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HARMONIC INDEX OF GRAPHS WITH ADDED EDGES

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Abstract. The harmonic index of a graph G is defined as the sum of the weights $\frac{2}{du+dv}$ of all edges uv of G, where du denotes the degree of a vertex u in G. Several operations including edge and vertex deletion and addition are utilized to ease the calculations related to some properties of graphs. In this work, we present the effect of adding a new edge to a connected simple graph on harmonic index is studied. In particular, some statements for the change of harmonic index of path, cycle, complete, star, complete bipartite and tadpole graphs are obtained.

1. Introduction

The Randić index is one of the most successful molecular descriptors in structure-property and structure-activity relationships studies. The Randić index $R(G)$ is defined as

$$R(G) = \sum_{uv\in E} \frac{1}{\sqrt{d(u)d(v)}},$$

[2]. Another variant of Randić index is the harmonic index denoted by $H(G)$.

Throughout this paper, all graphs are finite, simple, undirected and connected. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex $v \in V(G)$ is denoted by $d_G v$ or briefly by dv. A vertex of degree one will be called a pendant vertex. We will use P_n, C_n, S_n, K_n, $K_{r,s}$, and $T_{r,s}$ to denote the path, cycle, star, complete, complete bipartite and tadpole graphs of order n, respectively, where $n = r + s$ in the latter two.

2. Harmonic index

Let $e = uv$ be an edge in G. The number $\frac{2}{du+dv}$ will be called the weight of the edge e. The harmonic index of a graph G denoted by $H(G)$ is defined as the sum of the weights of all edges $e = uv$ of G, is given by

$$H(G) = \sum_{uv\in E} \frac{2}{du+dv}.$$

In this work, we find the change of the harmonic index of a given graph G when an edge e is added. This effect differs for adding a pendant edge or a non-pendant edge.

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3. Change in the harmonic index when a new edge is added

In combinatorial study of graphs, operations such as adding and deleting vertices and edges are frequently employed. In recent years, several such works have been published. In [1], the effect of edge and vertex deletion on omega invariant was considered. In [4], Zagreb indices of graphs having added edges are studied. In [3], the effects of edge deletion and addition on several Zagreb indices of graphs are calculated. In this section, we will determine the amount of change in the harmonic index when a new edge is added to a simple graph. We have

Theorem 3.1. *Let G be a simple graph and let $G+e$ be the graph obtained by adding an edge e to G. If a new edge e is added to G to join the vertex $v \in V(G)$ of degree d_Gv with a new pendant vertex u, then*

$$H(G+e) - H(G) = \frac{2}{2+d_Gv} - \sum_{vv_i \in E(G)} \frac{2}{(d_Gv_i + d_Gv)(d_Gv_i + d_Gv + 1)}.$$

Proof. Let the neighbouring vertices of v in the graph G be $v_1,\ v_2,\ \cdots,\ v_{d_Gv}$ of degrees $d_Gv_1, d_Gv_2, \cdots, d_Gv$, respectively. Let us add a new pendant edge e between the existing vertex $v \in G$ of degree d_Gv and a new vertex u which is not a vertex of G. We know that the harmonic index of a graph G can be stated as

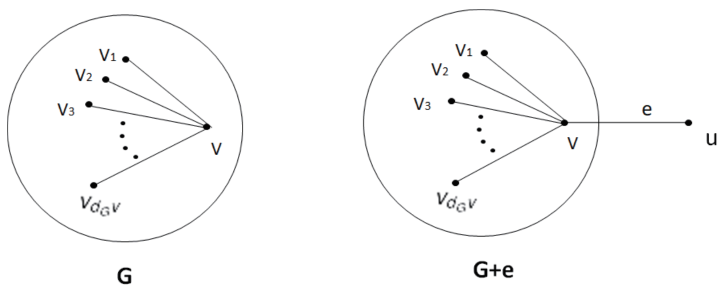

Figure 1. Adding a pendant edge e to G

$$H(G) = \sum_{vv_i \in E(G)} \frac{2}{d_Gv_i + d_Gv} + \sum_{\substack{v_iv_j \in E(G) \\ v_i, v_j \neq v}} \frac{2}{d_Gv_i + d_Gv_j}.$$

Therefore

$$\begin{aligned} H(G+e) &= \sum_{vv_i \in E(G)} \frac{2}{d_{G+e}v_i + d_{G+e}v} + \sum_{\substack{v_iv_j \in E(G) \\ v_i, v_j \neq v}} \frac{2}{d_{G+e}v_i + d_{G+e}v_j} + \frac{2}{d_{G+e}v + d_{G+e}u} \\ &= \sum_{vv_i \in E(G)} \frac{2}{d_Gv_i + d_Gv + 1} + \sum_{\substack{v_iv_j \in E(G) \\ v_i, v_j \neq v}} \frac{2}{d_Gv_i + d_Gv_j} + \frac{2}{1 + d_Gv + 1}. \end{aligned}$$

Hence

$$\begin{aligned} H(G+e)-H(G) &= \frac{2}{2+d_Gv}+\sum_{vv_i\in E(G)}\left[\frac{2}{d_Gv_i+d_Gv+1}-\frac{2}{d_Gv_i+d_Gv}\right] \\ &= \frac{2}{2+d_Gv}-\sum_{vv_i\in E(G)}\frac{2}{(d_Gv_i+d_Gv)(d_Gv_i+d_Gv+1)} \end{aligned}$$

which gives the result. □

Next we calculate the effect of a non-pendant edge addition on harmonic index:

Theorem 3.2. *Let G be a simple graph and let $u,\ v \in V(G)$ be two non-adjacent vertices. If we add a new edge e between u and v, then*

$$\begin{aligned} H(G+e)-H(G) &= \frac{2}{d_Gu+d_Gv+2}-\sum_{i=1}^{d_Gu}\frac{2}{(d_Gu+d_Gu_i)(d_Gu+d_Gu_i+1)} \\ &\quad -\sum_{j=1}^{d_Gv}\frac{2}{(d_Gv+d_Gv_i)(d_Gv+d_Gv_i+1)}. \end{aligned}$$

Proof. Let the vertices of graph G be $u_1, u_2, \cdots, u_{d_Gu}$ and $v_1, v_2, \cdots, v_{d_Gv}$. Let us add a new non-pendant edge e by joining two existing vertices u, v of G. We know

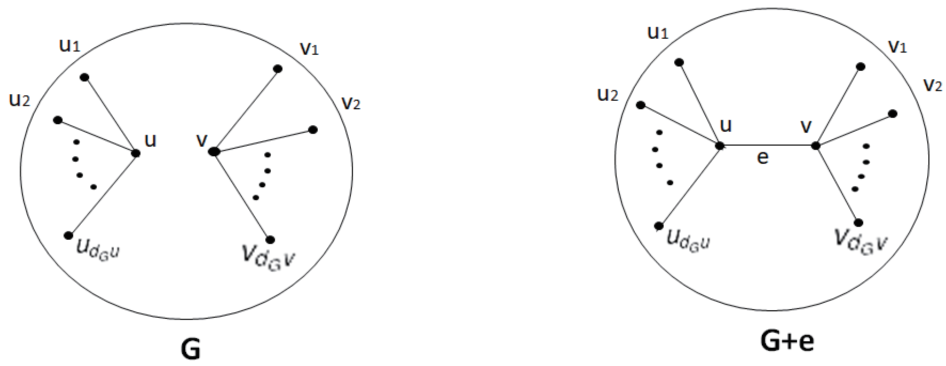

FIGURE 2. Adding a non-pendant edge e to G

that the harmonic index of a graph G can be stated as

$$H(G) = \sum_{i=1}^{d_Gu}\frac{2}{d_Gu+d_Gu_i}+\sum_{j=1}^{d_Gv}\frac{2}{d_Gv+d_Gv_j}+\sum_{\substack{w,t\in V(G)\\ w,t\notin\{u,v\}}}\frac{2}{d_Gw+d_Gt}.$$

Therefore

$$\begin{aligned}
H(G+e) &= \frac{2}{d_{G+e}u + d_{G+e}v} + \sum_{i=1}^{d_G u} \frac{2}{d_{G+e}u + d_{G+e}u_i} + \sum_{j=1}^{d_G v} \frac{2}{d_{G+e}v + d_{G+e}v_j} \\
&\quad + \sum_{\substack{w,t\in V(G)\\ w,t\notin\{u,v\}}} \frac{2}{d_{G+e}w + d_{G+e}t} \\
&= \frac{2}{d_G u + 1 + d_G v + 1} + \sum_{i=1}^{d_G u} \frac{2}{d_G u + 1 + d_G u_i} + \sum_{j=1}^{d_G v} \frac{2}{d_G v + 1 + d_G v_j} \\
&\quad + \sum_{\substack{w,t\in V(G)\\ w,t\notin\{u,v\}}} \frac{2}{d_G w + d_G t}.
\end{aligned}$$

Hence

$$\begin{aligned}
H(G+e) - H(G) &= \frac{2}{d_G u + d_G v + 2} - \sum_{i=1}^{d_G u} \frac{2}{(d_G u + d_G u_i)(d_G u + d_G u_i + 1)} \\
&\quad - \sum_{j=1}^{d_G v} \frac{2}{(d_G v + d_G v_j)(d_G v + d_G v_j + 1)}
\end{aligned}$$

which gives the result. □

4. Effect of edge addition on $H(G)$ for some classes of graphs

In this section, using the above results for adding a pendant or a non-pendant edge, we get easy results for graph classes $P_n,\ C_n,\ S_n,\ K_n,\ K_{r,s}$ and $T_{r,s}$.

4.1. Path graph P_n.

First we have

Lemma 4.1. *The harmonic index of a path graph P_n is*

$$H(P_n) = \frac{n}{2} - \frac{1}{6}.$$

Proof. From the definition of the harmonic index, we get the result as follows:

$$\begin{aligned}
H(P_n) &= \sum_{uv\in P_n} \frac{2}{du + dv} \\
&= \frac{n}{2} - \frac{1}{6}.
\end{aligned}$$

□

Therefore by Theorems 3.1 and 3.2, we can obtain the harmonic index of an edge added path graph considering the following possibilities for P_n:

a) If a pendant edge e is added to one of the two end points having degree 1 then,

$$H(P_n + e) = \frac{n}{2} + \frac{1}{3};$$

b) If a pendant edge e is added to a vertex which is next to end point of P_n of degree 2 then,

$$H(P_n + e) = \frac{n}{2} + \frac{1}{15};$$

c) If a pendant edge e is added to a midpoint of P_n of degree 2 which is not next to endpoints then,

$$H(P_n + e) = \frac{n}{2} + \frac{2}{15};$$

d) If a non-pendant edge e is added to the vertices v_i, v_j such that v_i is a endpoint and v_j is not next to the other endpoint. so that $d_i = 1, d_j = 2$ or vice versa then,

$$H(P_n + e) = \frac{n}{2} - \frac{2}{15};$$

e) If a non-pendant edge e is added to the vertices v_i, v_j such that v_i is a endpoint and v_j is next to other endpoint. so that $d_i = 1, d_j = 2$ or vice versa then,

$$H(P_n + e) = \frac{n}{2} - \frac{1}{5};$$

f) If a non-pendant edge e is added to both the end vertices v_i, v_j so that $d_i = d_j = 1$ then,

$$H(P_n + e) = \frac{n}{2};$$

g) If a non-pendant edge e is added to the vertices v_i, v_j such that both the vertices are the midpoints of P_n and one after the other. so that $d_i = d_j = 2$ then,

$$H(P_n + e) = \frac{n}{2} - \frac{4}{15};$$

h) If a non-pendant edge e is added to the vertices v_i, v_j such that both the vertices are next to the endpoints. so that $d_i = d_j = 2$ then,

$$H(P_n + e) = \frac{n}{2} - \frac{11}{30};$$

i) If a non-pendant edge e is added to the midpoints v_i, v_j such that both the vertices are not one after the other. so that $d_i = d_j = 2$ then,

$$H(P_n + e) = \frac{n}{2} - \frac{3}{10}.$$

Therefore, we determine the change in the harmonic index when a new edge is added to a path graph P_n:

Corollary 4.2. *When a new edge is added to a path graph P_n, its harmonic index either increases by 0.5, 0.2333, 0.3, 0.0333 or 0.1667 or decreases by 0.0333, 0.1, 0.2 or 0.1333.*

4.2. **Cycle graph C_n.** First we have

Lemma 4.3. *The harmonic index of a cycle graph C_n is*

$$H(C_n) = \frac{n}{2}.$$

Proof. By the definition, we easily get that

$$\begin{aligned} H(C_n) &= \sum_{uv \in C_n} \frac{2}{du + dv} \\ &= \frac{n}{2}. \end{aligned}$$

□

Either by Theorems 3.1 and 3.2 or by calculation of all possible cases, we can obtain the harmonic index of an edge added cycle graph. Since all the vertices are of degree 2 in C_n, we have,

a) If a pendant edge e is added to any one of the vertices v_i of C_n of degree 2, then

$$H(C_n + e) = \frac{n}{2} + \frac{3}{10};$$

b) If a non-pendant edge e is added to the vertices v_i, v_j, such that both the vertices are adjacent to each other. so that $d_i = d_j = 2$, then

$$H(C_n + e) = \frac{n}{2} - \frac{1}{30};$$

c) If a non-pendant edge e is added to the vertices v_i, v_j, such that both the vertices are not adjacent to each other. so that $d_i = d_j = 2$, then

$$H(C_n + e) = \frac{n}{2} - \frac{1}{15}.$$

So we proved the following result:

Corollary 4.4. *When a new edge is added to a cycle graph C_n, its harmonic index increases by either 0.3 or decreases by 0.0333 or 0.0667.*

4.3. **Star graph S_n.** First we have

Lemma 4.5. *The harmonic index of a star graph S_n is*

$$H(S_n) = \frac{2(n-1)}{n}.$$

Therefore by Theorems 3.1 and 3.2, we can obtain the harmonic index of an edge added star graph considering the following possibilities for S_n. Since the vertex of S_n is either pendant or of degree $n-1$, we have,

a) If a pendant edge e is added to one of the vertex v_i of S_n of degree 1 then,

$$H(S_n + e) = \frac{8}{3} - \frac{2}{n} - \frac{2}{n^2};$$

b) If a pendant edge e is added to the vertex v_i of S_n of degree $n-1$ then,

$$H(S_n + e) = \frac{2n}{1+n};$$

c) If a non-pendant edge e is added at the vertices v_i, v_j such that $d_i = d_j = 1$ then,

$$H(S_n + e) = \frac{4}{n+1} + \frac{1}{2} + \frac{2(n-3)}{n};$$

d) If a non-pendant edge e is added to the vertices v_i, v_j such that $d_i = 1$, $d_j = n-1$ or vice versa then,

$$H(S_n + e) = \frac{4}{2+n} + \frac{2(n-2)}{1+n}.$$

So we proved the following:

Corollary 4.6. *If an edge is added to a star graph S_n, its harmonic index increases by either $\frac{2}{3} - \frac{2}{n^2}$, $\frac{2}{n+n^2}$, $\frac{n^2+n-8}{2n(n+1)}$ or decreases by $\frac{2(n-2)}{n(n+1)(n+2)}$.*

4.4. **Complete graph K_n.** First, similarly to the above cases, we have

Lemma 4.7. *The harmonic index of a complete graph K_n is*

$$H(K_n) = \frac{n}{2}.$$

Hence Theorems 3.1 and 3.2 imply that the harmonic index of an edge added complete graph have the following possibilities. As all the vertices of K_n are of degree $n-1$, we have

a) If a pendant edge e is added to one of the vertex v_i of K_n of degree $n-1$ then,

$$H(K_n + e) = \frac{2n^3 + n^2 + 5n - 6}{(2n+2)(2n-1)};$$

b) If a non-pendant edge e is added at the vertices v_i, v_j such that $d_i = d_j = n-1$ then,

$$H(K_n + e) = \frac{2n^4 - 3n^3 + n^2 - 2n + 4}{2n(n-1)(2n-1)}.$$

So we proved the following result:

Corollary 4.8. *If an edge is added to a complete graph K_n, its harmonic index increases by either $\frac{3(n-1)}{(n+1)(2n-1)}$ or decreases by $\frac{(n-2)}{n(n-1)(2n-1)}$.*

4.5. **Complete bipartite graph $K_{r,s}$.** First we have

Lemma 4.9. *The harmonic index of a complete bipartite graph $K_{r,s}$ is*

$$H(K_{r,s}) = \frac{2rs}{r+s}.$$

By Theorems 3.1 and 3.2, we can obtain the harmonic index of an edge added complete bipartite graph considering the following possibilities for $K_{r,s}$. Recall that the degree sequence of $K_{r,s}$ is $\{r^s, s^r\}$. Hence the possibilities are as follows:

a) If a pendant edge e is added to a vertex of degree r then,

$$H(K_{r,s}+e)=\frac{2r^3s+2s^2+2r^2s^2+6r^2s+4rs^2+8rs+2r+2s-4r}{r^3+3r^2+2r^2s+rs^2+5rs+2s^2+2r+2s};$$

b) If a pendant edge e is added to a vertex of degree s then,

$$H(K_{r,s}+e)=\frac{2rs^3+2r^2+2r^2s^2+6rs^2+4r^2s+8rs+2r+2s-4s}{s^3+3s^2+2rs^2+r^2s+5rs+2r^2+2r+2s};$$

c) If a non-pendant edge e is added to two existing vertices v_i, v_j such that $d_i=d_j=r$ then,

$$H(K_{r,s}+e)=\frac{4r^3s+4r^2s^2+8r^2s+4rs^2+8rs+2r^2+2s^2-8r^2-6r+2s}{2r^3+4r^2s+2rs^2+4r^2+6rs+2r+2s+2s^2};$$

d) If a non-pendant edge e is added to the vertices v_i, v_j such that $d_i=d_j=s$ then,

$$H(K_{r,s}+e)=\frac{4rs^3+4r^2s^2+8rs^2+4r^2s+8rs+2r^2+2s^2-8s^2+2r-6s}{2s^3+4rs^2+2r^2s+4s^2+6rs+2r+2s+2r^2};$$

e) If a non-pendant edge e is added to two existing vertices v_i, v_j such that $d_i=r$ and $d_j=s$ or vice versa then,

$$H(K_{r,s}+e)=\frac{6r^2s++6rs^2+4r^2s^2+2r^3s+2rs^3+4rs-2r-2s+4}{r^3+s^3+3r^2+3s^2+2r+2s+3r^2s+3rs^2+6rs};$$

So we proved

Corollary 4.10. *When a new edge is added to a complete bipartite graph $K_{r,s}$, its harmonic index increases by either*

$$\frac{6rs^2+4r^2s-2r^2+2s^3+2s^2}{r^4+3r^3+3r^3s+2r^2+3r^2s^2+8r^2s+7rs^2+4rs+2s^3+rs^3+2s^2},$$

or

$$\frac{6r^2s+4rs^2+2r^3+2r^2-2s^2}{s^4+3s^3+3rs^3+2s^2+3r^2s^2+8rs^2+7r^2s+4rs+2r^3+r^3s+2r^2},$$

or

$$\frac{2r^3+2r^2-6s^3-6s^2+6r^2s-2rs^2-4rs}{2r^3+2r^2+2s^4+4s^3+2s^2+2r^3s+6rs^3+8r^2s+10rs^2+6r^2s^2+4rs},$$

or

$$\frac{-6r^3-6r^2+2s^3+2s^2-2r^2s+6rs^2-4rs}{2r^4+4r^3+2r^2+2s^3+2s^2+6r^3s}+2rs^3+6r^2s^2+10r^2s+8rs^2+4rs,$$

or

$$\frac{-2r^2-2s^2+4r+4s-4rs}{r^4+s^4+3r^3+3s^3+2r^2+2s^2+4r^3s+4rs^3+9r^2s+9rs^2+6r^2s^2+4rs}.$$

4.6. **Tadpole graph $T_{r,s}$.** We have

Lemma 4.11. *The harmonic index of a tadpole graph $T_{r,s}$ is*

$$H(T_{r,s}) = \frac{15(r+s)-4}{30}.$$

Therefore by Theorems 3.1 and 3.2, we can obtain all the changes in the harmonic index of an edge added tadpole graph considering the following possibilities for $T_{r,s}$:

By theorem 3.1, if we add a new pendant edge, we have the following cases to consider:

1) If a pendant edge e is added to a vertex v_{r+s} of degree 1, then

$$H(T_{r,s}+e) = \frac{15(r+s)+11}{30}$$

as the unique neighbour of v_{r+s} is v_{r+s-1} of degree 2.

2) If a pendant edge e is added to a vertex v_1 of degree 3, then

$$H(T_{r,s}+e) = \frac{15(r+s)+2}{30}$$

as the three neighbours of v_1 are all of degree 2.

3) All other vertices of $T_{r,s}$ are of degree 2 and each has exactly two neighbours. There are three subcases:

3a) If a pendant edge e is added to one of the vertices v_2, v_r or v_{r+1} of degree 2, then

$$H(T_{r,s}+e) = \frac{5(r+s)+2}{10}$$

as the two neighbours of each of these vertices are of degree 2 and 3.

3b) If a pendant edge e is added to one of the vertices v_3, v_4, $\cdots$, v_{r-1}, v_{r+2}, v_{r+3}, $\cdots$, v_{r+s-2} of degree 2, then

$$H(T_{r,s}+e) = \frac{3(r+s)+1}{6}$$

as both neighbours of these vertices are of degree 2.

3c) If a pendant edge e is added to the vertex v_{r+s-1} of degree 2, then

$$H(T_{r,s}+e) = \frac{5(r+s)+1}{10}$$

as the two neighbours of v_{r+s-1} are v_{r+s-2} and v_{r+s} of degrees 2 and 1, respectively.

This concludes the case where a pendant edge is added to a tadpole graph $T_{r,s}$.

Secondly, let us add a new non-pendant edge e to connect two existing vertices v_i and v_j of degrees d_i and d_j, respectively.

By Theorem 3.2, we have the following possibilities:

4) If a non-pendant edge e is added between v_{r+s-1} and one of the vertices v_3, v_4, $\cdots$, v_{r-1}, v_{r+2}, $\cdots$, v_{r+s-2} of degree 2, then

$$H(T_{r,s} + e) = \frac{15(r+s) - 8}{30}.$$

as the two neighbours of v_{r+s-1} are v_{r+s} and v_{r+s-2} of degrees 1 and 2 respectively and two neighbours of the vertices v_3, v_4, $\cdots$, v_{r-1}, v_{r+2}, $\cdots$, v_{r+s-2} are of degree 2.

5) If a non-pendant edge e is added between v_{r+s-1} and one of the vertices v_2, v_r or v_{r+1} of degree 2, then

$$H(T_{r,s} + e) = \frac{15(r+s) - 7}{30}.$$

as the two neighbours of v_{r+s-1} are v_{r+s} and v_{r+s-2} of degrees 1 and 2 respectively and two neighbours of the vertices v_2, v_r or v_{r+1} are of degrees 2 and 3.

6) If a non-pendant edge e is added between the vertices v_{r+s-1} and v_1 of degrees 2 and 3, respetively, then

$$H(T_{r,s} + e) = \frac{35(r+s) - 22}{70}.$$

as the two neighbours of v_{r+s-1} are v_{r+s} and v_{r+s-2} of degrees 1 and 2 respectively and three neighbours of v_1 are v_2, v_r and v_{r+1} of degree 2.

7) If a non-pendant edge e is added between two of the vertices v_3, v_4, $\cdots$, v_{r-1}, v_{r+2}, v_{r+3}, $\cdots$, v_{r+s-2} of degree 2, then

$$H(T_{r,s} + e) = \frac{15(r+s) - 6}{30}.$$

as the two neighbours of these vertices are of degree 2.

8) If a non-pendant edge e is added between two of the vertices v_2, v_r and v_{r+1} of degree 2, then

$$H(T_{r,s} + e) = \frac{15(r+s) - 4}{30}.$$

as the two neighbours of these vertices are of degrees 2 and 3.

9) If a non-pendant edge e is added between the vertices v_1 and v_{r+s} of degrees 3 and 1, respectively, then

$$H(T_{r,s} + e) = \frac{3(r+s) - 1}{6}.$$

as the neighbours of v_1 are v_r, v_2 and v_{r+1} of degree 2 and the unique neighbour of v_{r+s} is v_{r+s-1} of degree 2.

10) If a non-pendant edge e is added between v_1 of degree 3 and one of v_2, v_r and v_{r+1} of degree 2, then

$$H(T_{r,s}+e) = \frac{105(r+s)-41}{210}.$$

as the neighbours of v_1 are v_r, v_2 and v_{r+1} of degree 2 and the neighbours of v_2, v_r and v_{r+1} are of degrees 2 and 3.

11) If a non-pendant edge e is added between v_1 of degree 3 and one of the vertices $v_3, v_4, \cdots, v_{r-1}, v_{r+2}, v_{r+3}, \cdots, v_{r+s-2}$ of degree 2, then

$$H(T_{r,s}+e) = \frac{105(r+s)-52}{210}.$$

as the neighbours of v_1 are v_2, v_r and v_{r+1} of degree 2 and the neighbours of the vertices $v_3, v_4, \cdots, v_{r-1}, v_{r+2}, v_{r+3}, \cdots, v_{r+s-2}$ are of degree 2.

12) If a non-pendant edge e is added between the vertices v_{r+s} and v_{r+s-1} of degrees 1 and 2, respectively, then

$$H(T_{r,s}+e) = \frac{5(r+s)-1}{10}.$$

as the unique neighbour of v_{r+s} is v_{r+s-1} of degree 2 and two neighbours of v_{r+s-1} are v_{r+s} and v_{r+s-2} of degrees 1 and 2 respectively.

13) If a non-pendant edge e is added between v_{r+s} of degree 1 and one of $v_3, v_4, \cdots, v_{r-1}, v_{r+2}, \cdots, v_{r+s-2}$ of degree 2, then

$$H(T_{r,s}+e) = \frac{5(r+s)-1}{10}.$$

as the unique neighbour of v_{r+s} is v_{r+s-1} of degree 2 and two neighbours of the vertices $v_3, v_4, \cdots, v_{r-1}, v_{r+2}, \cdots, v_{r+s-2}$ are of degree 2.

14) If a non-pendant edge e is added between v_{r+s} of degree 1 and one of v_2, v_r and v_{r+1} of degree 2, then

$$H(T_{r,s}+e) = \frac{15(r+s)-2}{30}.$$

as the unique neighbour of v_{r+s} is v_{r+s-1} of degree 2 and two neighbours of the vertices v_2, v_r and v_{r+1} are of degrees 2 and 3.

So we proved the following result:

Corollary 4.12. *If an edge is added to a tadpole graph $T_{r,s}$, its harmonic index increases by either $\frac{1}{2}$, $\frac{1}{5}$, $\frac{1}{3}$, $\frac{3}{10}$, $\frac{7}{30}$, $\frac{1}{30}$, $\frac{1}{15}$, $\frac{1}{10}$, $\frac{1}{6}$, $\frac{2}{15}$ or decreases by $\frac{1}{15}$ or $\frac{1}{10}$ or it does not change.*

5. Summary and conclusions

In this work, the effect of adding a new edge to a connected simple graph on harmonic index is studied. This effect differs for adding a pendant edge or a non-pendant edge. Both effects are formulized and all possible differences are determined for six most frequently used graph classes P_n, C_n, S_n, K_n, $K_{r,s}$, and $T_{r,s}$. And there will be a continuation of this paper for other indices.

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