation, traffic networks, engineering sciences etc., can be modeled by an optimization problem coupled with a differential equation [1,8,9,11–14]. For example, recently, Rachah-Torres in [12] investigated the following optimal control model of the 2014 Ebola Outbreak in West Africa

$$(1.2)\qquad \begin{gathered} \text{Minimize } J(u) = \int_0^{t_f} I(t) + \frac{A}{2}u^2(t)dt, \\ \text{subject to } u(t)\in K := [0,0.9],\ t\in[0,t_f], \\ \begin{cases} \dfrac{dS(t)}{dt} = -\beta S(t) - u(t)S(t), \\ \dfrac{dI(t)}{dt} = \beta S(t)I(t) - \mu I(t), \\ \dfrac{dR(t)}{dt} = \mu I(t) + u(t)S(t), \end{cases} \end{gathered}$$

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where S, I and R stand for susceptible individuals, infected individuals and recovered individuals, respectively. It is obvious to see that problem (1.2) is an optimization problem constrained by an ordinary differential equation.

Although some significant results have been obtained for optimization problems coupled with differential equations (see [3, 5, 7]), there are still many interesting and unanswered problems. To our knowledge, only a few results on existence of solutions were obtained for differential optimization problems (1.1) (cf. [15]). The differential optimization problem (1.1) looks like an optimal control problem with the state variable x and control u. But both of them are completely different in nature. The main difference lies in the following:

- The control $u(t)$ in (1.1) is the solution of the differential optimization problem (1.1) at each time t. In other words, it is a pointwise optimization, whose objective function depends on the state and the control at time t.
- While the control u in the optimal control problems is to minimize a performance function that is an entire function in a subset of a function space.

Obviously, the differential optimization problem (1.1) is more complicated than optimal control problems. This is one of our motivations for the study of such differential optimization problem (1.1) in the present work.

It is worth mention that one of the most useful methods to study dynamical optimization problems is the gradient flow technique. However, in order to apply the gradient flow technique, ones had to assume that the objective functions involved (see (1.2)) should be Fréchet-differentiable. In fact, this hypothesis limits the applicability of the gradient flow method. In the present study, however, we consider the dynamical optimization problem (1.1) without the differentiability assumption for the objective functions involved. In this way, we expand the applicability of our results. This is our another motivation. Also, our results extend and generalize the ones in [15] in many respects (see Section 2).

This work is firstly devoted to explore the existence of solutions for (DOP) under suitable conditions, and then to establish a numerical approximating method to generate the approximate numerical value solution of (DOP).

For any $(t, x) \in [0, T] \times \mathbb{R}^n$, we denote the solution set of the optimization $(1.1)_1$ as $S(K, g(t, x, \cdot))$. Now, we introduce the solutions of problem (DOP) which are understood in the following weak Carathéodory sense.

Definition 1.1. A pair of functions (x, u), with $x \in C([0, T]; \mathbb{R}^n)$ and $u\colon [0, T] \to \mathbb{R}^m$ measurable, is said to be a weak Carathéodory solution of problem (DOP), if

$$x(t) = x_0 + \int_0^t [f(s, x(s)) + B(s, x(s))u(s)]ds,\ t \in [0, T],$$

and $u(s) \in S(K, g(s, x(s), \cdot))$ for $s \in [0, T]$.

In the end of this section, we will recall some basic definitions and preliminary facts needed in the sequel.

Definition 1.2. Let X, Y be topological spaces. A set-valued mapping $F : X \rightrightarrows Y$ is

(a) compact if its range $F(X)$ is relatively compact in Y;

(b) quasi-compact if its restriction to any compact subset $K \subset X$ is compact.

Theorem 1.3 ([4, Theorem 1.1.12])**.** *Let X and Y be metric spaces and $F\colon X \to K(Y)$ be a closed quasi-compact set-valued mapping, where $K(Y) := \{D \subseteq Y : D \text{ is compact}\}$. Then F is upper semicontinuous.*

Theorem 1.4 ([6, Corollary 2.26])**.** *Let $\Gamma\colon [0,T] \rightrightarrows X$ be measurable taking closed set values. Let $f\colon [0,T]\times X \to Y$ be a Souslin measurable function. For each $x \in X$, $f(\cdot,x)$ is measurable and for a.e. $t \in [0,T]$, $f(t,\cdot)$ is continuous. Let $y\colon [0,T] \to Y$ be Lebesgue measurable satisfying*

$$y(t) \in f(t,\Gamma(t)), \text{ a.e. } t \in [0,T].$$

Then there exists a measurable function $h : [0,T] \to Y$, such that

$$\begin{cases} h(t) \in \Gamma(t), \text{ a.e. } t \in [0,T], \\ y(t) = f(t,h(t)), \text{ a.e. } t \in [0,T]. \end{cases}$$

2. Existence of solutions for (DOP)

This section is devoted to explore the existence of solutions for (DOP) (1.1) in the weak Carathéodory sense, see Definition 1.1. In what follows, let the functions $f\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $B\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are of Carathéodory type. We also need the following assumptions:

(A_1) there exists $\rho_f \in L_+([0,T])$ such that

$$\begin{cases} \|f(t,x_1) - f(t,x_2)\|_{\mathbb{R}^n} \le \rho_f(t)\|x_1 - x_2\|_{\mathbb{R}^n}, \\ \|f(t,0)\|_{\mathbb{R}^n} \le \rho_f(t), \end{cases} \quad \text{for a.e. } \ t \in [0,T];$$

(A_2) there exists $\rho_B \in L_+([0,T])$ such that

$$\begin{cases} \|B(t,x_1) - B(t,x_2)\|_{\mathbb{R}^{n\times m}} \le \rho_B(t)\|x_1 - x_2\|_{\mathbb{R}^n}, \\ \|B(t,0)\|_{\mathbb{R}^{n\times m}} \le \rho_B(t), \end{cases} \quad \text{for a.e. } \ t \in [0,T];$$

(A_3) $g\colon [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is continuous and $g(t,x,\cdot)\colon K \to \mathbb{R}$ is convex for all $(t,x) \in [0,T] \times \mathbb{R}^n$.

Our main result is formulated as follows.

Theorem 2.1. *Let K be a bounded closed and convex subset of $\mathbb{R}^m$. If hypotheses (A_1)–(A_3) are satisfied, then (DOP) (1.1) has at least one weak Carathéodory solution.*

Proof. We will carry out the following steps to obtain the conclusion.

Step 1. The set-valued mapping $U\colon [0,T] \times \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ defined by

$$U(t,x) = S(K, g(t,x,\cdot)), \tag{2.1}$$

has nonempty, closed and convex values, for all $(t,x) \in [0,T] \times \mathbb{R}^n$.

Let $(t,x) \in [0,T] \times \mathbb{R}^n$ be fixed. Recall that $g(t,x,\cdot)\colon \mathbb{R}^m \to \mathbb{R}$ is continuous and convex (see (A_3). The later combines with the boundedness, closedness of K and

Weierstrass existence theorem [2, Theorem 4.12] deliver that there exists an element $u \in K$ solving the following optimization problem

$$\text{Minimize } g(t,x,u)$$
$$\text{subject to } u \in K.$$

This means that the set $U(t,x)$ is nonempty for any $(t,x) \in [0,T] \times \mathbb{R}^n$. Assume that $\{u_n\} \subset U(t,x)$ is such that $u_n \to u$ in $\mathbb{R}^m$, therefore we have

$$g(t,x,u_n) \leq g(t,x,v) \quad \text{for all } v \in K.$$

However, the continuity of $u \mapsto g(t,x,u)$ implies $u \in U(t,x)$, so, $U(t,x)$ is closed for any $(t,x) \in [0,T] \times \mathbb{R}^n$. It remains for us to verify the convexity of $U(t,x)$. Indeed, it is obtained readily from the convexity of $u \mapsto g(t,x,u)$.

Step 2. The mapping $U\colon [0,T] \times \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is upper semicontinuous.

To end this, we shall use Theorem 1.3 to prove the claim, this means we will show that $U\colon [0,T] \times \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is a closed and quasicompact set-valued mapping. Let $\{(t_n,x_n)\} \in [0,T] \times \mathbb{R}^n$ and $u_n \in U(t_n,x_n)$ be such that $(t_n,x_n) \to (t,x)$ and $u_n \to u$ as $n \to \infty$. Hence,

$$g(t_n,x_n,v) \geq g(t_n,x_n,u_n),\ \forall v \in K. \tag{2.2}$$

Taking $n \to \infty$ into (2.2), we have

$$g(t,x,v) \geq g(t,x,u),\ \forall v \in K,$$

which means that $u \in U(t,x)$, i.e.,. U is a closed set-valued mapping. In the following, for each compact set $D \subset [0,T] \times \mathbb{R}^n$, we will show $U(D) \subset K$ is compact. Notice that K is bounded closed subset in $\mathbb{R}^m$, so K is compact as well. It remains to prove that $U(D)$ is closed. Let $u_n \in U(D)$ and $u_n \to u$. Then there is $(t_n,x_n) \in D$ such that $u_n \in U(t_n,x_n)$. Therefore, there is a subsequence $(t_{n_k},x_{n_k}) \to (t,x) \in D$. By the discussion above, we know that U is a closed set-valued mapping. Hence, $u \in U(t,x) \subset U(D)$, i.e., U is quasicompact, which implies that U is upper semicontinuous by applying Theorem 1.3. We also get $U(\cdot,x) : [0,T] \rightrightarrows \mathbb{R}^m$ is measurable due to the upper semicontinuity of U.

Step 3. (DOP) (1.1) has at least one weak Carathéodory solution.

To do so, we now consider the following differential inclusion

$$\begin{cases} \dot{x}(t) \in F(t,x(t)), & \text{for a.e. } t \in [0,T], \\ x(0) = x_0, \end{cases} \tag{2.3}$$

where $F(t,x(t))$ is defined by

$$F(t,x(t)) = \{f(t,x(t)) + B(t,x(t))U(t,x(t))\}. \tag{2.4}$$

Since $f\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $B\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ satisfy Carathéodory conditions. Hence, for each $x \in \mathbb{R}^n$, $F(\cdot,x)\colon [0,T] \rightrightarrows \mathbb{R}^n$ is measurable and $F(t,\cdot)\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is upper semicontinuous by using [4, Theorem 1.2.8 and Theorem 1.3.4], which implies that the set $P_F(x)$ defined

$$P_F(x) = \{h\colon [0,T] \to \mathbb{R}^n \mid h \text{ is measurable and } h(t) \in F(t,x(t)) \text{ for a.e. } t \in [0,T]\}\}$$

is well-defined for each $x \in C([0,T];\mathbb{R}^n)$ from [10, Theorem 3.17]. Furthermore, we may obtain from (A_1) and (A_2)

$$\|F(t,x)\| \leq (\rho_f(t) + \rho_B(t)\|K\|)(1 + \|x\|_{\mathbb{R}^n}),\ \forall (t,x) \in [0,T] \times \mathbb{R}^n,$$

where $\|K\| := \sup\{\|u\|_{\mathbb{R}^m} :\ u \in K\}$.

Therefore, we verify all conditions of [4, Theorem 5.2.2] and then conclude that differential inclusion (2.3) has a mild solution $x \in C([0,T];\mathbb{R}^n)$ such that

$$x(t) = x_0 + \int_0^t k(s)ds,\ k \in P_F(x). \tag{2.5}$$

It follows from Filippov implicit function lemma, Theorem 1.4, that for every mild solution $x \in C([0,T];\mathbb{R}^n)$ of (2.3), there exists a measurable selection $u(t) \in U(t,x(t))$ such that

$$k(t) = f(t,x(t)) + B(t,x(t))u(t)\ (\textit{see}\ (2.5)),$$

i.e.

$$\dot{x}(t) = f(t,x(t)) + B(t,x(t))u(t) \text{ and } u(t) \in U(t,x(t)) \text{ a.e. } t \in [0,T].$$

Consequently, we prove that (DOP) has at least one weak solution in the sense of Carathéodory. □

3. A numerical approximation approach for (DOP)

In Section 2, we have proved that (DOP) has at least one weak Carathéodory solution. In fact, in many applied problems, it is difficult to obtain the exact solutions for (DOP). By this motivation, in this section, we will, further, introduce a numerical approximation approach for (DOP) as follows

$$\begin{cases} x_k(t) = x_0 + \displaystyle\int_0^t f(s,x_k(s)) + B(s,x_k(s))u_k(s)\,ds,\ t \in [0,T], \\ u_k(t) = \displaystyle\sum_{i=0}^{N_k-1} v_i \chi_{[t_i,t_{i+1}]}(t),\ t \in [0,T], \\ v_i \in U(t_i, x_k(ih)),\ 0 \leq i \leq N_k - 1, \end{cases} \tag{3.1}$$

where $N_k \in \mathbb{N}$, $h = \frac{T}{N_k}$ the time step, and $\chi_{[t_i,t_{i+1}]}$ stands the character function of $[t_i,t_{i+1}]$, thus is

$$\chi_{[t_i,t_{i+1}]} = \begin{cases} 1, & \text{if } t \in [t_i,t_{i+1}], \\ 0, & \text{otherwise.} \end{cases}$$

We now study the convergence of the Algorithm 1.

Theorem 3.1. *Under the hypotheses of Theorem 2.1, then there exists a sequence* $\{k\}$ *such that* $x_k \to x$ *in* $C([0,T];\mathbb{R}^n)$ *and* $u_k \rightharpoonup u$ *in* $L^2([0,T];\mathbb{R}^m)$. *Furthermore,* (x,u) *is a weak solution of (DOP) in the sense of Carathéodory.*

Algorithm 1 Framework of numerical approximation approach.

Require:

$g(t, x, u)$: The objective function;

x_0: The initial data;

$f(t, x), B(t, x)$: The dynamic functions;

T: The terminal time;

K: The constrain set;

N_k: The number of subdivisions, $N_k \in \mathbb{N}$ and $N_k > 0$.

Step 1. Divide the interval $[0, T]$ into N_k subintervals $[t_i, t_{i+1}], i = 0, 1, \cdots, N_k - 1$, where t_i is defined $t_i := ih$ and $h = \frac{T}{N_k}$;

Step 2. Compute $U(t_0, x(0))$ (see (2.1)), then take any element $v_0 \in U(t_0, x(0))$ and denote $u_k(t) = v_0$ for $t \in [0, \frac{T}{N_k}]$;

Step 3. Calculate $x_k(t) = x_0 + \int_0^t f(s, x_k(s)) + B(s, x_k(s))u_k(s)ds$ for $t \in [0, \frac{T}{N_k}]$ (x_k can be found in $[0, \frac{T}{N_k}]$, see Theorem 2.1);

Step 4. Take $v_1 \in U(t_1, x(t_1))$ and $u_k(t) = v_1$ for $t \in [\frac{T}{N_k}, \frac{2T}{N_k}]$, we can continue to obtain $x_k(t) = x_0 + \int_0^t f(s, x_k(s)) + B(s, x_k(s))u_k(s)\, ds$ for $t \in [t_1, t_2]$.

Step 5. By induction, we end up with (3.1).

Proof. Let sequences $\{u_k\}$ and $\{x_k\}$ be generalized by Algorithm 1, so, for each $k \in \mathbb{N}$ we have

$$\begin{cases} x_k(t) = x_0 + \int_0^t [f(s, x_k(s)) + B(s, x_k(s))u_k(s)]ds, \ t \in [0, T], \\ u_k(t) = \sum_{i=0}^{N_k - 1} v_i \chi_{[t_i, t_{i+1}]}(t), \ t \in [0, T], \\ v_i \in U(t_i, x_k(ih)), \ 0 \le i \le N_k - 1. \end{cases}$$

Note that K is bounded, so, we immediately deduce that the sequences $\{u_k\}$ is uniformly bounded in $L^2([0, T]; \mathbb{R}^m)$. Therefore, there exists a constant $r_1 > 0$ such that $\|u_k\|_{L^2([0,T],\mathbb{R}^m)} \le r_1$. Also, according to assumptions (A_1) and (A_2), we readily obtain that there exists $r_2 > 0$ such that $\|x_k\|_{C([0,T];\mathbb{R}^n)} \le r_2$ by using Gronwall inequality. It also implies that $\{x_k\}$ is equicontinuous. By applying Arzela-Ascoli theorem (see [16]), however, we may assume $x_k \to x$ in $C([0, T]; \mathbb{R}^n)$. In addition, using Alaoglu's theorem (see [16]), one implies that $\{u_k\}$ is weakly$*$ relatively compact. Without loss of generality, we also suppose $u_k \rightharpoonup u$ in $L^2([0, T]; \mathbb{R}^m)$. By conditions (A_1), (A_2) and upper semicontinuity of U, we easily get (x, u) satisfies

$$\begin{cases} x(t) = x_0 + \int_0^t f(s, x(s)) + B(s, x(s))u(s)ds, \quad \text{for a.e. } t \in [0, T], \\ u(t) \in U(t, x(t)), \quad \text{for a.e. } t \in [0, T]. \end{cases}$$

This completes the proof. $\square$

Finally, we give an example to illustrate the validity of Algorithm 1.

Example 3.2. Let $T = 3$. Find functions $u\colon [0,T] \to \mathbb{R}$ and $x\colon [0,T] \to \mathbb{R}$ such that for all

$$(3.2)\quad \begin{cases} \text{Minimize} & u^2(t) + u(t)x(t) - 10e^{-0.2\sqrt{t^2/2}} - 10e^{\frac{\cos(2\pi t)+\sin(2\pi t)}{2}} + 22.7, \\ \text{Subject to} & u(t) \in [-3,3], \\ & \dot{x}(t) = t\left(\frac{(x(t)+u(t))}{t^2+1}\right)^2, \\ & x(0) = 0.1. \end{cases}$$

It is obvious that problem (3.2) has a unique solution

$$x(t) = \frac{8(t^2+1)}{1+79(t^2+1)},$$

$$u(t) = \frac{-4(t^2+1)}{1+79(t^2+1)}.$$

Let $N_k = 6$. So, by Algorithm 1, we can get $h = 0.5$ and

$$\begin{cases} x_k(t) = x_0 + \displaystyle\int_0^t f(s, x_k(s)) + B(s, x_k(s))u_k(s)\, ds, \ t \in [0,T], \\ u_k(t) \in U(ih, x_k(ih)), \ 0 \le i \le 5. \end{cases}$$

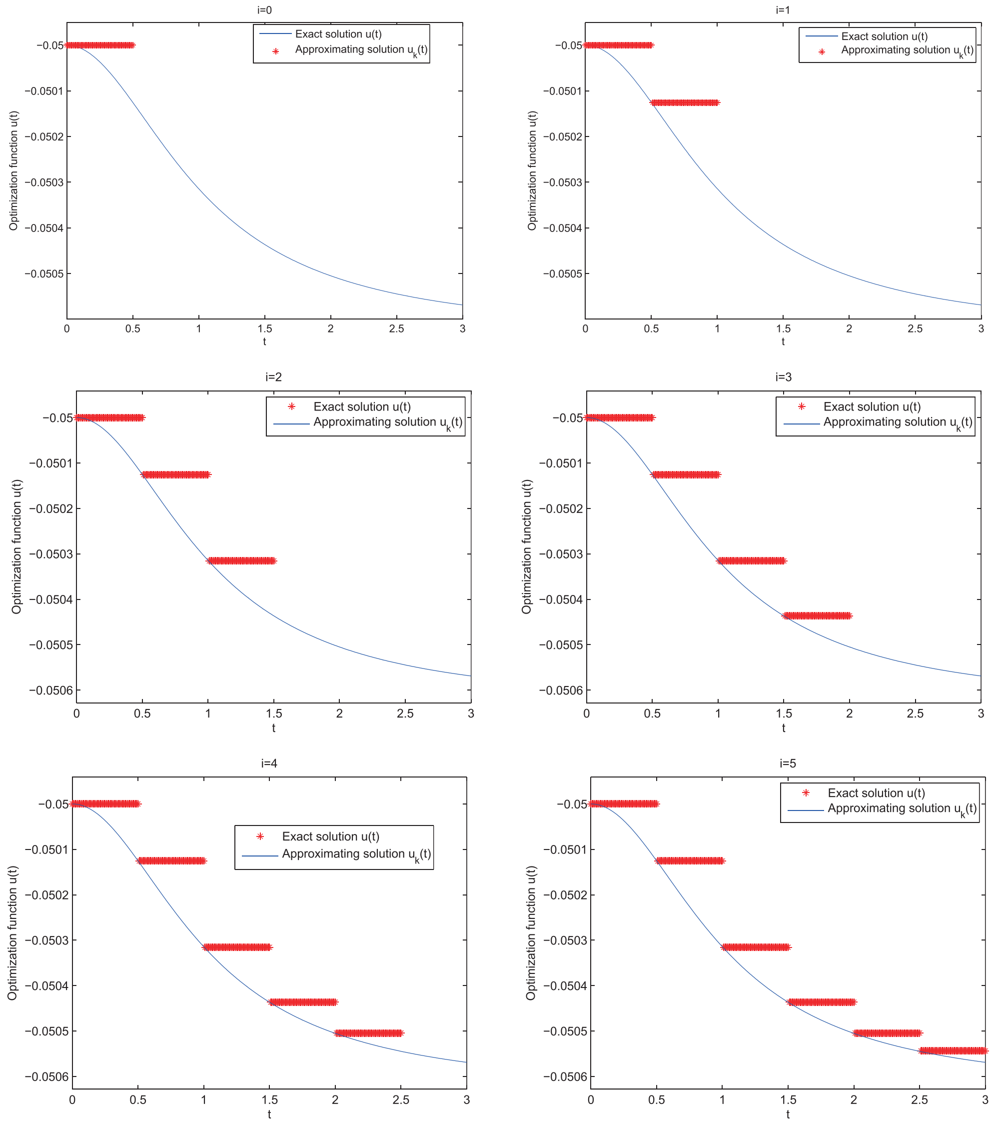

FIGURE 1. Optimization function $u_k(t)$

Therefore, we have the following results:

- Fig. 1 describes the approximating of optimization function $u_k(t)$, $t \in [t_i, t_{i+1}]$, for $i = 0, 1, \cdots, N_k - 1 = 5$;
- Fig. 2 shows the approximating of dynamic function $x_k(t)$, $t \in [t_i, t_{i+1}]$, for $i = 0, 1, \cdots, N_k - 1 = 5$;
- Fig. 3 illustrates the approximating of optimization function u_k, where $N_k = 10, 30$ and 50, respectively;
- Fig. 4 demonstrates the approximating of dynamic function x_k, where $N_k = 10, 30$ and 50, respectively.

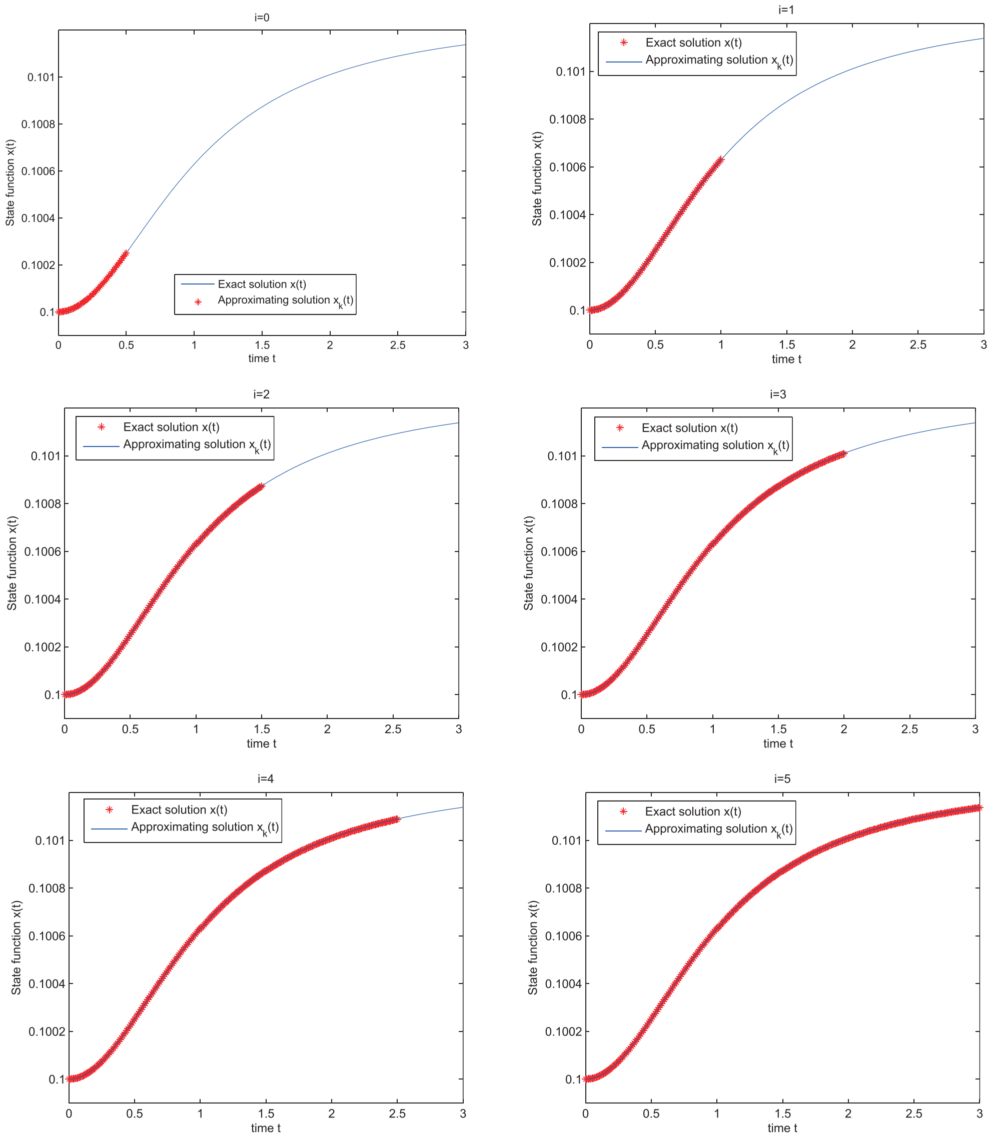

FIGURE 2. Dynamic function $x_k(t)$

Also, we get the error analysis between correct solution (x, u) and approximating solution (x_k, u_k) in Table 1.

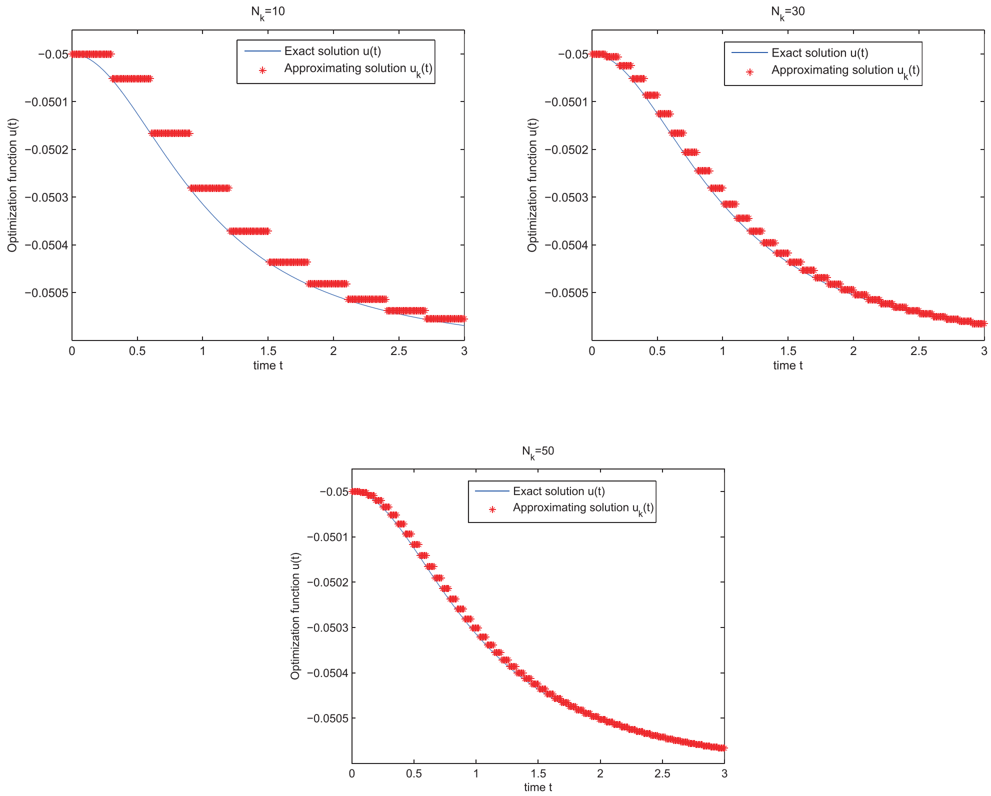

FIGURE 3. Optimization function $u_k(t)$ for $N_k = 10, 30$ and 50.

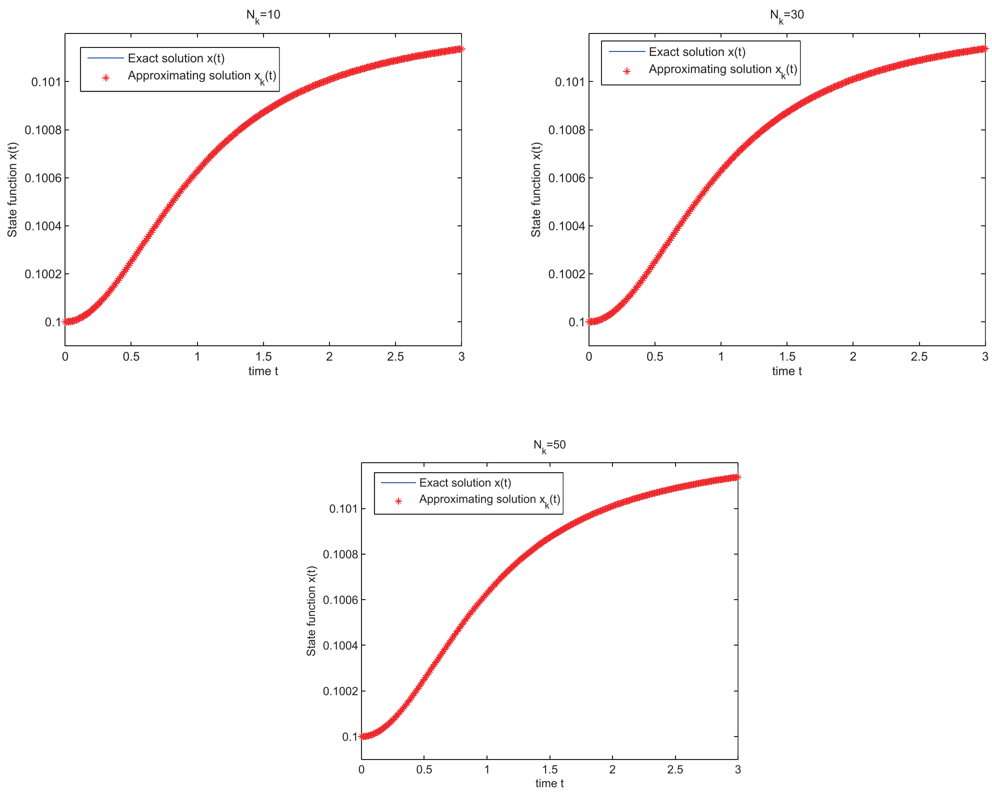

FIGURE 4. Dynamic function $x_k(t)$ for $N_k = 10, 30$ and 50.

Table 1. Error Analysis

Dynamic function	$\sup_{t\in[0,T]} \lvert x(t) - x_k(t)\rvert$	Optimization function	$\sup_{t\in[0,T]} \lvert u(t) - u_k(t)\rvert$
$N_k = 10$	1.6131×10^{-6}	$N_k = 10$	1.1428×10^{-4}
$N_k = 30$	6.2104×10^{-7}	$N_k = 30$	3.6607×10^{-5}
$N_k = 50$	3.7491×10^{-7}	$N_k = 50$	2.0372×10^{-5}
$N_k = 100$	1.8784×10^{-7}	$N_k = 100$	8.1410×10^{-6}
$N_k = 200$	9.3956×10^{-8}	$N_k = 200$	3.2789×10^{-7}

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