Applied Analysis and Optimization
Volume 5, Number 2, 2021, 123–144

Yokohama Publishers
ISSN 2189-1664 Online Journal

BOUNDARY VALUE PROBLEMS FOR NONLOCAL MULTI-POINT AND MULTI-TERM FRACTIONAL DIFFERENTIAL INCLUSIONS

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Abstract. In this paper, we discuss the existence of solutions for a new class of nonlocal, multi-point and integral boundary value problems of multi-term fractional differential inclusions by using standard fixed point theorems. We also demonstrate the application of the obtained results with the aid of examples.

1. Introduction

Fractional differential equations arise in the mathematical modelling of many engineering and scientific disciplines such as biophysics, bio-engineering, virology, control theory, signal and image processing, blood flow phenomena, etc. A huge amount of mathematically and physically interesting works published in recent years, including several excellent monographs, clearly reflects the overwhelming interest in the topic. For details we refer the reader the texts [12, 17, 22, 25, 26, 29] and references cited therein.

Nonlocal boundary value problems of fractional order differential equations and inclusions also received significant attention. One can witness a great deal of work on the topic involving different kinds of boundary conditions in the literature, for example, see [1, 2, 5, 14, 23] and the references cited therein.

There is another class of fractional differential equations containing more than one fractional order differential operators. Such equations appear in the modelling of the motion of a rigid plate immersed in a Newtonian fluid. Other typical examples include Bagley-Torvik [28] and Basset equation [21]. Some recent results on multi-term fractional differential equations can be found in the articles [3, 4, 19, 20, 27].

The objective of the present work is to develop the existence theory for multi-term fractional differential inclusions equipped with nonlocal, multi-point and integral boundary conditions. Precisely, we investigate the following boundary value problem:

$$(q_2\ {}^cD^{\sigma+2} + q_1\ {}^cD^{\sigma+1} + q_0\ {}^cD^{\sigma})x(t) \in F(t, x(t)),\ \ 0 < t < 1, \tag{1.1}$$

$$x(0) = g(x),\ \ x(\xi) = \sum_{i=1}^{n} j_i x(\eta_i),\ \ x(1) = \lambda \int_0^{\delta} x(s)ds, \tag{1.2}$$

where ${}^cD^{\sigma}$ denotes the Caputo fractional derivative of order σ, $0 < \sigma < 1$, $F : [0,1]\times \mathbb{R} \to \mathcal{P}(\mathbb{R})$, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty substs of $\mathbb{R}$, $g : C([0,1],\mathbb{R}) \to \mathbb{R}$

2020 *Mathematics Subject Classification.* 34A08, 34A60, 34B10.
Key words and phrases. Caputo fractional derivative, multi-term fractional differential inclusions, existence, fixed point.

is a given continuous functions, $0 < \delta < \xi < \eta_1 < \eta_2 < \cdots < \eta_n < 1$, $\lambda \in \mathbb{R}$, $q_0, q_1, and\ q_2$ are real constants with $q_2 \neq 0$.

The concept of nonlocal problems introduced by Bicadze and Samarskii [6] is found to be more practical than the classical problems with the initial conditions, see also Byszewski [8–10]. In the last few decades, several kinds of nonlocal problems have been studied. One can notice that $g(x)$ given in (1.2) may be expressed as $g(x) = \sum_{j=1}^{p} \alpha_j x(t_j)$ where $\alpha_j, j = 1, \dots, p$, are given constants and $0 < t_1 < \cdots < t_p \leq 1$. For more details, we refer to the work by Byszewski [8, 10].

We discuss the existence of solutions for the problem (1.1)-(1.2) by means of the nonlinear alternative for contractive multi-valued maps and the combination of the nonlinear alternative for contractive single-valued maps and a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values. An example is constructed for the illustration of main results.

2. Basic results

Before presenting some auxiliary results, let us recall some preliminary concepts of fractional calculus [17, 29].

Definition 2.1. Let ϕ be a locally integrable real-valued function on $-\infty \leq a < t < b \leq +\infty$. The Riemann–Liouville fractional integral I_a^α of order $\alpha \in \mathbb{R}$ $(\alpha > 0)$ for ϕ is defined as

$$I_a^\alpha \phi(t) = (\phi * K_\alpha)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \phi(s) ds,$$

where $K_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, Γ denotes the Euler gamma function.

Definition 2.2. Let $\phi \in L^1[a, b]$, $-\infty \leq a < t < b \leq +\infty$ and $\phi * K_{m-\alpha} \in W^{m,1}[a, b], m = [\alpha] + 1, \alpha > 0$, where $W^{m,1}[a, b]$ is the Sobolev space defined as

$$W^{m,1}[a, b] = \left\{ \phi \in L^1[a, b] : \frac{d^m}{dt^m} \phi \in L^1[a, b] \right\}.$$

The Riemann–Liouville fractional derivative D_a^α of order $\alpha > 0$ $(m - 1 < \alpha < m,\ m \in \mathbb{N})$ for ϕ is defined as

$$D_a^\alpha \phi(t) = \frac{d^m}{dt^m} I_a^{1-\alpha} \phi(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-s)^{m-1-\alpha} \phi(s) ds.$$

Definition 2.3. Let $\phi \in L^1[a, b]$, $-\infty \leq a < t < b \leq +\infty$ and $\phi * K_{m-\alpha} \in W^{m,1}[a, b], m = [\alpha], \alpha > 0$. The Caputo fractional derivative ${}^cD_a^\alpha$ of order $\alpha \in \mathbb{R}$ $(m - 1 < \alpha < m,\ m \in \mathbb{N})$ for ϕ is defined as

$${}^cD_a^\alpha \phi(t) = D_a^\alpha \left[\phi(t) - \phi(a) - \phi'(a) \frac{(t-a)}{1!} - \cdots - \phi^{(m-1)}(a) \frac{(t-a)^{m-1}}{(m-1)!} \right].$$

If $\phi \in C^m[a,b]$, then the Caputo fractional derivative ${}^cD_a^\alpha$ of order $\alpha \in \mathbb{R}$ $(m-1 < \alpha < m,\ m \in \mathbb{N})$ for ϕ is defined as

$$ {}^cD_a^\alpha [\phi](t) = I_a^{1-\alpha}\phi^{(m)}(t) = \frac{1}{\Gamma(m-\alpha)} \int\limits_a^t (t-s)^{m-1-\alpha}\phi^{(m)}(s)ds. $$

In the sequel, the Riemann–Liouville fractional integral I_a^α and the Caputo fractional derivative ${}^cD_a^\alpha$ with $a=0$ are respectively denoted by I^α and ${}^cD^\alpha$.

Property 2.4 ([17])**.** With the given notations, the following equality holds:

$$ I^\alpha({}^cD^\alpha\varphi(t)) = \varphi(t) - c_0 - c_1 t - \cdots - c_{n-1}t^{n-1},\ t>0,\ n-1<\alpha<n, \tag{2.1} $$

where c_i $(i=1,\dots,n-1)$ are arbitrary constants.

The following lemmas associated with the linear variant of problem (1.1)-(1.2) play an important role in the sequel.

Lemma 2.5. *For any $y \in C([0,1],\mathbb{R})$ and $q_1^2 - 4q_0q_2 > 0$, the solution of linear multi-term fractional differential equation*

$$ (q_2\ {}^cD^{\sigma+2} + q_1\ {}^cD^{\sigma+1} + q_0\ {}^cD^\sigma)x(t) = y(t),\ \ 0<\sigma<1,\ \ 0<t<1, \tag{2.2} $$

supplemented with the boundary conditions (1.2) is given by

$$ \begin{aligned} x(t) &= \frac{1}{q_2(m_2-m_1)}\bigg\{ \int_0^t\int_0^s \mathcal{A}(t)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}y(u)du\ ds \\ &\quad +\rho_1(t)\Big[\int_0^\xi\int_0^s \mathcal{A}(\xi)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}y(u)du\ ds \\ &\quad -\sum_{i=1}^n j_i\int_0^{\eta_i}\int_0^s \mathcal{A}(\eta_i)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}y(u)du\ ds\Big] \\ &\quad +\rho_2(t)\Big[\int_0^1\int_0^s \mathcal{A}(1)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}y(u)du\ ds \\ &\quad -\lambda\int_0^\delta\int_0^s \Big(\frac{(e^{m_2(\delta-s)}-1)}{m_2} - \frac{(e^{m_1(\delta-s)}-1)}{m_1}\Big) \\ &\quad \times\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}y(u)du\ ds\Big]\bigg\} \\ &\quad +g(x)\Big[e^{m_2t} + \rho_1(t)\Big(e^{m_2\xi} - \sum_{i=1}^n j_i + \sum_{i=1}^n j_ie^{m_2\eta_i}\Big) \\ &\quad +\rho_2(t)\Big(\frac{m_2e^{m_2} - \lambda e^{m_2\delta} - \lambda}{m_2}\Big)\Big], \end{aligned} \tag{2.3} $$

where

$$ \begin{aligned} \mathcal{A}(\kappa) &= e^{m_2(\kappa-s)} - e^{m_1(\kappa-s)},\ \ \kappa = t,\ 1,\ \xi \text{ and } \eta_i, \\ m_1 &= \frac{-q_1 - \sqrt{q_1^2-4q_0q_2}}{2q_2},\ \ m_2 = \frac{-q_1 + \sqrt{q_1^2-4q_0q_2}}{2q_2}, \end{aligned} $$

$$
\begin{aligned}
\rho_1(t) &= \frac{\omega_4\varrho_1(t)-\omega_3\varrho_2(t)}{\mu_1}, \quad \rho_2(t)=\frac{\omega_1\varrho_2(t)-\omega_2\varrho_1(t)}{\mu_1},\\
\varrho_1(t) &= \frac{m_1(1-e^{m_2t})-m_2(1-e^{m_1t})}{m_1m_2},\\
\varrho_2(t) &= q_2(m_2-m_1)(e^{m_2t}-e^{m_1t}),\\
\mu_1 &= \omega_1\omega_4-\omega_2\omega_3\neq 0, \qquad (2.4)\\
\omega_1 &= \frac{1}{m_1m_2}\Big[m_2\Big(1-\sum_{i=1}^{n}j_i-e^{m_1\xi}+\sum_{i=1}^{n}j_ie^{m_1\eta_i}\Big)\\
&\quad -m_1\Big(1-\sum_{i=1}^{n}j_i-e^{m_2\xi}+\sum_{i=1}^{n}j_ie^{m_2\eta_i}\Big)\Big],\\
\omega_2 &= q_2\Big(m_2-m_1)(e^{m_1\xi}-e^{m_2\xi}-\sum_{i=1}^{n}j_ie^{m_1\eta_i}+\sum_{i=1}^{n}j_ie^{m_2\eta_i}\Big),\\
\omega_3 &= \frac{1}{m_1m_2}\Big[m_2\Big(1-e^{m_1}-\lambda\delta+\lambda/m_1(e^{m_1\delta}-1)\Big)\\
&\quad -m_1\Big(1-e^{m_2}-\lambda\delta+\lambda/m_2(e^{m_2\delta}-1)\Big)\Big],\\
\omega_4 &= q_2(m_2-m_1)\Big((e^{m_1}+\lambda/m_1(1-e^{m_1\delta}))\\
&\quad -(e^{m_2}+\lambda/m_2(1-e^{m_2\delta}))\Big).
\end{aligned}
$$

Proof. Applying the operator I^σ on (2.2) and using (2.1), we get

$$
(q_2D^2+q_1D+q_0)x(t)=\int_0^t\frac{(t-s)^{\sigma-1}}{\Gamma(\sigma)}y(s)ds+c_1, \tag{2.5}
$$

where c_1 is an arbitrary constant. By the method of variation of parameters, the solution of (2.5) can be written as

$$
\begin{aligned}
x(t) &= c_1\left[\frac{m_2(1-e^{m_1t})-m_1(1-e^{m_2t})}{q_2m_1m_2(m_2-m_1)}\right]+c_2e^{m_1t}+c_3e^{m_2t}\\
&\quad -\frac{1}{q_2(m_2-m_1)}\int_0^t e^{m_1(t-s)}\left(\int_0^s\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}y(u)du\right)ds\\
&\quad +\frac{1}{q_2(m_2-m_1)}\int_0^t e^{m_2(t-s)}\left(\int_0^s\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}y(u)du\right)ds, \qquad (2.6)
\end{aligned}
$$

where m_1 and m_2 are given by (2.4). Using $x(0)=g(x)$ in (2.6), we get

$$
\begin{aligned}
x(t) &= c_1\left[\frac{m_2(1-e^{m_1t})-m_1(1-e^{m_2t})}{q_2m_1m_2(m_2-m_1)}\right]+c_2\Big(e^{m_1t}-e^{m_2t}\Big)+g(x)e^{m_2t}\\
&\quad +\frac{1}{q_2(m_2-m_1)}\left[\int_0^t\Big(e^{m_2(t-s)}-e^{m_1(t-s)}\Big)\right.
\end{aligned}
$$

$$\times\left(\int_0^s \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}y(u)du\right)ds\Bigg], \tag{2.7}$$

which together with the conditions $x(\xi)=\sum_{i=1}^n j_i x(\eta_i)$, $x(1)=\lambda\int_0^\delta x(s)ds$ yields the following system of equations in the unknown constants c_1 and c_2:

$$c_1\omega_1 + c_2\omega_2 = V_1, \tag{2.8}$$

$$c_1\omega_3 + c_2\omega_4 = V_2. \tag{2.9}$$

where

$$\begin{aligned} V_1 &= -\int_0^\xi\int_0^s \mathcal{A}(\xi)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}y(u)du\, ds \\ &\quad +\sum_{i=1}^n j_i\int_0^{\eta_i}\int_0^s \mathcal{A}(\eta_i)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}y(u)du\, ds \\ &\quad +g(x)\Big(\sum_{i=1}^n j_i e^{m_2\eta_i} - e^{m_2\xi}\Big), \\ V_2 &= -\int_0^1\int_0^s \mathcal{A}(1)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}y(u)du\, ds + g(x)\Big(\frac{\lambda e^{m_2\delta}-\lambda-m_2e^{m_2}}{m_2}\Big) \\ &\quad +\lambda\int_0^\delta\int_0^s\Big[\frac{(e^{m_1(\delta-s)}-1)}{m_1}-\frac{(e^{m_2(\delta-s)}-1)}{m_2}\Big]\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}y(u)du\, ds. \end{aligned}$$

Solving the system (2.8)-(2.9) together with the notations (2.4), we find that

$$c_1=\frac{V_1\omega_4-V_2\omega_2}{\mu_1},\quad c_2=\frac{V_2\omega_1-V_1\omega_3}{\mu_1}.$$

Substituting the value of c_1 and c_2 in (2.7), we obtain the solution (2.3). The converse of the lemma follows by direct computation. This completes the proof. □

We do not provide the proofs of the following lemmas, as they are similar to that of Lemma 2.5.

Lemma 2.6. *For any $y\in C([0,1],\mathbb{R})$ and $q_1^2-4q_0q_2=0$, the solution of linear multi-term fractional differential equation*

$$(q_2\,{}^cD^{\sigma+2}+q_1\,{}^cD^{\sigma+1}+q_0\,{}^cD^{\sigma})x(t)=y(t),\ \ 0<\sigma<1,\ \ 0<t<1, \tag{2.10}$$

supplemented with the boundary conditions (1.2) is given by

$$\begin{aligned} x(t) &= \frac{1}{q_2}\Bigg\{\int_0^t\int_0^s \mathcal{B}(t)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}y(u)du\, ds \\ &\quad +\chi_1(t)\Big[\int_0^\xi\int_0^s \mathcal{B}(\xi)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}y(u)du\, ds \\ &\quad -\sum_{i=1}^n j_i\int_0^{\eta_i}\int_0^s \mathcal{B}(\eta_i)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}y(u)du\, ds\Big] \\ &\quad +\chi_2(t)\Big[\int_0^1\int_0^s \mathcal{B}(1)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}y(u)du\, ds \end{aligned} \tag{2.11}$$

$$
\begin{aligned}
&-\lambda\int_0^\delta\int_0^s\Big(\frac{m(\delta-s)e^{m(\delta-s)}-e^{m(\delta-s)}+1}{m^2}\Big)\\
&\times\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}y(u)du\ ds\Big]\Bigg\}\\
&+g(x)\Big[e^{mt}+\chi_1(t)\Big(e^{m\xi}-\sum_{i=1}^n j_ie^{m\eta_i}\Big)\\
&+\chi_2(t)\Big(\frac{me^m-\lambda e^{m\delta}+\lambda}{m}\Big)\Big],
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{B}(\kappa) &= (\kappa-s)e^{m(\kappa-s)},\ \ \kappa=t,\,1,\,\xi \text{ and } \ \eta_i,\\
m &= \frac{-q_1}{2q_2},\\
\chi_1(t) &= \frac{\varpi_3v_2(t)-\varpi_4v_1(t)}{\mu_2},\ \ \chi_2(t)=\frac{\varpi_2v_1(t)-\varpi_1v_2(t)}{\mu_2},\\
v_1(t) &= \frac{mte^{mt}-e^{mt}+1}{m^2},\ \ v_2(t)=q_2te^{mt},\\
\varpi_1 &= \frac{m\xi e^{m\xi}-e^{m\xi}+1-\sum_{i=1}^n j_i(m\eta_ie^{m\eta_i}-e^{m\eta_i}+1)}{m^2}, \qquad (2.12)\\
\varpi_2 &= q_2\Big(\xi e^{m\xi}-\sum_{i=1}^n j_i\eta_ie^{m\eta_i}\Big),\\
\varpi_3 &= \frac{m^2e^m-me^m+m-m\lambda\delta e^{m\delta}+2\lambda e^{m\delta}-2\lambda-m\lambda\delta}{m^3},\\
\varpi_4 &= q_2\Big(\frac{m^2e^m-\lambda m\delta e^{m\delta}+\lambda e^{m\delta}-\lambda}{m^2}\Big),\\
\mu_2 &= \varpi_1\varpi_4-\varpi_2\varpi_3\neq 0.
\end{aligned}
$$

Lemma 2.7. *For any $y\in C([0,1],\mathbb{R})$ and $q_1^2-4q_0q_2<0$, the solution of linear multi-term fractional differential equation*

$$
(q_2\ {}^cD^{\sigma+2}+q_1\ {}^cD^{\sigma+1}+q_0\ {}^cD^{\sigma})x(t)=y(t),\ \ 0<\sigma<1,\ \ 0<t<1, \qquad (2.13)
$$

supplemented with the boundary conditions (1.2) is given by

$$
\begin{aligned}
x(t) &= \frac{1}{q_2b}\Bigg\{\int_0^t\int_0^s\mathcal{F}(t)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}y(u)du\ ds\\
&+\tau_1(t)\Big[\int_0^\xi\int_0^s\mathcal{F}(\xi)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}y(u)du\ ds\\
&-\sum_{i=1}^n j_i\int_0^{\eta_i}\int_0^s\mathcal{F}(\eta_i)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}y(u)du\ ds\Big]\\
&+\tau_2(t)\Big[\int_0^1\int_0^s\mathcal{F}(1)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}y(u)du\ ds \qquad (2.14)
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{\lambda}{a^2+b^2}\int_0^\delta\int_0^s\Big(b-be^{-a(\delta-s)}\cos b(\delta-s)\\
& -ae^{-a(\delta-s)}\sin b(\delta-s)\Big)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}y(u)du\ ds\Big]\Bigg\}\\
& +g(x)\Big[e^{-at}\cos bt+\tau_1(t)\Big(e^{-a\xi}\cos b\xi-\sum_{i=1}^n j_ie^{-a\eta_i}\cos b\eta_i\Big)\\
& +\tau_2(t)\Big(e^{-a}\cos b-\frac{\lambda}{a^2+b^2}(a-ae^{-a\delta}\cos b\delta+be^{-a\delta}\sin b\delta)\Big)\Big],
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{F}(\kappa) &= e^{-a(\kappa-s)}\sin b(\kappa-s),\ \ \kappa=t,\ 1,\ \xi\ \text{and}\ \ \eta_i,\\
m_{1,2} &= -a\pm bi,\ \ a=\frac{q_1}{2q_2},\ \ b=\frac{\sqrt{4q_0q_2-{q_1}^2}}{2q_2},\\
\tau_1(t) &= \frac{p_3\nu_2(t)-p_4\nu_1(t)}{\mu_3},\ \ \tau_2(t)=\frac{p_2\nu_1(t)-p_1\nu_2(t)}{\mu_3},\\
\nu_1(t) &= \frac{b-be^{-at}\cos bt-ae^{-at}\sin bt}{a^2+b^2},\ \ \nu_2(t)=q_2be^{-at}\sin bt\\
p_1 &= \frac{1}{a^2+b^2}\Big[b-be^{-a\xi}\cos b\xi-ae^{-a\xi}\sin b\xi\\
&\quad -\sum_{i=1}^n j_i(b-be^{-a\eta_i}\cos b\eta_i-ae^{-a\eta_i}\sin b\eta_i)\Big],\\
p_2 &= q_2b\Big(e^{-a\xi}\sin b\xi-\sum_{i=1}^n j_ie^{-a\eta_i}\sin b\eta_i\Big),\\
p_3 &= \frac{1}{a^2+b^2}\Big[b-be^{-a}\cos b-ae^{-a}\sin b-b\lambda\delta\\
&\quad +\frac{b\lambda}{a^2+b^2}(a-ae^{-a\delta}\cos b\delta+be^{-a\delta}\sin b\delta)\\
&\quad -\frac{a\lambda}{a^2+b^2}(b-be^{-a\delta}\cos b\delta-ae^{-a\delta}\sin b\delta)\Big],\\
p_4 &= q_2b\Big[e^{-a}\sin b-\frac{\lambda}{a^2+b^2}(b-be^{-a\delta}\cos b\delta-ae^{-a\delta}\sin b\delta)\Big],\\
\mu_3 &= p_1p_4-p_2p_3\neq 0.
\end{aligned}
\tag{2.15}
$$

3. Existence results for problem (1.1)-(1.2) with $q_1^2-4q_0q_2>0$

Let us recall some basic definitions on multi-valued maps [11,16].

For a normed space $(X,\|\cdot\|)$, let $\mathcal{P}_{cl}(X)=\{Y\in\mathcal{P}(X): Y \text{ is closed}\}$, $\mathcal{P}_b(X)=\{Y\in\mathcal{P}(X): Y \text{ is bounded}\}$, and $\mathcal{P}_{cp,c}(X)=\{Y\in\mathcal{P}(X): Y \text{ is compact and convex}\}$. A multi-valued map $G: X\to\mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x\in X$. The map G is bounded on bounded sets if $G(\mathbb{B})=\cup_{x\in\mathbb{B}}G(x)$ is bounded in X for all $\mathbb{B}\in P_b(X)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{|y|: y\in G(x)\}\}<\infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0\in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(X)$. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$. G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The set of fixed points of the multivalued operator G will be denoted by $FixG$. A multivalued map $G : [0;1] \to \mathcal{P}_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.

Definition 3.1. Let $q_1^2 - 4q_0q_2 > 0$. A function $x \in C([0,1],\mathbb{R})$ possessing a Caputo fractional derivative of order $\sigma + 2$ is a solution of the problem (1.1)-(1.2) if $x(0) = g(x)$, $x(\xi) = \sum_{i=1}^{n} j_i x(\eta_i)$, $x(1) = \lambda \int_0^\delta x(s)ds$, and there exists a function $v \in L^1([0,1],\mathbb{R})$ such that $v(t) \in F(t, x(t))$ a.e. on $[0,1]$ and

$$\begin{aligned}
x(t) \quad = \quad & \frac{1}{q_2(m_2 - m_1)}\Big\{ \int_0^t \int_0^s \mathcal{A}(t)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} v(u)du\, ds \\
& +\rho_1(t)\Big[\int_0^\xi \int_0^s \mathcal{A}(\xi)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} v(u)du\, ds \\
& -\sum_{i=1}^{n} j_i \int_0^{\eta_i} \int_0^s \mathcal{A}(\eta_i)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} v(u)du\, ds\Big] \\
& +\rho_2(t)\Big[\int_0^1 \int_0^s \mathcal{A}(1)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} v(u)du\, ds \\
& -\lambda \int_0^\delta \int_0^s \Big(\frac{(e^{m_2(\delta-s)} - 1)}{m_2} - \frac{(e^{m_1(\delta-s)} - 1)}{m_1}\Big) \\
& \times \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} v(u)du\, ds\Big]\Big\} \\
& +g(x)\Big[e^{m_2 t} + \rho_1(t)\Big(e^{m_2\xi} - \sum_{i=1}^{n} j_i + \sum_{i=1}^{n} j_i e^{m_2\eta_i}\Big) \\
& +\rho_2(t)\Big(\frac{m_2 e^{m_2} - \lambda e^{m_2\delta} - \lambda}{m_2}\Big)\Big].
\end{aligned} \tag{3.1}$$

3.1. **The Carathéodory case.**

Definition 3.2. A multivalued map $F : [0,1]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(*i*) $t \longmapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$;

(*ii*) $x \longmapsto F(t,x)$ is upper semicontinuous (u.s.c.) for almost all $t \in [0,1]$;

Further a Carathéodory function F is called L^1-Carathéodory if

(*iii*) for each $\alpha > 0$, there exists $\varphi_\alpha \in L^1([0,1],\mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \le \varphi_\alpha(t)$$

for all $x \in \mathbb{R}$ with $\|x\| \leq \alpha$ and for a.e. $t \in [0,1]$.

For each $y \in C([0,1],\mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1([0,1],\mathbb{R}) : v(t) \in F(t,y(t)) \text{ for a.e. } t \in [0,1]\}.$$

We define the graph of G to be the set $Gr(G) = \{(x,y) \in X \times Y, y \in G(x)\}$ and recall a result for closed graphs and upper-semicontinuity.

Lemma 3.3 ([11, Proposition 1.2]). *If $G : X \to \mathcal{P}_{cl}(Y)$ is u.s.c., then $Gr(G)$ is a closed subset of $X \times Y$; i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if when $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$ and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if G is completely continuous and has a closed graph, then it is upper semi-continuous.*

The following lemma will be used in the sequel.

Lemma 3.4 ([18]). *Let X be a separable Banach space. Let $F : [0,1]\times\mathbb{R} \to \mathcal{P}_{cp,c}(X)$ be an L^1- Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([0,1],X)$ to $C([0,1],X)$. Then the operator*

$$\Theta \circ S_F : C([0,1],X) \to \mathcal{P}_{cp,c}(C([0,1],X)),\ x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C([0,1],X) \times C([0,1],X)$.

We apply the following form of the Nonlinear Alternative for contractive maps [24, Corollary 3.8] to prove our main result in this section.

Theorem 3.5. *Let X be a Banach space, and D a bounded neighborhood of $0 \in X$. Let $Z_1 : X \to \mathcal{P}_{cp,c}(X)$ and $Z_2 : \bar{D} \to \mathcal{P}_{cp,c}(X)$ two multi-valued operators satisfying*

(a) *Z_1 is contraction, and*
(b) *Z_2 is u.s.c and compact.*

Then, if $G = Z_1 + Z_2$, either

(i) *G has a fixed point in $\bar{D}$ or*
(ii) *there is a point $u \in \partial D$ and $\theta \in (0,1)$ with $u \in \theta G(u)$.*

Theorem 3.6. *Assume that:*

(H_1) *$F : [0,1] \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is L^1-Carathéodory multivalued map;*
(H_2) *$g : C([0,1],\mathbb{R}) \to \mathbb{R}$ is a continuous function satisfying the condition:*

$$|g(u) - g(v)| \leq \ell\|u - v\|,\ \ell < \Delta_1^{-1},\ \forall\ u,v \in C([0,1],\mathbb{R}),\ \ell > 0;$$

(H_3) *there exists a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in C([0,1],\mathbb{R}^+)$ such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \leq p(t)\psi(\|x\|) \text{ for each } (t,x) \in [0,1] \times \mathbb{R};$$

(H_4) *there exists a number $M > 0$ such that*

$$\frac{(1 - \ell\Delta_1)M}{\|p\|\psi(M)\alpha + \ell_0\Delta_1} > 1, \tag{3.2}$$

where $\ell_0 = |g(0)|$ and

$$\widehat{\rho}_1 = \max_{t\in[0,1]} |\rho_1(t)|,\ \widehat{\rho}_2 = \max_{t\in[0,1]} |\rho_2(t)|,$$

$$
\begin{aligned}
\varepsilon &= \max_{t\in[0,1]} \Big|m_2(1-e^{m_1 t}) - m_1(1-e^{m_2 t})\Big|, \\
\alpha &= \frac{1}{|q_2 m_1 m_2 (m_2-m_1)|\Gamma(\sigma+1)}\Big\{\varepsilon + \widehat{\rho}_1\Big[\xi^\sigma |m_2(1-e^{m_1\xi}) \\
&\quad -m_1(1-e^{m_2\xi})| + \sum_{i=1}^{n} |j_i|\eta_i^\sigma |m_2(1-e^{m_1\eta_i}) - m_1(1-e^{m_2\eta_i})|\Big] \\
&\quad +\widehat{\rho}_2\Big[|m_2(1-e^{m_1}) - m_1(1-e^{m_2})| \qquad (3.3)\\
&\quad +\frac{\delta^\sigma|\lambda|}{|m_1 m_2|}|m_2^2(m_1\delta - e^{m_1\delta}+1) - m_1^2(m_2\delta - e^{m_2\delta}+1)|\Big]\Big\}, \\
\Delta_1 &= \max_{t\in[0,1]} |e^{m_2 t}| + \widehat{\rho}_1\Big(|e^{m_2\xi}| + \sum_{i=1}^{n}|j_i||e^{m_2\eta_i}+1|\Big) \\
&\quad +\widehat{\rho}_2\Big(\frac{|m_2 e^{m_2}| + |\lambda||e^{m_2\delta}+1|}{|m_2|}\Big).
\end{aligned}
$$

Then the boundary value problem (1.1)-(1.2), with $q_1^2 - 4q_0q_2 > 0$, *has at least one solution on* $[0,1]$.

Proof. To transform the problem (1.1)-(1.2) into a fixed point problem, we introduce an operator $\mathcal{N} : C([0,1],\mathbb{R}) \longrightarrow \mathcal{P}(C([0,1],\mathbb{R}))$ defined by

$$\mathcal{N}(x) = \{h \in C([0,1],\mathbb{R}) : h(t) = \mathcal{M}(x)(t)\},$$

where

$$
\begin{aligned}
\mathcal{M}(x)(t) &= \frac{1}{q_2(m_2-m_1)}\Big\{\int_0^t\int_0^s \mathcal{A}(t)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}v(u)du\,ds \\
&\quad +\rho_1(t)\Big[\int_0^\xi\int_0^s \mathcal{A}(\xi)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}v(u)du\,ds \\
&\quad -\sum_{i=1}^{n} j_i\int_0^{\eta_i}\int_0^s \mathcal{A}(\eta_i)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}v(u)du\,ds\Big] \\
&\quad +\rho_2(t)\Big[\int_0^1\int_0^s \mathcal{A}(1)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}v(u)du\,ds \\
&\quad -\lambda\int_0^\delta\int_0^s \Big(\frac{(e^{m_2(\delta-s)}-1)}{m_2} - \frac{(e^{m_1(\delta-s)}-1)}{m_1}\Big) \\
&\quad \times\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}v(u)du\,ds\Big]\Big\} \\
&\quad +g(x)\Big[e^{m_2 t} + \rho_1(t)\Big(e^{m_2\xi} - \sum_{i=1}^{n} j_i + \sum_{i=1}^{n} j_i e^{m_2\eta_i}\Big) \\
&\quad +\rho_2(t)\Big(\frac{m_2 e^{m_2} - \lambda e^{m_2\delta} - \lambda}{m_2}\Big)\Big],
\end{aligned}
$$

for $v \in S_{F,x}$.

Now, we define two operators $\bar{\mathcal{A}} : C([0,1],\mathbb{R}) \longrightarrow C([0,1],\mathbb{R})$ by

$$\begin{aligned}\bar{\mathcal{A}}x(t) &= g(x)\Big[e^{m_2 t} + \rho_1(t)\Big(e^{m_2\xi} - \sum_{i=1}^{n} j_i + \sum_{i=1}^{n} j_i e^{m_2\eta_i}\Big)\\ &\quad +\rho_2(t)\Big(\frac{m_2e^{m_2} - \lambda e^{m_2\delta} - \lambda}{m_2}\Big)\Big], \end{aligned} \tag{3.4}$$

and a multi-valued operator $\bar{\mathcal{B}} : C([0,1],\mathbb{R}) \longrightarrow \mathcal{P}(C([0,1],\mathbb{R}))$ by

$$\bar{\mathcal{B}}(x) = \{h \in C([0,1],\mathbb{R}) : h(t) = \bar{\mathcal{B}}_1(x)(t),\ v \in S_{F,x}\},$$

where

$$\begin{aligned}\bar{\mathcal{B}}_1(x)(t) &= \frac{1}{q_2(m_2-m_1)}\bigg\{\int_0^t\int_0^s \mathcal{A}(t)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}v(u)du\ ds\\ &\quad +\rho_1(t)\Big[\int_0^{\xi}\int_0^s \mathcal{A}(\xi)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}v(u)du\ ds\\ &\quad -\sum_{i=1}^{n} j_i\int_0^{\eta_i}\int_0^s \mathcal{A}(\eta_i)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}v(u)du\ ds\Big]\\ &\quad +\rho_2(t)\Big[\int_0^{1}\int_0^s \mathcal{A}(1)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}v(u)du\ ds\\ &\quad -\lambda\int_0^{\delta}\int_0^s \Big(\frac{(e^{m_2(\delta-s)}-1)}{m_2} - \frac{(e^{m_1(\delta-s)}-1)}{m_1}\Big)\\ &\quad \times\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}v(u)du\ ds\Big]\bigg\}. \end{aligned} \tag{3.5}$$

Observe that $\mathcal{N} = \bar{\mathcal{A}} + \bar{\mathcal{B}}$. We shall show that the operators $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ satisfy all the conditions of Theorem 3.5 on $[0,1]$. The proof consists of several steps and claims.

Step 1: We show that $\bar{\mathcal{A}}$ *is a contraction on* $C([0,1],\mathbb{R})$. For $x,y \in C([0,1],\mathbb{R})$, we have

$$\begin{aligned}|\bar{\mathcal{A}}x(t) - \bar{\mathcal{A}}y(t)| &\le |g(x)-g(y)|\Big[|e^{m_2t}| + \rho_1(t)\Big(|e^{m_2\xi}| + \sum_{i=1}^{n}|j_i||e^{m_2\eta_i}+1|\Big)\\ &\quad +\rho_2(t)\Big(\frac{|m_2e^{m_2}| + |\lambda||e^{m_2\delta}+1|}{|m_2|}\Big)\Big]\\ &\le \ell\Delta_1\|x-y\|, \end{aligned}$$

which, on taking supremum over $t \in [0,1]$, yields

$$\|\bar{\mathcal{A}}x - \bar{\mathcal{A}}y\| \le L_0\|x-y\|, \quad L_0 = \ell\Delta_1 < 1.$$

This shows that $\bar{\mathcal{A}}$ is a contraction as $L_0 < 1$.

Step 2: $\bar{\mathcal{B}}$ *is compact and convex valued and it is completely continuous.* This will be established in several claims.

Claim I: $\bar{\mathcal{B}}$ *maps bounded sets into bounded sets in* $C([0,1],\mathbb{R})$. Let $\mathcal{E}_\zeta = \{x \in C([0,1],\mathbb{R}) : \|x\| \le \zeta\}$ be a bounded set in $C([0,1],\mathbb{R})$. Then, for each $h \in \bar{\mathcal{B}}(x), x \in$

$\mathcal{E}_\zeta$, there exists $v \in S_{F,x}$ such that

$$\begin{aligned} h(t) &= \frac{1}{q_2(m_2-m_1)}\bigg\{\int_0^t\int_0^s \mathcal{A}(t)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}v(u)du\ ds \\ &+\rho_1(t)\Big[\int_0^\xi\int_0^s \mathcal{A}(\xi)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}v(u)du\ ds \\ &-\sum_{i=1}^n j_i\int_0^{\eta_i}\int_0^s \mathcal{A}(\eta_i)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}v(u)du\ ds\Big] \\ &+\rho_2(t)\Big[\int_0^1\int_0^s \mathcal{A}(1)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}v(u)du\ ds \\ &-\lambda\int_0^\delta\int_0^s \Big(\frac{(e^{m_2(\delta-s)}-1)}{m_2}-\frac{(e^{m_1(\delta-s)}-1)}{m_1}\Big) \\ &\times\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}v(u)du\ ds\Big]\bigg\}. \end{aligned}$$

Then, for $t \in [0,1]$, we have

$$\begin{aligned} |h(t)| &\leq \frac{1}{|q_2(m_2-m_1)|}\sup_{t\in[0,1]}\bigg\{\int_0^t\int_0^s \mathcal{A}(t)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}|v(u)|du\ ds \\ &+|\rho_1(t)|\Big[\int_0^\xi\int_0^s \mathcal{A}(\xi)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}|v(u)|du\ ds \\ &+\sum_{i=1}^n |j_i|\int_0^{\eta_i}\int_0^s \mathcal{A}(\eta_i)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}|v(u)|du\ ds\Big] \\ &+|\rho_2(t)|\Big[\int_0^1\int_0^s \mathcal{A}(1)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}|v(u)|du\ ds \\ &+|\lambda|\int_0^\delta\int_0^s \Big(\frac{(e^{m_2(\delta-s)}-1)}{m_2}-\frac{(e^{m_1(\delta-s)}-1)}{m_1}\Big) \\ &\times\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}|v(u)|du\ ds\Big]\bigg\} \\ &\leq \frac{\|p\|\psi(\zeta)}{|q_2(m_2-m_1)|}\sup_{t\in[0,1]}\bigg\{\int_0^t \Big|e^{m_2(t-s)}-e^{m_1(t-s)}\Big|\frac{s^\sigma}{\Gamma(\sigma+1)}ds \\ &+|\rho_1(t)|\Big[\int_0^\xi \Big|e^{m_2(\xi-s)}-e^{m_1(\xi-s)}\Big|\frac{s^\sigma}{\Gamma(\sigma+1)}ds \\ &+\sum_{i=1}^n |j_i|\int_0^{\eta_i} \Big|e^{m_2(\eta_i-s)}-e^{m_1(\eta_i-s)}\Big|\frac{s^\sigma}{\Gamma(\sigma+1)}ds\Big] \\ &+|\rho_2(t)|\Big[\int_0^1 \Big|e^{m_2(1-s)}-e^{m_1(1-s)}\Big|\frac{s^\sigma}{\Gamma(\sigma+1)}ds \end{aligned}$$

$$
\begin{aligned}
&+|\lambda|\int_0^\delta \Big|\frac{(e^{m_2(\delta-s)}-1)}{m_2}-\frac{(e^{m_1(\delta-s)}-1)}{m_1}\Big|\frac{s^\sigma}{\Gamma(\sigma+1)}ds\Big]\Big\}\\
\leq\ & \frac{\|p\|\psi(\zeta)}{|q_2m_1m_2(m_2-m_1)|\Gamma(\sigma+1)}\big\{\varepsilon+\widehat{\rho}_1[\xi^\sigma|m_2(1-e^{m_1\xi})\\
&-m_1(1-e^{m_2\xi})|+\sum_{i=1}^n|j_i|\eta_i^\sigma|m_2(1-e^{m_1\eta_i})-m_1(1-e^{m_2\eta_i})|]\\
&+\widehat{\rho}_2[|m_2(1-e^{m_1})-m_1(1-e^{m_2})|\\
&+\frac{\delta^\sigma|\lambda|}{|m_1m_2|}|m_2^2(m_1\delta-e^{m_1\delta}+1)-m_1^2(m_2\delta-e^{m_2\delta}+1)|]\big\},
\end{aligned}
$$

which yields

$$
\begin{aligned}
\|h\|\ \leq\ & \frac{\|p\|\psi(\zeta)}{|q_2m_1m_2(m_2-m_1)|\Gamma(\sigma+1)}\Big\{\varepsilon+\widehat{\rho}_1\xi^\sigma|m_2(1-e^{m_1\xi})\\
&-m_1(1-e^{m_2\xi})|+\sum_{i=1}^n|j_i|\eta_i^\sigma|m_2(1-e^{m_1\eta_i})-m_1(1-e^{m_2\eta_i})|]\\
&+\widehat{\rho}_2[|m_2(1-e^{m_1})-m_1(1-e^{m_2})|\\
&+\frac{\delta^\sigma|\lambda|}{|m_1m_2|}|m_2^2(m_1\delta-e^{m_1\delta}+1)-m_1^2(m_2\delta-e^{m_2\delta}+1)|]\big\}.
\end{aligned}
$$

Claim II: *$\bar{\mathcal{B}}$ maps bounded sets into equi-continuous sets.*

Let $t_1,t_2\in[0,1]$ with $t_1<t_2$ and $y\in\mathcal{E}_\zeta$, where $\mathcal{E}_\zeta$ is a bounded set of $\mathcal{C}$. Then we obtain

$$
\begin{aligned}
&|(\bar{\mathcal{B}}x)(t_2)-(\bar{\mathcal{B}}x)(t_1)|\\
\leq\ & \frac{1}{|q_2(m_2-m_1)|}\Bigg\{\Bigg|\int_0^{t_1}\int_0^s\Big[\mathcal{A}(t_2)-\mathcal{A}(t_1)\Big]\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}v(u)du\ ds\\
&+\int_{t_1}^{t_2}\int_0^s\mathcal{A}(t_2)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}v(u)du\ ds\Bigg|\\
&+|\rho_1(t_2)-\rho_1(t_1)|\Big[\int_0^\xi\int_0^s\mathcal{A}(\xi)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}|v(u)|du\ ds\\
&+\sum_{i=1}^n|j_i|\int_0^{\eta_i}\int_0^s\mathcal{A}(\eta_i)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}|v(u)|du\ ds\Big]\\
&+|\rho_2(t_2)-\rho_2(t_1)|\Big[\int_0^1\int_0^s\mathcal{A}(1)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}|v(u)|du\ ds\\
&+|\lambda|\int_0^\delta\int_0^s\Big(\frac{(e^{m_2(\delta-s)}-1)}{m_2}-\frac{(e^{m_1(\delta-s)}-1)}{m_1}\Big)\\
&\times\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}|v(u)|du\ ds\Big]\Bigg\}
\end{aligned}
$$

$$\begin{aligned}
\leq\quad & \frac{\|p\|\psi(\zeta)}{|q_2 m_1 m_2(m_2-m_1)|\Gamma(\sigma+1)}\Big\{\Big(t_1^\sigma - t_2^\sigma\Big)\Big|m_1(1-e^{m_2(t_2-t_1)}) \\
& -m_2(1-e^{m_1(t_2-t_1)})\Big| + t_1^\sigma\Big|m_1(e^{m_2t_2}-e^{m_2t_1}) - m_2(e^{m_1t_2}-e^{m_1t_1})\Big| \\
& +|\rho_1(t_2)-\rho_1(t_1)|[\xi^\sigma|m_2(1-e^{m_1\xi})-m_1(1-e^{m_2\xi})| \\
& +\sum_{i=1}^n |j_i|\eta_i^\sigma|m_2(1-e^{m_1\eta_i})-m_1(1-e^{m_2\eta_i})|] \\
& +|\rho_2(t_2)-\rho_2(t_1)|[|m_2(1-e^{m_1})-m_1(1-e^{m_2})| \\
& +\frac{\delta^\sigma|\lambda|}{|m_1m_2|}|m_1^2(m_2\delta-e^{m_2\delta}+1)-m_2^2(m_1\delta-e^{m_1\delta}+1)|]\Big\},
\end{aligned}$$

which tends to zero independently of $x \in \mathcal{E}_\zeta$ as $t_2 - t_1 \to 0$. As $\bar{\mathcal{B}}$ satisfies the above assumptions, therefore it follows by the Arzelá-Ascoli theorem that $\bar{\mathcal{B}} : C([0,1],\mathbb{R}) \longrightarrow \mathcal{P}(C([0,1],\mathbb{R}))$ is completely continuous.

Since $\mathcal{F}$ is completely continuous, it is enough to show that it is upper semicontinuous, which will be achieved via a closed graph result given by Lemma 3.3 in the following step.

Claim III: $\bar{\mathcal{B}}$ *has a closed graph.* As a consequence of Lemma 3.4, it follows that $\bar{\mathcal{B}} = \Theta \circ S_F$ is a closed graph operator (for details, see [2]), where $\Theta : L^1([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is the linear operator defined by

$$\begin{aligned}
v \mapsto \Theta(v)(t) \quad = \quad & \frac{1}{q_2(m_2-m_1)}\Big\{\int_0^t\int_0^s \mathcal{A}(t)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}v(u)du\, ds \\
& +\rho_1(t)\Big[\int_0^\xi\int_0^s \mathcal{A}(\xi)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}v(u)du\, ds \\
& -\sum_{i=1}^n j_i\int_0^{\eta_i}\int_0^s \mathcal{A}(\eta_i)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}v(u)du\, ds\Big] \\
& +\rho_2(t)\Big[\int_0^1\int_0^s \mathcal{A}(1)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}v(u)du\, ds \\
& -\lambda\int_0^\delta\int_0^s\Big(\frac{(e^{m_2(\delta-s)}-1)}{m_2}-\frac{(e^{m_1(\delta-s)}-1)}{m_1}\Big) \\
& \times\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}v(u)du\, ds\Big]\Big\}.
\end{aligned}$$

Hence $\bar{\mathcal{B}}$ has a closed graph (and therefore has closed values). In consequence, the operator $\bar{\mathcal{B}}$ is compact valued.

Thus the operators $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ satisfy all the conditions of Theorem 3.5 and hence its conclusion implies either condition (i) or condition (ii) holds. We show that the conclusion (ii) is not possible. If $x \in \theta\bar{\mathcal{A}}(x) + \theta\bar{\mathcal{B}}(x)$ for $\theta \in (0,1)$, then there exists

$v \in S_{F,x}$ such that

$$\begin{aligned} x(t) &= \frac{\theta}{q_2(m_2-m_1)}\Big\{ \int_0^t \int_0^s \mathcal{A}(t)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}v(u)du\ ds \\ &+\rho_1(t)\Big[\int_0^\xi \int_0^s \mathcal{A}(\xi)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}v(u)du\ ds \\ &-\sum_{i=1}^n j_i \int_0^{\eta_i}\int_0^s \mathcal{A}(\eta_i)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}v(u)du\ ds\Big] \\ &+\rho_2(t)\Big[\int_0^1 \int_0^s \mathcal{A}(1)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}v(u)du\ ds \\ &-\lambda\int_0^\delta\int_0^s \Big(\frac{(e^{m_2(\delta-s)}-1)}{m_2} - \frac{(e^{m_1(\delta-s)}-1)}{m_1}\Big) \\ &\times\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}v(u)du\ ds\Big]\Big\} \\ &+\theta g(x)\Big[e^{m_2 t}+\rho_1(t)\Big(e^{m_2\xi}-\sum_{i=1}^n j_i + \sum_{i=1}^n j_i e^{m_2\eta_i}\Big) \\ &+\rho_2(t)\Big(\frac{m_2e^{m_2}-\lambda e^{m_2\delta}-\lambda}{m_2}\Big)\Big]. \end{aligned}$$

Following the method for proof of Claim I, and using the fact that

$$|g(x)| \le |g(x)-g(0)| + |g(0)| \le \ell\|x\| + \ell_0,\ \ \ell_0 = |g(0)|,$$

we can obtain

$$\begin{aligned} |x(t)| &\le \frac{\|p\|\psi(\|x\|)}{|q_2m_1m_2(m_2-m_1)|\Gamma(\sigma+1)}\{\varepsilon+\widehat{\rho}_1[\xi^\sigma|m_2(1-e^{m_1\xi}) \\ &-m_1(1-e^{m_2\xi})| + \sum_{i=1}^n |j_i|\eta_i^\sigma|m_2(1-e^{m_1\eta_i})-m_1(1-e^{m_2\eta_i})|] \\ &+\widehat{\rho}_2[|m_2(1-e^{m_1})-m_1(1-e^{m_2})| \\ &+\frac{\delta^\sigma|\lambda|}{|m_1m_2|}|m_2^2(m_1\delta-e^{m_1\delta}+1)-m_1^2(m_2\delta-e^{m_2\delta}+1)|]\} \\ &+|g(x)|\Big[|e^{m_2t}|+\rho_1(t)\Big(|e^{m_2\xi}|+\sum_{i=1}^n|j_i||e^{m_2\eta_i}+1|\Big) \\ &+\rho_2(t)\Big(\frac{|m_2e^{m_2}|+|\lambda||e^{m_2\delta}+1|}{|m_2|}\Big)\Big] \\ &\le \|p\|\psi(\|x\|)\alpha + \Delta_1(\ell\|x\|+\ell_0). \end{aligned}$$

Thus

$$\|x\| \le \|p\|\psi(\|x\|)\alpha + \Delta_1(\ell\|x\|+\ell_0). \tag{3.6}$$

If condition (ii) of Theorem 3.5 holds, then there exists $\lambda \in (0,1)$ and $x \in \partial B_r$ with $x = \lambda\mathcal{N}(x)$. Then, x is a solution of (1.1)-(1.2) with $\|x\| = M$. Now, by the inequality (3.6), we get

$$\frac{(1-\ell\Delta_1)M}{\|p\|\psi(M)\alpha+\ell_0\Delta_1} \leq 1,$$

which contradicts (3.2). Hence, $\mathcal{N}$ has a fixed point in $[0,1]$ by Theorem 3.5, and consequently the problem (1.1)-(1.2) has a solution. This completes the proof. □

3.2. **The lower semi-continuous case.** In this section, we study the case when F is not necessarily convex valued by applying the nonlinear alternative contractive single-valued maps and a selection theorem due to Bressan and Colombo [7] for lower semi-continuous maps with decomposable values.

Let us mention some auxiliary facts. Let X be a nonempty closed subset of a Banach space E and $G : X \to \mathcal{P}(E)$ be a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{y \in X : G(y) \cap B \neq \emptyset\}$ is open for any open set B in E. Let A be a subset of $[0,1] \times \mathbb{R}$. A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the $\sigma-$algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in $[0,1]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$. A subset $\mathcal{A}$ of $L^1([0,1],\mathbb{R})$ is decomposable if for all $u, v \in \mathcal{A}$ and measurable $\mathcal{J} \subset [0,1] = J$, the function $u\chi_{\mathcal{J}} + v\chi_{J-\mathcal{J}} \in \mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

Definition 3.7. Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ be a multivalued operator. We say N has a property (BC) if N is lower semi-continuous (l.s.c.) and has nonempty closed and decomposable values.

Let $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Define a multivalued operator $\mathcal{F} : C([0,1] \times \mathbb{R}) \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ associated with F as

$$N_F(x) = \{w \in L^1([0,1],\mathbb{R}) : w(t) \in F(t,x(t)) \text{ for a.e. } t \in [0,1]\},$$

which is called the Nemytskii operator associated with F.

Definition 3.8. Let $F : [0,1]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Nemytskii operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Lemma 3.9 ([13]). *Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ be a multivalued operator satisfying the property (BC). Then N has a continuous selection, that is, there exists a continuous function (single-valued) $g : Y \to L^1([0,1],\mathbb{R})$ such that $g(x) \in N(x)$ for every $x \in Y$.*

Theorem 3.10. *Assume that $(H_2), (H_3), (H_4)$ and the following condition hold:*

- (H_5) *$F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a nonempty compact-valued multivalued map such that*
  - (a) *$(t,x) \longmapsto F(t,x)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable;*
  - (b) *$x \longmapsto F(t,x)$ is lower semicontinuous for each $t \in [0,1]$.*

Then the boundary value problem (1.1)-(1.2), with $q_1^2 - 4q_0q_2 > 0$, has at least one solution on $[0,1]$.

Proof. It follows from (H_2) and (H_5) that F is of l.s.c. type. Then, by Lemma 3.9, there exists a continuous function $f : C([0,1],\mathbb{R}) \to L^1([0,1],\mathbb{R})$ such that $f(x) \in \mathcal{F}(x)$ for all $x \in C([0,1],\mathbb{R})$.

Consider the problem

$$(q_2\, {}^cD^{\sigma+2} + q_1\, {}^cD^{\sigma+1} + q_0\, {}^cD^{\sigma})x(t) = f(x(t)),\ \ 0 < t < 1, \tag{3.7}$$

$$x(0) = g(x),\ \ x(\xi) = \sum_{i=1}^{n} j_i x(\eta_i),\ \ x(1) = \lambda \int_0^{\delta} x(s)ds. \tag{3.8}$$

Observe that if $x \in AC([0,1],\mathbb{R})$ is a solution of (3.7)-(3.8), then x is a solution to the problem (1.1)-(1.2). Now, we define two operators, namely $\mathcal{A}' : C([0,1],\mathbb{R}) \longrightarrow C([0,1],\mathbb{R})$ by

$$\begin{aligned} \mathcal{A}'x(t) &= g(x)\Big[e^{m_2 t} + \rho_1(t)\Big(e^{m_2\xi} - \sum_{i=1}^{n} j_i + \sum_{i=1}^{n} j_i e^{m_2\eta_i}\Big) \\ &\quad + \rho_2(t)\Big(\frac{m_2 e^{m_2} - \lambda e^{m_2\delta} - \lambda}{m_2}\Big)\Big], \end{aligned}$$

and $\mathcal{B}' : C([0,1],\mathbb{R}) \longrightarrow C([0,1],\mathbb{R})$ by

$$\begin{aligned} \mathcal{B}'x(t) &= \frac{1}{q_2(m_2 - m_1)}\Bigg\{\int_0^t \int_0^s \mathcal{A}(t)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} f(x(u))du\ ds \\ &\quad + \rho_1(t)\Big[\int_0^{\xi}\int_0^s \mathcal{A}(\xi)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} f(x(u))du\ ds \\ &\quad - \sum_{i=1}^{n} j_i \int_0^{\eta_i}\int_0^s \mathcal{A}(\eta_i)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} f(x(u))du\ ds\Big] \\ &\quad + \rho_2(t)\Big[\int_0^1\int_0^s \mathcal{A}(1)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} f(x(u))du\ ds \\ &\quad - \lambda\int_0^{\delta}\int_0^s \Big(\frac{(e^{m_2(\delta-s)} - 1)}{m_2} - \frac{(e^{m_1(\delta-s)} - 1)}{m_1}\Big) \\ &\quad \times \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} f(x(u))du\ ds\Big]\Bigg\}. \end{aligned}$$

Clearly $\mathcal{A}', \mathcal{B}' : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ are continuous. The arguments used in the proof of Theorem 3.6 apply and hence guarantee that $\mathcal{A}'$ and $\mathcal{B}'$ satisfy all the conditions of the Nonlinear Alternative for contractive maps in the single valued setting [15] and hence the problem (3.7) has a solution. □

4. Existence results for problem (1.1)-(1.2) with $q_1^2 - 4q_0q_2 = 0$

Definition 4.1. Let $q_1^2 - 4q_0q_2 = 0$. A function $x \in C([0,1],\mathbb{R})$ possessing a Caputo fractional derivative of order $\sigma+2$ is a solution of the problem if $x(0) = g(x),\ \ x(\xi) = \sum_{i=1}^{n} j_i x(\eta_i),\ \ x(1) = \lambda\int_0^{\delta} x(s)ds$, and there exists a function $v \in L^1([0,1],\mathbb{R})$ such that $v(t) \in F(t, x(t))$ a.e. on $[0,1]$ and

$$\begin{aligned} x(t) &= \frac{1}{q_2}\Big\{ \int_0^t \int_0^s \mathcal{B}(t)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}v(u)du\ ds \\ &+\chi_1(t)\Big[\int_0^{\xi}\int_0^s \mathcal{B}(\xi)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}v(u)du\ ds \\ &-\sum_{i=1}^{n} j_i \int_0^{\eta_i}\int_0^s \mathcal{B}(\eta_i)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}v(u)du\ ds\Big] \\ &+\chi_2(t)\Big[\int_0^1\int_0^s \mathcal{B}(1)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}v(u)du\ ds \\ &-\lambda\int_0^{\delta}\int_0^s\Big(\frac{m(\delta-s)e^{m(\delta-s)}-e^{m(\delta-s)}+1}{m^2}\Big) \\ &\times\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}v(u)du\ ds\Big]\Big\} \\ &+g(x)\Big[e^{mt}+\chi_1(t)\Big(e^{m\xi}-\sum_{i=1}^{n} j_i e^{m\eta_i}\Big) \\ &+\chi_2(t)\Big(\frac{me^m-\lambda e^{m\delta}+\lambda}{m}\Big)\Big], \end{aligned}$$

The proof of the following theorem is similar to that of Theorem 3.6. So it is omitted.

Theorem 4.2. *Assume that $(H_1), (H_2)$ and (H_3) hold. In addition we suppose that*

$(H_4)'$ *there exists a number $M_1 > 0$ such that*

$$\frac{(1-\ell\Delta_2)M_1}{\|g\|\psi(M_1)\beta+\ell_0\Delta_2} > 1,$$

where $\ell_0 = |g(0)|$ and

$$\begin{aligned} \widehat{\chi}_1 &= \max_{t\in[0,1]}|\chi_1(t)|,\ \widehat{\chi}_2 = \max_{t\in[0,1]}|\chi_2(t)|, \\ \beta &= \frac{1}{|q_2|m^2\Gamma(\sigma+1)}\Big\{(1+\widehat{\chi}_2)|me^m-e^m+1| \\ &+\widehat{\chi}_1\Big[\xi^{\sigma}|m\xi e^{m\xi}-e^{m\xi}+1|+\sum_{i=1}^{n}|j_i|\eta_i^{\sigma}|m\eta_i e^{m\eta_i}-e^{m\eta_i}+1|\Big] \\ &+\frac{|\lambda|\delta^{\sigma}\widehat{\chi}_2}{|m|}\Big|m\delta(e^{m\delta}+1)+2(1-e^{m\delta})\Big|\Big\}, \\ \Delta_2 &= \max_{t\in[0,1]}|e^{mt}|+\widehat{\chi}_1\Big(|e^{m\xi}|+\sum_{i=1}^{n}|j_i||e^{m\eta_i}|\Big) \\ &+\widehat{\chi}_2\Big(\frac{|me^m|+|\lambda||e^{m\delta}+1|}{|m|}\Big). \end{aligned}$$

Then the boundary value problem (1.1)-(1.2), with $q_1^2 - 4q_0q_2 = 0$, *has at least one solution on* $[0, 1]$.

5. Existence results for problem (1.1)-(1.2) with $q_1^2 - 4q_0q_2 < 0$

Definition 5.1. Let $q_1^2 - 4q_0q_2 < 0$. A function $x \in C([0,1],\mathbb{R})$ possessing a Caputo fractional derivative of order $\sigma + 2$ is a solution of the problem (1.1)-(1.2) if $x(0) = g(x)$, $x(\xi) = \sum_{i=1}^{n} j_i x(\eta_i)$, $x(1) = \lambda \int_0^\delta x(s)ds$, and there exists a function $v \in L^1([0,1],\mathbb{R})$ such that $v(t) \in F(t, x(t))$ a.e. on $[0,1]$ and

$$\begin{aligned} x(t) &= \frac{1}{q_2 b}\Bigg\{ \int_0^t \int_0^s \mathcal{F}(t)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} v(u)du\ ds \\ &+\tau_1(t)\Big[\int_0^\xi \int_0^s \mathcal{F}(\xi)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} v(u)du\ ds \\ &-\sum_{i=1}^{n} j_i \int_0^{\eta_i}\int_0^s \mathcal{F}(\eta_i)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} v(u)du\ ds\Big] \\ &+\tau_2(t)\Big[\int_0^1 \int_0^s \mathcal{F}(1)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} v(u)du\ ds \\ &-\frac{\lambda}{a^2+b^2}\int_0^\delta\int_0^s \Big(b - be^{-a(\delta-s)}\cos b(\delta-s) \\ &-ae^{-a(\delta-s)}\sin b(\delta-s)\Big)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} v(u)du\ ds\Big]\Bigg\} \\ &+g(x)\Big[e^{-at}\cos bt + \tau_1(t)\Big(e^{-a\xi}\cos b\xi - \sum_{i=1}^{n} j_i e^{-a\eta_i}\cos b\eta_i\Big) \\ &+\tau_2(t)\Big(e^{-a}\cos b - \frac{\lambda}{a^2+b^2}(a - ae^{-a\delta}\cos b\delta + be^{-a\delta}\sin b\delta)\Big)\Big]. \end{aligned}$$

Theorem 5.2. *Assume that* $(H_1), (H_2)$ *and* (H_3) *hold. In addition we suppose that*

$(H_4)'$ *there exists a number* $M_1 > 0$ *such that*

$$\frac{(1-\ell\Delta_3)M_2}{\|g\|\psi(M_2)\gamma + \ell_0\Delta_3} > 1,$$

where $\ell_0 = |g(0)|$ *and*

$$\begin{aligned} \widehat{\tau}_1 &= \max_{t\in[0,1]}|\tau_1(t)|, \quad \widehat{\tau}_2 = \max_{t\in[0,1]}|\tau_2(t)| \\ \gamma &= \frac{1}{|q_2 b(a^2+b^2)|\Gamma(\sigma+1)}\Big\{(1+\widehat{\tau}_2)\Big[|b - be^{-a}\cos b - ae^{-a}\sin b|\Big] \\ &+\widehat{\tau}_1\Big[\xi^\sigma|b - be^{-a\xi}\cos b\xi - ae^{-a\xi}\sin b\xi| \\ &+\sum_{i=1}^{n}|j_i|\eta_i^\sigma|b - be^{-a\eta_i}\cos b\eta_i - ae^{-a\eta_i}\sin b\eta_i|\Big] \\ &+|\lambda|\delta^\sigma\widehat{\tau}_2\Big[|b\delta - e^{-a\delta}\sin b\delta|\Big]\Big\}, \end{aligned}$$

$$\begin{aligned}\Delta_3 &= \max_{t\in[0,1]} |e^{-at}\cos bt| + \widehat{\tau}_1(|e^{-a\xi}\cos b\xi| + \sum_{i=1}^{n} |j_i||e^{-a\eta_i}\cos b\eta_i|) \\ &\quad + \widehat{\tau}_2\Big(|e^{-a}\cos b| + \frac{|\lambda|}{a^2+b^2}(|a - ae^{-a\delta}\cos b\delta + be^{-a\delta}\sin b\delta|)\Big).\end{aligned}$$

Then the boundary value problem (1.1)-(1.2), with $q_1^2 - 4q_0q_2 < 0$, *has at least one solution on* $[0,1]$.

We do not provide the proof of the above theorem, as it is similar to that of Theorem 3.6.

6. Example

Example 6.1. Consider the following boundary value problem of fractional differential inclusions

$$(2\ {}^cD^{12/5} + 3\ {}^cD^{7/5} + {}^cD^{2/5})x(t) \in F(t, x(t)),\ \ 0 < t < 1, \tag{6.1}$$

$$x(0) = \frac{1}{9}\sin x,\ \ x(1/5) = x(1/4) + 2x(1/3) + x(1/2),$$

$$x(1) = 2\int_0^{1/6} x(s)ds, \tag{6.2}$$

where

$$F(t, x(t)) = \left[\frac{1}{2}e^{-t}\sin x,\ \ \frac{|x|}{8(|x|+6)} + \frac{1}{20}\cos^2 t\right].$$

Here, $\sigma = 2/5$, $\xi = 1/5$, $\eta_1 = 1/4$, $\eta_2 = 1/3$, $\eta_3 = 1/2$, $\delta = 1/6$, $j_1 = 1$, $j_2 = 2$, $j_3 = 1$, $\lambda = 2$, $q_1^2 - 4q_0q_2 = 1 > 0$, $\ell = 1/9$, and $\ell_0 = 1$.

Clearly $\|F(t, x(t))\| \le p(t)\psi(\|x\|)$, where $p(t) = e^{-t}$ and $\psi(\|x\|) = \frac{1}{2}$. Using the value $\alpha \approx 6.9171$, we find that $M > 0.584302$. Since the hypothesis of Theorem 3.6 is satisfied, the problem (6.1)-(6.2) has at least one solution on $[0,1]$.

Acknowledgment

The authors are grateful to the anonymous referee for his/her valuable comments that led to the improvement of the original manuscript.

References

[1] R. P. Agarwal, B. Ahmad and A. Alsaedi, *Fractional-order differential equations with anti-periodic boundary conditions: a survey*, Bound. Value Probl. **2017** (2017): 173.

[2] B. Ahmad, A. Alsaedi, S. K. Ntouyas and J. Tariboon, *Hadamard-type Fractional Differential Equations, Inclusions and Inequalities*, Springer, Cham, 2017.

[3] B. Ahmad, M. M. Matar and O. M. El-Salmy, *Existence of solutions and Ulam stability for Caputo type sequential fractional differential equations of order* $\alpha \in (2,3)$, Inter. J. Anal. Appl. **15** (2017), 86–101.

[4] B. Ahmad and S. K. Ntouyas, *A higher-order nonlocal three-point boundary value problem of sequential fractional differential equations*, Miscolc Math. Notes **15** (2014), 265–278.

[5] Z. B. Bai and W. Sun, *Existence and multiplicity of positive solutions for singular fractional boundary value problems*, Comput. Math. Appl. **63** (2012) 1369–1381.

[6] A. V. Bicadze and A. A. Samarskii, *Some elementary generalizations of linear elliptic boundary value problems (Russian)*, Anal. Dokl. Akad. Nauk SSSR, **185** (1969), 739–740.

[7] A. Bressan and G. Colombo, *Extensions and selections of maps with decomposable values*, Studia Math. **90** (1988), 69–86.
[8] L. Byszewski, *Theorems about existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem*, J. Math. Anal. Appl. **162** (1991), 494–505.
[9] L. Byszewski, *Existence and uniqueness of mild and classical solutions of semilinear functional-differential evolution nonlocal Cauchy problem*, Selected problems of mathematics, 50th Anniv. Cracow Univ. Technol. Anniv., Issue 6, Cracow Univ. Technol., Krakow, 1995, pp. 25–33.
[10] L. Byszewski and V. Lakshmikantham, *Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space*, Appl. Anal. **40** (1991), 11–19.
[11] K. Deimling, *Multivalued Differential Equations*, Walter De Gruyter, Berlin-New York, 1992.
[12] K. Diethelm, *The Analysis of Fractional Differential Equations. An Application-oriented Exposition Using Differential Operators of Caputo Type. Lecture Notes in Mathematics 2004*, Springer-Verlag, Berlin, 2010.
[13] M. Frigon, *Théorèmes d'existence de solutions d'inclusions différentielles*, *Topological Methods in Differential Equations and Inclusions*, A. Granas and M. Frigon (eds.), NATO ASI Series C, Vol. 472, Kluwer Acad. Publ., Dordrecht, 1995, pp. 51–87.
[14] J. R. Graef, L. Kong and Q. Kong, *Application of the mixed monotone operator method to fractional boundary value problems*, Fract. Calc. Differ. Calc. **2** (2011), 554–567.
[15] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.
[16] Sh. Hu and N. Papageorgiou, *Handbook of Multivalued Analysis, Theory I*, Kluwer, Dordrecht, 1997.
[17] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.
[18] A. Lasota and Z. Opial, *An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. **13** (1965), 781–786.
[19] C.-G. Li, M. Kostic and M. Li, *Abstract multi-term fractional differential equations*, Kragujevac J. Math. **38** (2014), 51–71.
[20] Y. Liu, *Boundary value problems of singular multi-term fractional differential equations with impulse effects*, Math. Nachr. **289** (2016), 1526–1547.
[21] F. Mainardi, *Some basic problems in continuum and statistical mechanics, in: Fractals and Fractional Calculus in Continuum Mechanics*, A. Carpinteri, F. Mainardi (Eds.), Springer, Berlin, 1997, pp. 291–348.
[22] K.S. Miller, B.Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley, NewYork, 1993.
[23] S. K. Ntouyas, J. Tariboon and W. Sudsutad, *Boundary value problems for Riemann-Liouville fractional differential inclusions with nonlocal Hadamard fractional integral conditions*, Mediter. J. Math. **13** (2016), 939–954.
[24] W. V. Petryshyn and P. M. Fitzpatric, *A degree theory, fixed point theorems, and mapping theorems for multivalued noncompact maps*, Trans. Amer. Math. Soc. **194** (1974), 1–25.
[25] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
[26] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, Yverdon, 1993.
[27] S. Stanek, *Periodic problem for two-term fractional differential equations*, Fract. Calc. Appl. Anal. **20** (2017), 662–678.
[28] P.J. Torvik, R.L. Bagley, *On the appearance of the fractional derivative in the behavior of real materials*, J. Appl. Mech. **51** (1984), 294–298.
[29] Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2014.

Manuscript received May 23 2020
revised September 13 2020

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