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UPPER BOUND OF FIFTH HANKEL DETERMINANT FOR A CLASS OF ANALYTIC FUNCTION

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Abstract. In this present article, we drive a bound of the fifth Hankel determinant for the class of analytic function f such that $Re\frac{f(z)}{z} > \alpha$ for some $\alpha, (0 \leq \alpha < 1)$ and $z \in \mathbf{U} = \{z; |z| < 1\}$. Moreover, the upper bounds of this problem for symmetric analytic functions class are also obtained.

1. Introduction

The goal of this specific section is to include some simple notions regarding Geometric Function Theory that will allow us to understand our main findings in a precise way. In this regard, first we start to define the most basic class $\mathcal{A}$ which represents the set of all analytic(holomorphic) functions f in region $\mathbf{U} = \{z; |z| < 1\}$ having the Taylor series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

In addition, let $\mathcal{S} \subset \mathcal{A}$ be the class of all functions which are univalent in $\mathbf{U}$.

Let $\mathcal{P}$ denote the class of analytic functions p whose real parts are positive in $\mathbf{U}$ having the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \tag{1.2}$$

In 1916, Bieberbach [7] claimed the coefficient structure for $f \in \mathcal{S}$ and it became a challenge to all the mathematician. Finally in 1985 de-Branges [10] proved it completely. In(1916-1985) this period, while tackling with this conjecture, several subfamilies of $\mathcal{S}$ connected with different image domains were defined such as Starlike, Convex, Close to Convex with nice geometric properties. These families are defined as

$$\mathcal{S}^* = \left\{ f \in \mathcal{S}, Re\left\{ \frac{zf'(z)}{f(z)} \right\} > 0, z \in \mathbf{U} \right\}$$

$$\mathcal{C} = \left\{ f \in \mathcal{S}, Re\left\{ \frac{(zf'(z))'}{f'(z)} \right\} > 0, z \in \mathbf{U} \right\}$$

$$\mathcal{K} = \left\{ f \in \mathcal{S}, Re\left\{ \frac{zf'(z)}{g(z)} \right\} > 0, g(z) \in \mathcal{S}^*, z \in \mathbf{U} \right\}$$

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Now for $\alpha \in [0,1)$, the class $\mathcal{T}(\alpha)$ is defined as

$$\mathcal{T}(\alpha) = \left\{ f \in \mathcal{A}, Re\left\{\frac{f(z)}{z}\right\} > \alpha, z \in \mathbf{U} \right\} \tag{1.3}$$

Also $\mathcal{T}(0) = \mathcal{T}$. The families $\mathcal{T}$ and $\mathcal{T}\left(\frac{1}{2}\right)$ play an important role in the theory of univalent functions although their elements are functions which are not necessarily univalent. One of the important results given by Marx [21] and Strohhäcker [32] is

$$\mathcal{C} \subset \mathcal{S}^*\left(\frac{1}{2}\right) \subset \mathcal{T}\left(\frac{1}{2}\right)$$

where $\mathcal{C}$ is a class of convex function, $\mathcal{S}^*\left(\frac{1}{2}\right)$ is class of starlike function of order $\frac{1}{2}$. The interesting fact is that the function $f(z) = \frac{z}{1-z},\ z \in \mathbf{U}$ is extremal function for many computational problems in above three classes. The class $\mathcal{T}$ plays a fundamental role in the theory of semigroups of analytic functions as a generator of one-parameter continues semigroups studied by Berkson, Porta, Shoikhet, Elin and others (see [31], [11]). For other classical results concerning the classes $\mathcal{T}$ and $\mathcal{T}\left(\frac{1}{2}\right)$ see [[20], [27]]. Kowalczyk et al. [14] proved third Hankel determinant for class $\mathcal{T}(\alpha)$.

The Hankel determinant $H_{q,n}(f)$ where $q \geq 0, n \geq 1$ for a function $f \in \mathbf{S}$ was defined by Pommerenke [24, 25] as

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \tag{1.4}$$

Computing the upper bound of $H_{q,n}$ over subfamilies of $\mathbf{A}$ is an interesting problem to study. The growth rate of $H_{q,n}$ as $n \to \infty$ has been studied by Noonan and Noor [22, 23] for different univalent functions. Sharp upper bound of $H_{2,2}(f) = a_2a_4 - a_3^2$ of order 2 were obtained by various authors. It is worth citing a few of them [9, 12–14, 18].

The estimation of $H_{3,1}(f)$ is much more difficult than the case of $H_{2,2}(f)$. Very few papers have been devoted to third Hankel determinant. The first paper on $H_{3,1}(f)$ was given in 2010 by Babalola [5] in which he obtained the upper bound of $H_{3,1}(f)$ for the families of $\mathcal{S}^*, \mathcal{C}$ and $\mathcal{R}$. Later on some other authors [[29], [30], [30], [6], [8], [28], [15]] published their work concerning $H_{3,1}(f)$ for different subfamilies of analytic and univalent functions. In 2017, Zaprawa [33] improved the results of Babalola [5] by proving

$$|H_{3,1}(f)| = \begin{cases} 1 & for \quad f \in \mathcal{S}^* \\ 0.090 & for \quad f \in \mathcal{C} \\ 0.683 & for \quad f \in \mathcal{R} \end{cases}$$

and claimed that these bounds are still not sharp. For the sharpness, he considered the subfamilies of $\mathcal{S}^*, \mathcal{C}$ and $\mathcal{R}$ consisting of function with m fold symmetry and obtained the sharp upper bounds. But in 2018, Kwon et al. [16] improved the zaprawa results for starlike function and proved $H_{3,1}(f) \leq \frac{8}{9}$. Different authors have studied the third Hankel determinant for different subfamilies of $\mathcal{S}$ but till

2018, no one succeeded. In 2018 Kowalczyk et al. [14] and Lecko et al. [17] proved the sharp bounds $H_{3,1}(f) \leq \frac{4}{135}$ and $H_{3,1}(f) \leq \frac{1}{9}$ for the sets Convex function and Starlike function of order half respectively. Research work on 2nd and 3rd Hankel determinant is still continuing for various classes and subclasses of univalent and analytic function while the research on fourth Hankel determinant has also started. The first paper on $H_{4,1}(f)$ for function with bounded turning has been obtained by Arif et al. [3] and they proved $f \in \mathcal{R}$ then $|H_{4,1}(f)| \leq 0.78050$. Here, in this paper we contribute in the fifth Hankel determinant for a class of analytic function $\mathcal{T}(\alpha)$ with the motivation from recent paper of Arif et al. [4] fifth Hankel determinant for function with bounded turning.

2. Preliminary lemmas

In order to find the sharp upper bound of $H_{(5,1)}(f)$ we need the following lemmas and results

Lemma 2.1. *If $p \in \mathbf{P}$ is of the form (1.2) , then*

$$|c_n| \leq 2 \quad for \quad n \in \mathbb{N} \tag{2.1}$$

$$|c_{n+k} - \lambda c_n c_k| < 2 \quad for \quad 0 \leq \lambda \leq 1 \tag{2.2}$$

$$|c_m c_n - c_k c_l| \leq 4 \quad for \quad m + n = k + l \tag{2.3}$$

For the inequalities in (2.1) ,(2.2) , (2.3) see [26]

Lemma 2.2. *If $p \in \mathbf{P}$ is of the form 1.2, then $2c_2 = c_1^2 + x(4 - c_1^2)$ for some x with $|x| \leq 1$ This result is due to Libera [19]*

Theorem 2.3. *If $f \in \mathcal{T}$ then*

$$|a_n| \leq 2 \tag{2.4}$$

Let $f \in \mathcal{T}$ according to definition,$\frac{f(z)}{z} = p(z)$ where $p \in \mathbf{P}$ of the form (1.2) we can easily obtain that $a_n = c_{n-1}$ by using (2.1) , we get the result.

Theorem 2.4. *If $f \in \mathcal{T}\left(\frac{1}{2}\right)$ then*

$$|a_n| \leq 1 \tag{2.5}$$

If $f \in \mathcal{T}(\frac{1}{2})$ then $\frac{f(z)}{z} = \frac{1}{2}(p(z) + 1)$ where $p \in \mathbf{P}$ of the form (1.2) we can easily find the coefficients $a_n = \frac{c_{n-1}}{2}$ Now apply (2.1) , we get our desired result.

3. Bounds of fifth Hankel determinant

First, $H_{5,1}(f)$ can be written in the form

$$H_{5,1}(f) = a_5 H_{4,2}(f) - a_4\lambda_1 + a_3\lambda_2 - a_2\lambda_3 + H_{4,3}(f) \tag{3.1}$$

where $a_i's$ are coefficients of function $f \in \mathbf{A}$ of the form (1.1) and $\lambda_1, \lambda_2, \lambda_3, H_{4,2}(f)$ and $H_{4,3}(f)$ are determinants of order 4 given by

$$H_{4,2}(f) = a_8 H_{3,2}(f) - a_7\Delta_1 + a_6\Delta_2 - a_5\Delta_3; \tag{3.2}$$

$$H_{4,3}(f) = a_9 H_{3,3}(f) - a_8\Delta_4 + a_7\Delta_5 - a_6\Delta_6; \tag{3.3}$$

$$\lambda_1 = a_2\Delta_7 - a_3\Delta_8 + a_4\Delta_9 - a_6\Delta_{10}; \tag{3.4}$$

$$\lambda_2 = a_2\Delta_{11} - a_3\Delta_{12} + a_4\Delta_{13} - a_5\Delta_{14}; \tag{3.5}$$

$$\lambda_3 = a_2\Delta_{15} - a_3\Delta_{16} + a_4\Delta_{17} - a_5\Delta_{18}; \tag{3.6}$$

where

$$H_{3,2}(f) = a_2(a_4a_6 - a_5^2) - a_3(a_3a_6 - a_4a_5) + a_4(a_3a_5 - a_4^2); \tag{3.7}$$

$$H_{3,3}(f) = a_3(a_5a_7 - a_6^2) - a_4(a_4a_7 - a_5a_6) + a_5(a_4a_6 - a_5^2); \tag{3.8}$$

$$\Delta_1 = a_2(a_4a_7 - a_5a_6) - a_3(a_3a_7 - a_5^2) + a_4(a_3a_6 - a_4a_5); \tag{3.9}$$

$$\Delta_2 = a_2(a_5a_7 - a_6^2) - a_3(a_4a_7 - a_5a_6) + a_4(a_4a_6 - a_5^2); \tag{3.10}$$

$$\Delta_3 = a_3(a_5a_7 - a_6^2) - a_4(a_4a_7 - a_5a_6) + a_5(a_4a_6 - a_5^2); \tag{3.11}$$

$$\Delta_4 = a_3(a_5a_8 - a_6a_7) - a_4(a_4a_8 - a_6^2) + a_5(a_4a_7 - a_5a_6); \tag{3.12}$$

$$\Delta_5 = a_3(a_6a_8 - a_7^2) - a_4(a_5a_8 - a_6a_7) + a_5(a_5a_7 - a_6^2); \tag{3.13}$$

$$\Delta_6 = a_4(a_6a_8 - a_7^2) - a_5(a_5a_8 - a_6a_7) + a_6(a_5a_7 - a_6^2); \tag{3.14}$$

$$\Delta_7 = a_4(a_6a_9 - a_7a_8) - a_5(a_5a_9 - a_7^2) + a_6(a_5a_8 - a_6a_7); \tag{3.15}$$

$$\Delta_8 = a_3(a_6a_9 - a_7a_8) - a_4(a_5a_9 - a_7^2) + a_5(a_5a_8 - a_6a_7); \tag{3.16}$$

$$\Delta_9 = a_3(a_5a_9 - a_8a_6) - a_4(a_4a_9 - a_6a_7) + a_5(a_4a_8 - a_5a_7); \tag{3.17}$$

$$\Delta_{10} = a_3(a_5a_7 - a_6^2) - a_4(a_4a_7 - a_5a_6) + a_5(a_4a_6 - a_5^2); \tag{3.18}$$

$$\Delta_{11} = a_4(a_7a_9 - a_8^2) - a_5(a_6a_9 - a_7a_8) + a_6(a_6a_8 - a_7^2); \tag{3.19}$$

$$\Delta_{12} = a_3(a_7a_9 - a_8^2) - a_5(a_5a_9 - a_6a_8) + a_6(a_5a_8 - a_6a_7); \tag{3.20}$$

$$\Delta_{13} = a_3(a_6a_9 - a_7a_8) - a_4(a_5a_9 - a_6a_8) + a_6(a_5a_7 - a_6^2); \tag{3.21}$$

$$\Delta_{14} = a_3(a_6a_8 - a_7^2) - a_4(a_5a_8 - a_6a_7) + a_5(a_5a_7 - a_6^2); \tag{3.22}$$

$$\Delta_{15} = a_5(a_7a_9 - a_8^2) - a_6(a_6a_9 - a_7a_8) + a_7(a_6a_8 - a_7^2); \tag{3.23}$$

$$\Delta_{16} = a_4(a_7a_9 - a_8^2) - a_6(a_5a_9 - a_6a_8) + a_7(a_5a_8 - a_6a_7); \tag{3.24}$$

$$\Delta_{17} = a_4(a_6a_9 - a_7a_8) - a_5(a_5a_9 - a_8a_6) + a_7(a_5a_7 - a_6^2); \tag{3.25}$$

and

$$\Delta_{18} = a_4(a_6a_8 - a_7^2) - a_5(a_5a_8 - a_6a_7) + a_6(a_5a_7 - a_6^2). \tag{3.26}$$

In (1.4) it can be easily checked that $H_{5,1}(f)$ is a polynomial of eight successive coefficients $a_2, a_3, a_4, a_5, a_6, a_7$ and a_8 of function f in the concerned class. These coefficients are connected with the coefficients of p from class $\mathcal{P}$.

Theorem 3.1. *If $f \in \mathcal{T}$ of the form (1.1) then*

$$|H_{4,2}(f)| \leq 192 \tag{3.27}$$

Proof Let $f \in \mathcal{T}$ of the form (1.3) , we have $\frac{f(z)}{z} = p(z)$
where $p \in \mathbf{P}$ of the form (1.2) by identifying the coefficients we can easily obtain that

$$a_n = c_{n-1} \tag{3.28}$$

using (3.28) in (3.7) , (3.9) , (3.10) and (3.11), it follows that

$$H_{3,2}(f) = c_1c_3c_5 - c_1c_4^2 - c_2^2c_5 + 2c_2c_3c_4 - c_3^3$$
$$\Delta_1 = c_1c_3c_6 - c_1c_4c_5 - c_2^2c_6 + c_2c_4^2 + c_2c_3c_5 - c_3^2c_4$$
$$\Delta_2 = c_1c_4c_6 - c_1c_5^2 - c_2c_3c_6 + c_2c_4c_5 + c_3^2c_5 - c_3c_4^2$$
$$\Delta_3 = c_2c_4c_6 - c_2c_5^2 - c_3^2c_6 + 2c_3c_4c_5 - c_4^3$$

by using (2.1) and (2.3) in above equations, we obtain

$$|H_{3,2}(f)| \leq 24, \ \ |\Delta_1| \leq 24, \ \ |\Delta_2| \leq 24, \ \ |\Delta_3| \leq 24 \tag{3.29}$$

Now by using the (3.29) along with (2.4) in (3.2) we obtain the desired result. Hence it completes the proof of theorem.

Theorem 3.2. *If $f \in \mathcal{T}$ of the form (1.1) then*

$$|H_{4,3}(f)| \leq 192 \tag{3.30}$$

Proof Let $f \in \mathcal{T}$ of the form (1.3) and using (3.28) in (3.8) , (3.12) , (3.13) and (3.14), it follows that

$$H_{3,3}(f) = c_2c_4c_6 - c_2c_5^2 - c_3^2c_6 + 2c_3c_4c_5 - c_4^3$$
$$\Delta_4 = c_2c_4c_7 - c_2c_5c_6 - c_3^2c_7 + c_3c_5^2 + c_3c_4c_6 - c_4^2c_5$$
$$\Delta_5 = c_2c_5c_7 - c_2c_6^2 - c_3c_4c_7 + c_3c_5c_6 + c_4^2c_6 - c_4c_5^2$$
$$\Delta_6 = c_3c_5c_7 - c_3c_6^2 - c_4^2c_7 + 2c_4c_5c_6 - c_5^3$$

by using (2.1) and (2.3) in above equations, we obtain

$$|H_{3,3}(f)| \leq 24, \ \ |\Delta_4| \leq 24, |\Delta_5| \leq 24, \ \ |\Delta_6| \leq 24. \tag{3.31}$$

Now by using the (3.31) along with (2.4) in (3.3) we obtain the desired result. Hence it completes the proof of theorem.

Theorem 3.3. *If $f \in \mathcal{T}$ of the form (1.1) then*

$$|\lambda_1| \leq 192 \tag{3.32}$$

Proof Let $f \in \mathcal{T}$ of the form (1.3) and using (3.28) in (3.15) , (3.16) , (3.17) and (3.18), it follows that

$$\Delta_7 = c_3c_5c_8 + c_4c_6^2 - c_4^2c_8 - c_3c_6c_7 + c_4c_5c_7 - c_5^2c_6$$
$$\Delta_8 = c_2c_5c_8 - c_2c_7c_6 + c_4^2c_7 + c_3c_6^2 - c_3c_4c_8 - c_4c_5c_6$$
$$\Delta_9 = c_2c_4c_8 - c_2c_7c_5 - c_3^2c_8 + c_3c_5c_6 + c_4c_3c_7 - c_4^2c_6$$
$$\Delta_{10} = c_2c_4c_6 - c_2c_5^2 - c_3^2c_6 + 2c_3c_4c_5 - c_4^3$$

by using (2.1) and (2.3) in above equations, we obtain

$$|\Delta_7| \le 24, \ \ |\Delta_8| \le 24, \ \ |\Delta_9| \le 24, \ \ \Delta_{10}| \le 24 \tag{3.33}$$

Now by using the (3.33) along with (2.4) in (3.4) , we obtain the desired result. Hence it completes the proof of theorem.

Theorem 3.4. *If $f \in \mathcal{T}$ of the form (1.1) then*

$$|\lambda_2| \le 192 \tag{3.34}$$

Proof Let $f \in \mathcal{T}$ of the form (1.3) and using (3.28) in (3.19) , (3.20) , (3.21) and (3.22), it follows that

$$\Delta_{11} = c_3c_6c_8 - c_3c_7^2 - c_4c_5c_8 + c_4c_6c_7 + c_5^2c_7 - c_5c_6^2$$

$$\Delta_{12} = c_2c_6c_8 - c_2c_7^2 - c_4^2c_8 + 2c_4c_5c_7 - c_5^2c_6`$$

$$\Delta_{13} = c_2c_5c_8 - c_2c_7c_6 - c_3c_4c_8 + c_3c_5c_7 + c_4c_5c_6 - c_5^3$$

$$\Delta_{14} = c_2c_5c_7 - c_2c_6^2 - c_3c_4c_7 + c_3c_5c_6 + c_4^2c_6 - c_4c_5^2$$

by using (2.1) and (2.3) in above equations, we obtain

$$|\Delta_{11}| \le 24, \ \ |\Delta_{12}| \le 24, \ \ |\Delta_{13}| \le 24, \ \ |\Delta_{14}| \le 24. \tag{3.35}$$

Now by using the(3.35) along with (2.4) in (3.5) , we obtain the desired result. Hence it completes the proof of theorem.

Theorem 3.5. *If $f \in \mathcal{T}$ of the form (1.1) then*

$$|\lambda_3| \le 192 \tag{3.36}$$

Proof Let $f \in \mathcal{T}$ of the form (1.3) and using (3.28) in (3.23) , (3.24), (3.25) and (3.26) , it follows that

$$\Delta_{15} = c_4c_6c_8 - c_7^2c_4 - c_5^2c_8 + 2c_5c_6c_7 - c_6^3$$

$$\Delta_{16} = c_3c_6c_8 - c_3c_7^2 - c_4c_5c_8 + c_5^2c_7 + c_4c_6c_7 - c_5c_6^2$$

$$\Delta_{17} = c_3c_5c_8 - c_3c_7c_6 - c_4^2c_8 + c_4c_5c_7 + c_4c_6^2 - c_5^2c_6$$

$$\Delta_{18} = c_3c_5c_7 - c_3c_6^2 - c_4^2c_7 + 2c_4c_5c_6 - c_5^3$$

by using (2.1) and (2.3) in above equations, we obtain

$$|\Delta_{15}| \le 24, \ \ |\Delta_{16}| \le 24, \ \ |\Delta_{17}| \le 24, \ \ |\Delta_{18}| \le 24. \tag{3.37}$$

Now by using the (3.37) along with (2.4) in (3.6) , we obtain the desired result. Hence it completes the proof of theorem.

Theorem 3.6. *If $f \in \mathcal{T}$ of the form (1.1) then*

$$|H_{5,1}(f)| \le 1728. \tag{3.38}$$

By putting the bounds found in theorem (3.1) , (3.2) , (3.3) , (3.4) and (3.5) along with (2.4) in (3.1), we get our desired result.

Theorem 3.7. *If $f \in \mathcal{T}\left(\frac{1}{2}\right)$ then*

$$|H_{4,2}(f)| \le 12 \tag{3.39}$$

Proof Let $f \in \mathcal{T}\left(\frac{1}{2}\right)$ we have

$$\frac{f(z)}{z} = \frac{1}{2}(p(z)+1) \tag{3.40}$$

where $p \in \mathbf{P}$ of the form (1.2) by identifying the coefficients we can easily obtain that

$$a_n = \frac{c_{n-1}}{2} \tag{3.41}$$

using (3.41) in (3.7) , (3.9 , (3.10) and in (3.11) , it can be easily assemble as

$$H_{3,2}(f) = \frac{1}{8}c_1c_3c_5 - \frac{1}{8}c_1c_4^2 - \frac{1}{8}c_2^2c_5 + \frac{2}{8}c_2c_3c_4 - \frac{1}{8}c_3^3$$

$$\Delta_1 = \frac{1}{8}c_1c_3c_6 - \frac{1}{8}c_1c_4c_5 - \frac{1}{8}c_2^2c_6 + \frac{1}{8}c_2c_4^2 + \frac{1}{8}c_2c_3c_5 - \frac{1}{8}c_3^2c_4$$

$$\Delta_2 = \frac{1}{8}c_1c_4c_6 - \frac{1}{8}c_1c_5^2 - \frac{1}{8}c_2c_3c_6 + \frac{1}{8}c_2c_4c_5 + \frac{1}{8}c_3^2c_5 - \frac{1}{8}c_3c_4^2$$

$$\Delta_3 = \frac{1}{8}c_2c_4c_6 - \frac{1}{8}c_2c_5^2 - \frac{1}{8}c_3^2c_6 + \frac{2}{8}c_3c_4c_5 - \frac{1}{8}c_4^3$$

by using (2.1) and (2.3) in above equations, we obtain

$$|H_{3,2}(f)| \leq 3, \ \ |\Delta_1| \leq 3, \ \ |\Delta_2| \leq 3, \ \ |\Delta_3| \leq 3. \tag{3.42}$$

Now by using the (3.42) along with (2.5) in (3.2) , we obtain the desired result. Hence it completes the proof of theorem.

Theorem 3.8. *If $f \in \mathcal{T}\left(\frac{1}{2}\right)$ of the form (1.1) then*

$$|H_{4,3}(f)| \leq 12 \tag{3.43}$$

Proof Let $f \in \mathcal{T}\left(\frac{1}{2}\right)$ of the form (1.3) and using (3.41) in (3.8) , (3.12) , (3.13) and (3.14) , it follows that

$$H_{3,3}(f) = \frac{1}{8}\left(c_2c_4c_6 - c_2c_5^2 - c_3^2c_6 + 2c_3c_4c_5 - c_4^3\right)$$

$$\Delta_4 = \frac{1}{8}\left(c_2c_4c_7 - c_2c_5c_6 - c_3^2c_7 + c_3c_5^2 + c_3c_4c_6 - c_4^2c_5\right)$$

$$\Delta_5 = \frac{1}{8}\left(c_2c_5c_7 - c_2c_6^2 - c_3c_4c_7 + c_3c_5c_6 + c_4^2c_6 - c_4c_5^2\right)$$

$$\Delta_6 = \frac{1}{8}\left(c_3c_5c_7 - c_3c_6^2 - c_4^2c_7 + 2c_4c_5c_6 - c_5^3\right)$$

by using (2.1) and (2.3) in above equations, we obtain

$$|H_{3,3}(f)| \leq 3, \ \ |\Delta_4| \leq 3, \ \ |\Delta_5| \leq 3, \ \ |\Delta_6| \leq 3 \tag{3.44}$$

Now by using (3.44) along with (2.5) in (3.3) , we obtain the desired result. Hence it completes the proof of theorem.

Theorem 3.9. *If $f \in \mathcal{T}\left(\frac{1}{2}\right)$ of the form (1.1) then*

$$|\lambda_1| \leq 12 \tag{3.45}$$

Proof Let $f \in \mathcal{T}\left(\frac{1}{2}\right)$ of the form (1.3) and using (3.41) in (3.15) , (3.16) , (3.17) and (3.18), it follows that

$$\Delta_7 = \frac{1}{8}\left(c_3c_5c_8 + c_4c_6^2 - c_4^2c_8 - c_3c_6c_7 + c_4c_5c_7 - c_5^2c_6\right)$$

$$\Delta_8 = \frac{1}{8}\left(c_2c_5c_8 - c_2c_7c_6 + c_4^2c_7 + c_3c_6^2 - c_3c_4c_8 - c_4c_5c_6\right)$$

$$\Delta_9 = \frac{1}{8}\left(c_2c_4c_8 - c_2c_7c_5 - c_3^2c_8 + c_3c_5c_6 + c_4c_3c_7 - c_4^2c_6\right)$$

$$\Delta_{10} = \frac{1}{8}\left(c_2c_4c_6 - c_2c_5^2 - c_3^2c_6 + 2c_3c_4c_5 - c_4^3\right)$$

by using (2.1) and (2.3) in above equations, we obtain

$$|\Delta_7| \leq 3, \quad |\Delta_8| \leq 3, \quad |\Delta_9| \leq 3, \quad |\Delta_{10}| \leq 3. \tag{3.46}$$

Now by using the (3.46) , along with (2.5) in (3.4) , we obtain the desired result. Hence it completes the proof of theorem.

Theorem 3.10. *If $f \in \mathcal{T}\left(\frac{1}{2}\right)$ of the form (1.1) then*

$$|\lambda_2| \leq 12 \tag{3.47}$$

Proof Let $f \in \mathcal{T}\left(\frac{1}{2}\right)$ of the form (1.3) and using (3.41) in (3.19) , (3.20) , (3.21) and (3.22) , it follows that

$$\Delta_{11} = \frac{1}{8}\left(c_3c_6c_8 - c_3c_7^2 - c_4c_5c_8 + c_4c_6c_7 + c_5^2c_7 - c_5c_6^2\right)$$

$$\Delta_{12} = \frac{1}{8}\left(c_2c_6c_8 - c_2c_7^2 - c_4^2c_8 + 2c_4c_5c_7 - c_5^2c_6\right)$$

$$\Delta_{13} = \frac{1}{8}\left(c_2c_5c_8 - c_2c_7c_6 - c_3c_4c_8 + c_3c_5c_7 + c_4c_5c_6 - c_5^3\right)$$

$$\Delta_{14} = \frac{1}{8}\left(c_2c_5c_7 - c_2c_6^2 - c_3c_4c_7 + c_3c_5c_6 + c_4^2c_6 - c_4c_5^2\right)$$

by using (2.1) and (2.3) in above equations, we obtain

$$|\Delta_{11}| \leq 3, \quad |\Delta_{12}| \leq 3, \quad |\Delta_{13}| \leq 3, \quad |\Delta_{14}| \leq 3. \tag{3.48}$$

Now by using the (3.48) along with (2.5) in (3.5) , we obtain the desired result. Hence it completes the proof of theorem.

Theorem 3.11. *If $f \in \mathcal{T}\left(\frac{1}{2}\right)$ of the form (1.1) then*

$$|\lambda_3| \leq 12 \tag{3.49}$$

Proof Let $f \in \mathcal{T}\left(\frac{1}{2}\right)$ of the form (1.3) and using (3.41) in (3.23) , (3.24) , (3.25) and (3.26) , it follows that

$$\Delta_{15} = \frac{1}{8}\left(c_4c_6c_8 - c_7^2c_4 - c_5^2c_8 + 2c_5c_6c_7 - c_6^3\right)$$

$$\Delta_{16} = \frac{1}{8}\left(c_3c_6c_8 - c_3c_7^2 - c_4c_5c_8 + c_5^2c_7 + c_4c_6c_7 - c_5c_6^2\right)$$

$$\Delta_{17} = \frac{1}{8}\left(c_3c_5c_8 - c_3c_7c_6 - c_4^2c_8 + c_4c_5c_7 + c_4c_6^2 - c_5^2c_6\right)$$

$$\Delta_{18} = \frac{1}{8}\left(c_3c_5c_7 - c_3c_6^2 - c_4^2c_7 + 2c_4c_5c_6 - c_5^3\right)$$

by using (2.1) and (2.3) in above equations, we obtain

$$|\Delta_{15}| \leq 3, \ \ |\Delta_{16}| \leq 3, \ \ |\Delta_{17}| \leq 3, \ \ |\Delta_{18}| \leq 3. \tag{3.50}$$

Now by using the (3.50) along with (2.5) in (3.6) , we obtain the desired result. Hence it completes the proof of theorem.

Theorem 3.12. *If* $f \in \mathcal{T}\left(\frac{1}{2}\right)$ *of the form (1.1) then*

$$|H_{5,1}(f)| \leq 60 \tag{3.51}$$

By putting the bounds found in theorem (3.7), (3.8) , (3.9) , (3.10) and (3.11) along with (2.5) in (3.1) , we get our desired result.

4. Bounds of $H_{5,1}(f)$ for twofold and fourfold Symmetric Functions

A function f is said to be n-fold symmetric if $f(\varepsilon z) = \varepsilon f(z)$ holds for all $z \in \mathbf{U}$, where $\varepsilon = \exp(\frac{2\Pi\iota}{n})$ means the principal n-th root of 1. The set of all n-fold symmetric functions belonging to $\mathcal{S}$ is denoted by $\mathcal{S}^{(n)}$, i.e. n fold univalent function having the following expansion

$$f(z) = z + \sum_{k=1}^{\infty} a_{nk+1}z^{nk+1}, \ \ z \in \mathbf{U} \tag{4.1}$$

An analytic function f of the form (4.1) belongs to the family $\mathcal{T}^{(n)}$ if and only if

$$\frac{f(z)}{z} = p(z) \quad with \quad p \in \mathcal{P}^{(n)}$$

where

$$\mathcal{P}^{(n)} = \{p(z) : p(z) = 1 + \sum_{k=1}^{\infty} c_{nk}z^{nk}\} \tag{4.2}$$

Observe that if $f \in \mathcal{S}^{(4)}$ then $f(z) = z + a_5z^5 + a_9z^9 + ...$ and consequently $H_{5,1}(f) = a_5^3(a_5^2 - a_9)$ and if $f \in \mathcal{S}^{(2)}$ consists all function of $\mathcal{S}$ which are odd and of the form $f(z) = z + a_3z^3 + a_5z^5 + ...$ so $H_{5,1}(f) = (a_5^2 - a_3a_7)(a_5^3 + a_9a_3^2 + a_7^2 - 2a_3a_5a_7 - a_5a_9)$.

Theorem 4.1. *In four fold Symmetric function*

1 $f \in \mathcal{T}^{(4)}$ *then* $|H_{5,1}(f)| \leq 16$.
2 $f \in (\mathcal{T}(\frac{1}{2}))^{(4)}$ *then* $|H_{5,1}(f)| \leq \frac{3\sqrt{12}}{25\sqrt{5}}$.

Proof 1: Let $f \in \mathcal{T}^{(4)}$ then there exist a function $p \in \mathcal{P}^{(n)}$ such that

$$\frac{f(z)}{z} = p(z)$$

using the series (4.1) and (4.2) for $n = 4$, we can write

$$a_5 = c_4, \qquad a_9 = c_8 \tag{4.3}$$

now $H_{5,1}(f) = a_5^3(a_5^2 - a_9)$.

Therefore $H_{5,1}(f) = c_4^3(c_4^2 - c_8)$

as $max\{c_4^3(c_4^2 - c_8); p \in \mathcal{P}^{(n)}\}$ which is same as $max\{c_1^3(c_1^2 - c_2); p \in \mathbf{P}\}$

because of the obvious equivalence $p \in \mathcal{P}^{(4)} \Leftrightarrow q \in \mathcal{P}$ provided that $p(z) = q(z^4)$ By Lemma (2.2)

$$K = c_1^3(c_1^2 - c_2) = \frac{1}{2}c_1^3(c_1^2 - x(4 - c_1^2))$$

where $|x| \leq 1$. Since K is invariant under rotation, we can assume that $c = c_1$ is a non-negative real number; so $c \in [0, 2]$. Hence

$$|K| = \frac{1}{2}c^3|c^2 - (4 - c^2)x| \leq 2|c|^3$$

then $max\{2c^3; c \in (0, 2)\} = 16$ Hence we get our desired result.

2. Let $f \in \mathcal{T}(\frac{1}{2})^{(4)}$ then there exist a function $p \in \mathcal{P}^{(n)}$ such that

$$\frac{f(z)}{z} = \frac{1}{2}(p(z) + 1)$$

using the series (4.1) and (4.2) for $n = 4$, we can write

$$2a_5 = c_4, \quad 2a_9 = c_8 \tag{4.4}$$

now $H_{5,1}(f) = a_5^3(a_5^2 - a_9)$.

Therefore $H_{5,1}(f) = \frac{1}{32}c_4^3(c_4^2 - 2c_8)$ as $max\{\frac{1}{32}c_4^3(c_4^2 - 2c_8); p \in \mathcal{P}^{(n)}\}$ which is same as $max\{\frac{1}{32}c_1^3(c_1^2 - 2c_2); p \in \mathcal{P}\}$

because of the obvious equivalence $p \in \mathcal{P}^{(4)} \Leftrightarrow q \in \mathcal{P}$ provided that $p(z) = q(z^4)$ By Lemma (2.2)

$$K = \frac{1}{32}c_1^3(c_1^2 - 2c_2) = \frac{1}{32}c_1^3(-x(4 - c_1^2))$$

where $|x| \leq 1$. Since K is invariant under rotation, we can assume that $c = c_1$ is a non-negative real number; so $c \in [0, 2]$. Hence

$$|K| = \frac{1}{32}c^3|-(4 - c^2)x| \leq \frac{1}{32}c^3(4 - c^2)$$

then it is easy to show that $max\{\frac{1}{32}c^3(4 - c^2); c \in (0, 2)\} = \frac{3\sqrt{12}}{25\sqrt{5}}$. Hence we get our desired result.

Theorem 4.2. *For Two fold symmetry*

1 $f \in \mathcal{T}^{(2)}$ *then* $|H_{5,1}(f)| \leq 80$.
2 $f \in (\mathcal{T}(\frac{1}{2}))^{(2)}$ *then* $|H_{5,1}(f)| \leq 3$.

1 Let $f \in \mathcal{T}^{(2)}$ and it is clear that $a_2 = a_4 = a_6 = a_8 = 0$, Consequently

$$H_{5,1}(f) = (a_5^2 - a_3a_7)(a_5^3 + a_9a_3^2 + a_7^2 - 2a_3a_5a_7 - a_5a_9)$$

Since $f \in \mathcal{T}^{(2)}$ there exists a function $p \in \mathcal{P}^{(2)}$ such that $\frac{f(z)}{z} = p(z)$ by using (4.1) and (4.2) for $n = 2$, we get

$$1 + a_3z^2 + a_5z^4 + a_7z^6 + ... = 1 + c_2z^2 + c_4z^4 + c_6z^6 + ...$$

$$\therefore a_3 = c_2, \quad a_5 = c_4, \quad a_7 = c_6, \quad a_9 = c_8$$

$$|H_{5,1}(f)| = |(c_4^2 - c_2c_6)((c_4c_8 - c_6^2) + c_2(c_2c_8 - c_4c_6) + c_4(c_4^2 - c_2c_6))|$$

By using triangular inequality and Lemma (2.3) we get our desired result.

2 For $f \in (\mathcal{T}(\frac{1}{2}))^{(2)}$ there exists a function $p \in \mathcal{P}^{(2)}$ such that $\frac{2f(z)-z}{z} = p(z)$ by using (4.1) and (4.2) for $n = 2$, we get we can easily assemble the estimates

$$\therefore 2a_3 = c_2, \quad 2a_5 = c_4, \quad 2a_7 = c_6, \quad 2a_9 = c_8$$

$$|H_{5,1}(f)| = |\frac{1}{4}(c_4^2 - c_2c_6)(\frac{1}{4}(c_4c_8 - c_6^2) + \frac{1}{8}c_2(c_2c_8 - c_4c_6) + \frac{1}{8}c_4(c_4^2 - c_2c_6))|$$

By using triangular inequality Lemma (2.3) we get our desired result.

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