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APPELL HYPERGEOMETRIC FUNCTION AND ITS APPLICATION TO THE POTENTIAL THEORY FOR A GENERALIZED BI-AXIALLY SYMMETRIC ELLIPTIC EQUATION

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Abstract. Fundamental solutions of the generalized bi-axially symmetric elliptic equation are expressed in terms of the well-known Appell hypergeometric function in two variables, the properties of which are required to study boundary value problems for the above equation. In this paper, by using some properties of the Appell hypergeometric function F_2, we prove limiting theorems and derive integral equations concerning a denseness of the double- and simple-layer potentials. We apply the results of the constructed potential theory to the study of the Holmgren problem for the generalized bi-axially symmetric elliptic equation in the domain bounded in the first quarter of the plane.

1. Introduction

The great success of the theory of hypergeometric series in one variable has stimulated the development of a corresponding theory in two and more variables. Appell has defined, in 1880, four series, F_1 to F_4 which are all analogous to Gauss' $F(a,b;c;z)$ (cf. equations (2.1) and (2.4) for definition of F_2 and F, respectively, *infra*). A great interest in the theory of multiple hypergeometric functions is motivated essentially by the fact that the solutions of many applied problems involving (for example) partial differential equations are obtainable with the help of such hypergeometric functions (see, for details, [24, p.47 et seq. Section 1.7]; see also the works [18, 19] and the references cited therein). For instance, the energy absorbed by some nonferromagnetic conductor sphere included in an internal magnetic field can be calculated with the help of such functions [15]. Hypergeometric functions of several variables are used in physical and quantum chemical applications as well [17, 18] . Especially, many problems in gas dynamics lead to solutions of degenerate second-order partial differential equations which are then solvable in terms of multiple hypergeometric functions. Among examples, we can cite the problem of adiabatic flat-parallel gas flow without whirlwind, the flow problem of supersonic current from vessel with flat walls, and a number of other problems connected with gas flow [8].

In the last century, to construct a potential theory for equation

$$u_{xx} + u_{yy} + \frac{2\alpha}{x} u_x = 0,\ 0 < 2\alpha < 1,\ \ x > 0$$

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in the domain bounded in the half-plane $x > 0$, the properties of the Gaussian hypergeometric function $F(a, b; c; z)$ were used [8, 9, 13, 20].

An exposition of the results on the potential theory for this two-dimensional singular elliptic equation together with references to the original literature are to be found in the monograph by Smirnov [21], which is the standard work on the subject. This work also contains an extensive bibliography of all relevant papers up to 1966; the list of references given in the present work is largely supplementary to Smirnov's bibliography. Various interesting problems associated with the two-dimensional singular elliptic equations were studied by many authors (see [1, 2, 10, 11, 14, 16, 25]).

The recent work [23] is devoted to the construction of a potential theory for a multidimensional elliptic equation with one singular coefficient

$$\sum_{k=1}^{m} u_{x_k x_k} + \frac{2\alpha}{x_1} u_{x_1} = 0, \ 0 < 2\alpha < 1, \ \ m \geq 2$$

in the domain bounded in the half-space $x_1 > 0$ and with the help of this theory the solution of the Holmgren problem is obtained in a convenient form for further research.

Relatively few works are devoted to the potential theory for an elliptic equation with two and more singular coefficients. In the works [3, 22] the authors studied only some properties of the double-layer potential for the generalized bi-axially symmetric elliptic equation

$$E(u) \equiv u_{xx} + u_{yy} + \frac{2\alpha}{x} u_x + \frac{2\beta}{y} u_y = 0, \ 0 < 2\alpha, \ 2\beta < 1. \tag{1.1}$$

In the present work, using some properties of Appell hypergeometric function F_2, we shall give the potential theory for equation (1.1). We also apply this theory to the finding of the regular solution of the Holmgren problem for equation (1.1) in the domain, which is bounded in the first quarter $R_2^{2+} := \{(x, y) : x > 0, \ y > 0\}$ of the xOy-plane.

2. Appell hypergeometric function F_2

Below we give some formulae for Appell hypergeometric function in two variables, which will be used in the next sections.

The Appell hypergeometric function in two variables has a form [7, eq.5.7(7)]

$$\begin{aligned} F_2\left(a; b_1, b_2; c_1, c_2; x, y\right) &= F_2 \left[\begin{matrix} a, b_1, b_2; \\ c_1, c_2; \end{matrix} \ x, y \right] \\ &= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b_1)_m (b_2)_n}{m! n! (c_1)_m (c_2)_n} x^m y^n \\ &\quad [c_1, c_2 \neq 0, -1, -2, ...; \ |x| + |y| < 1] . \end{aligned} \tag{2.1}$$

Here $(\lambda)_k$ is the Pochhammer's symbol for which an equality $(\lambda)_{k+l} = (\lambda)_k (a + k)_l$ is true [7, eq.1.21(5)].

We give some elementary relations for F_2 necessary in this study:

$$\begin{aligned}\frac{\partial^{m+n}}{\partial x^m \partial y^n} F_2\left(a; b_1, b_2; c_1, c_2; x, y\right) &= \frac{(a)_{m+n}(b_1)_m(b_2)_n}{m!n!(c_1)_m(c_2)_n} \\ &\times F_2\left(a+m+n; b_1+m, b_2+n; c_1+m, c_2+n; x, y\right),\end{aligned}$$

$$\begin{aligned}&\frac{b_1}{c_1} x F_2\left(a+1; b_1+1, b_2; c_1+1, c_2; x, y\right) \\ &\qquad + \frac{b_2}{c_2} y F_2\left(a+1; b_1, b_2+1; c_1, c_2+1; x, y\right) \\ &= F_2\left(a+1; b_1, b_2; c_1, c_2; x, y\right) - F_2\left(a; b_1, b_2; c_1, c_2; x, y\right),\end{aligned}$$

$$\begin{aligned}&F_2(a, b_1, b_2; c_1, c_2; x, y) = (1-x-y)^{-a} \\ &\times F_2\left(a, c_1-b_1, c_2-b_2; c_1, c_2; \frac{x}{x+y-1}, \frac{y}{x+y-1}\right).\end{aligned} \tag{2.2}$$

For a given Appell hypergeometric function F_2, it is useful to find a decomposition formula which would express the double hypergeometric function in terms of products of several simpler hypergeometric functions involving fewer variables. The following expansion formula [5]

$$\begin{aligned}F_2\left(a; b_1, b_2; c_1, c_2; x, y\right) &= \sum_{k=0}^{\infty} \frac{(a)_k(b_1)_k(b_2)_k}{k!(c_1)_k(c_2)_k} x^k y^k \\ &\times F\left(a+k, b_1+k; c_1+k; x\right) F\left(a+k, b_2+k; c_2+k; y\right).\end{aligned} \tag{2.3}$$

is valid. Here $F(a, b; c; z)$ is a famous Gaussian hypergeometric function [7, eq.2.1(2)] :

$$F(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{k!(c)_k} z^k, \;\; c \neq 0, -1, -2, ...; |z| < 1. \tag{2.4}$$

Note that each point of the line $x + y = 1 (x > 0, \; y > 0)$ can be a point of the logarithmic singularity of the function F_2.

Lemma 2.1 ([6]). *If x and y are positive and $\alpha > 0$, $\beta > 0$, then*

$$F_2\left(\alpha+\beta, \alpha, \beta; 2\alpha, 2\beta; x, y\right) \sim -\frac{\Gamma(2\alpha)\Gamma(2\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta)} \frac{\ln(1-x-y)}{x^{\alpha} y^{\beta}} \tag{2.5}$$

as $x + y \to 1 - 0$.

Let $c_1 > b_1$, $c_2 > b_2$ and $a + b_1 + b_2 = c_1 + c_2$. If $x > 0$ and $y > 0$, then

$$F_2\left(a, b_1, b_2; c_1, c_2; x, y\right) \sim -\frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(a)\Gamma(b_1)\Gamma(b_2)} \frac{\ln(1-x-y)}{x^{c_1-b_1} y^{c_2-b_2}} \tag{2.6}$$

as $x + y \to 1 - 0$.

If $c_1 + c_2 < a + b_1 + b_2$, then

$$\begin{aligned}F_2\left(a, b_1, b_2; c_1, c_2; x, y\right) &\sim \frac{\Gamma(c_1)\Gamma(c_2)\Gamma(a+b_1+b_2-c_1-c_2)}{\Gamma(a)\Gamma(b_1)\Gamma(b_2)} \\ &\times x^{b_1-c_1} y^{b_2-c_2} (1-x-y)^{c_1+c_2-a-b_1-b_2}.\end{aligned} \tag{2.7}$$

In addition, the fundamental solutions of the equation (1.1) are expressed in terms of the Appell hypergeometric function F_2, one of which has the form [6, 12]:

$$q\left(x, y; \xi, \eta\right) = \kappa r^{-2\alpha-2\beta} F_2\left(\alpha+\beta, \alpha, \beta; 2\alpha, 2\beta; \sigma_1, \sigma_2\right), \tag{2.8}$$

where

$$\sigma_1 = 1 - \frac{r_1^2}{r^2}, \;\; \sigma_2 = 1 - \frac{r_2^2}{r^2}; \;\; r^2 = (x-\xi)^2 + (y-\eta)^2,$$
$$r_1^2 = (x+\xi)^2 + (y-\eta)^2, \; r_2^2 = (x-\xi)^2 + (y+\eta)^2,$$
$$\kappa = \frac{2^{2\alpha+2\beta}}{4\pi} \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta)}{\Gamma(2\alpha)\Gamma(2\beta)}.$$

The function $q\left(x, y; \xi, \eta\right)$ satisfies the equation (1.1), and by virtue of the formula (2.5), it has a logarithmic singularity at $r \to 0$ $(x > 0, \, y > 0)$ and, therefore, the function $q\left(x, y; \xi, \eta\right)$ is a fundamental solution to the equation (1.1).

The fundamental solution given by (2.8) possesses the following potentially useful property:

$$\left. \left(x^{2\alpha} \frac{\partial q\left(x, y; \xi, \eta\right)}{\partial x} \right) \right|_{x=0} = \left. \left(y^{2\beta} \frac{\partial q\left(x, y; \xi, \eta\right)}{\partial y} \right) \right|_{y=0} = 0. \tag{2.9}$$

3. Green's formula

We consider the following identity:

$$\begin{aligned} & x^{2\alpha} y^{2\beta} \left[uE(v) - vE(u) \right] \\ & = y^{2\beta} \frac{\partial}{\partial x} \left[x^{2\alpha} \left(u \frac{\partial v}{\partial x} - v \frac{\partial u}{\partial x} \right) \right] + x^{2\alpha} \frac{\partial}{\partial y} \left[y^{2\beta} \left(u \frac{\partial v}{\partial y} - v \frac{\partial u}{\partial y} \right) \right]. \end{aligned} \tag{3.1}$$

Integrating both sides of this identity in a domain D, which is located and bounded in the quarter-plane $x > 0$, $y > 0$, and using the Gauss-Ostrogradsky formula, we obtain

$$\begin{aligned} & \int\int_D x^{2\alpha} y^{2\beta} \left[uE(v) - vE(u) \right] dxdy \\ & \quad = \int_\gamma x^{2\alpha} y^{2\beta} \left[-\left(u \frac{\partial v}{\partial y} - v \frac{\partial u}{\partial y} \right) dx + \left(u \frac{\partial v}{\partial x} - v \frac{\partial u}{\partial x} \right) dy \right], \end{aligned} \tag{3.2}$$

where γ is a contour of D.

The Green's formula (3.2) is derived under the following assumptions: (a) The functions $u(x, y)$ and $v(x, y)$, and their first-order derivatives, are continuous in the closed domain $\overline{D}$; (b) The second-order partial derivatives are continuous inside the domain D.

The integrals over D, consisting of $E(u)$ and $E(v)$, have a meaning. If $E(u)$ and $E(v)$ are not continuous up to S, then they are improper integrals obtained as limits on any sequence of domains D_n contained inside D when these domains D_n tend to D, so that any point in this D_n will be inside of D, starting with some number n.

If u and v are solutions of equation (1.1), then we find from formula (3.2) that

$$\int_\gamma \left(u A_n^{\alpha,\beta}[v] - v A_n^{\alpha,\beta}[u] \right) ds = 0, \tag{3.3}$$

where $A_n^{\alpha,\beta}[\ ]$ is the conormal derivative with respect to (x, y):

$$A_n^{\alpha,\beta}[\ ] \equiv x^{2\alpha}y^{2\beta}\left(\frac{dy}{ds}\frac{\partial}{\partial x} - \frac{dx}{ds}\frac{\partial}{\partial y}\right).$$

Here $\dfrac{dy}{ds} = \cos(n, x)$, $\dfrac{dx}{ds} = -\cos(n, y)$, n is the outer normal to the curve γ.

Assuming that $v \equiv 1$ in (3.2) and replacing u by u^2, we obtain

$$\int_D x^{2\alpha}y^{2\beta}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]dxdy = \int_\gamma uA_n^{\alpha,\beta}[u]ds, \tag{3.4}$$

where $u\,(x, y)$ is the solution of equation (1.1).

The special case of (3.3) when $v \equiv 1$ reduces to the following form:

$$\int_\gamma A_n^{\alpha,\beta}[u]ds = 0. \tag{3.5}$$

We note from (3.5) that the integral of the conormal derivative of the solution of equation (1.1) along the boundary γ of the domain is equal to zero.

4. A double-layer potential

Let D be a domain bounded by two segments $[0, a]$ of the axes x and y, and a curve Γ with the ends at the points $A(a, 0)$ and $B(0, a)$ lying in the quarter-plane $x > 0$, $y > 0$.

Let the parametric equation of the curve Γ be $x = x(s)$, $y = y(s)$, where s is the length of the arc measured from the point A. With respect to the curve Γ, we will assume that:

(i) the functions $x(s)$ and $y(s)$ have the continuous derivatives $x'(s)$ and $y'(s)$ on the segment $[0, l]$, which do not vanish simultaneously; the derivatives $x''(s)$ and $y''(s)$ satisfy the Holder condition on $[0, l]$, where l is the length of the curve Γ;

(ii) in a neighborhoods of the points A and B on the curve Γ the following conditions are satisfied

$$\left|\frac{dx}{ds}\right| \le C_1 y(s), \quad \left|\frac{dy}{ds}\right| \le C_2 x(s), \tag{4.1}$$

respectively.

The coordinates of a variable point on the curve Γ will be denoted by (ξ, η).

We now consider the following integral:

$$w(x, y) = \int_0^l \mu\,(s)\,A_\nu^{\alpha,\beta}\left[q\,(\xi, \eta; x, y)\right]ds, \tag{4.2}$$

where the density $\mu(s) \in C\left(\overline{\Gamma}\right)$, $q\,(\xi, \eta; x, y)$ is given by (2.8), ν is outer normal to the curve Γ and $A_\nu^{\alpha,\beta}[\ ]$ is the conormal derivative with respect to (ξ, η):

$$A_\nu^{\alpha,\beta}[\ ] = \xi^{2\alpha}\eta^{2\beta}\left[cos(\nu, \xi)\cdot\frac{\partial[\ ]}{\partial\xi} + cos(\nu, \eta)\cdot\frac{\partial[\ ]}{\partial\eta}\right].$$

We call the integral (4.2) *a double-layer potential with density* $\mu\,(s)$. When $\mu(s) = 1$, we denote the double-layer potential (4.2) by $w_1(x, y)$.

We now investigate some properties of the double-layer potential $w_1\,(x, y)$.

Lemma 4.1. *The following formula holds true:*

$$w_1(x,y)=\begin{cases} -1, & (x,y)\in \overline{D}\setminus\left\{\overline{\Gamma}\cup\{O\}\right\}\\ -\dfrac{1}{2}, & (x,y)\in\overline{\Gamma}\cup\{O\}\,,\\ 0, & (x,y)\notin\overline{D},\end{cases}$$

where O is a origin of the coordinate system.

Lemma 4.1 was proved in [22].

Lemma 4.2. *If $(x,y)\in\Gamma$, then the following inequality holds true:*

$$\left|A_\nu^{\alpha,\beta}\left[q\left(\xi,\eta;x,y\right)\right]\right|\le\frac{B_1}{r_1^{2\alpha}r_2^{2\beta}}\left(\ln\frac{r_1r_2}{r_{12}r}+1\right). \tag{4.3}$$

where B_1 is a constant.

Proof. The estimate (4.3) follows from the formula

$$\begin{aligned} &A_\nu^{\alpha,\beta}\left[q\left(\xi,\eta;x,y\right)\right]\\ &=-(\alpha+\beta)\kappa\frac{1}{r^{2\alpha+2\beta}}F_2\left[\begin{matrix}1+\alpha+\beta,\alpha,\beta;\\ 2\alpha,2\beta;\end{matrix}\ \sigma_1,\sigma_2\right]A_\nu^{\alpha,\beta}\left[\ln r^2\right]\\ &-2(\alpha+\beta)\kappa\frac{x\xi^{2\alpha}\eta^{2\beta}}{r^{2+2\alpha+2\beta}}F_2\left[\begin{matrix}1+\alpha+\beta,1+\alpha,\beta;\\ 1+2\alpha,2\beta;\end{matrix}\ \sigma_1,\sigma_2\right]\frac{d\eta(s)}{ds}\\ &+2(\alpha+\beta)\kappa\frac{y\xi^{2\alpha}\eta^{2\beta}}{r^{2+2\alpha+2\beta}}F_2\left[\begin{matrix}1+\alpha+\beta,\alpha,1+\beta;\\ 2\alpha,1+2\beta;\end{matrix}\ \sigma_1,\sigma_2\right]\frac{d\xi(s)}{ds}.\end{aligned} \tag{4.4}$$

and lemma 2.1. □

Lemma 4.3. *If a curve Γ satisfies the conditions (i) and (ii), then the following inequality holds true:*

$$\int_0^l\left|A_\nu^{\alpha,\beta}[q\left(\xi,\eta;x,y\right)]\right|ds\le\frac{B_2}{x^\alpha y^\beta},$$

where B_2 is a constant.

Proof. We begin our proof of Lemma 4.3 by transforming the right-hand side of equality (4.4). Using the formula (2.2), we obtain

$$A_\nu^{\alpha,\beta}\left[q\left(\xi,\eta;x,y\right)\right]=P_0(s;x,y)+P_1(s;x,y)+P_2(s;x,y), \tag{4.5}$$

where

$$P_0(s;x,y)=-\frac{(\alpha+\beta)\kappa r^2}{r_{12}^{2+2\alpha+2\beta}}F_2\left[\begin{matrix}1+\alpha+\beta,\alpha,\beta;\\ 2\alpha,2\beta;\end{matrix}\ \bar\sigma_1,\,\bar\sigma_2\right]A_\nu^{\alpha,\beta}\left[\ln r^2\right],$$

$$P_1(s;x,y)=2(\alpha+\beta)\kappa\frac{y\xi^{2\alpha}\eta^{2\beta}}{r_{12}^{2+2\alpha+2\beta}}F_2\left[\begin{matrix}1+\alpha+\beta,\alpha,\beta;\\ 2\alpha,1+2\beta;\end{matrix}\ \bar\sigma_1,\,\bar\sigma_2\right]\frac{d\xi(s)}{ds},$$

$$P_2(s;x,y)=-2(\alpha+\beta)\kappa\frac{x\xi^{2\alpha}\eta^{2\beta}}{r_{12}^{2+2\alpha+2\beta}}F_2\left[\begin{matrix}1+\alpha+\beta,\alpha,\beta;\\ 1+2\alpha,2\beta;\end{matrix}\ \bar\sigma_1,\,\bar\sigma_2\right]\frac{d\eta(s)}{ds},$$

$$\bar{\sigma}_1 = \frac{4x\xi}{r_{12}^2},\ \bar{\sigma}_2 = \frac{4y\eta}{r_{12}^2},\ r_{12}^2 = (x+\xi)^2 + (y+\eta)^2,\ 0 \le \bar{\sigma}_1 + \bar{\sigma}_1 \le 1.$$

By virtue of (2.7), we have

$$\begin{aligned}\int_0^l |P_0(s;x,y)|\,ds \le C_2 \int_0^l \frac{r^2}{r_{12}^{2+2\alpha+2\beta}} \left(\frac{x\xi}{r_{12}^2}\right)^{-\alpha} \left(\frac{y\eta}{r_{12}^2}\right)^{-\beta} \\ \times \left(\frac{r^2}{r_{12}^2}\right)^{-1} \xi^{2\alpha}\eta^{2\beta} \left|\frac{\partial}{\partial\nu}\left(\ln\frac{1}{r}\right)\right| ds \\ \le \frac{C_2}{x^\alpha y^\beta} \int_0^l \xi^\alpha \eta^\beta \left|\frac{\partial}{\partial\nu}\left(\ln\frac{1}{r}\right)\right| ds \le \frac{C_3}{x^\alpha y^\beta} \int_0^l \frac{|cos\vartheta|}{r} ds,\end{aligned} \tag{4.6}$$

where ϑ is an angle between r and the outer normal ν to the curve Γ.

From the theory of the logarithmic potential we have

$$\int_0^l \frac{|cos\vartheta|}{r} ds < C_4. \tag{4.7}$$

Now we will estimate $P_1(s;x,y)$ and $P_2(s;x,y)$. It is easy to see that

$$\int_{\varepsilon_k}^{l-\varepsilon_k} |P_k(s;x,y)|\,ds \le \frac{D_k}{x^\alpha y^\beta} \quad (\varepsilon_k > 0,\ k = 1,2), \tag{4.8}$$

where D_1 and D_2 are independent of (x,y).

Integrals $\int_0^{\varepsilon_k} |P_k(s;x,y)|\,ds$ and $\int_{l-\varepsilon_k}^l |P_k(s;x,y)|\,ds$ are estimated similarly. Let us estimate the first of them for $k=1$. Using the estimate (2.6), taking into account the first of the conditions (4.1), we get

$$\int_0^{\varepsilon_1} |P_1(s;x,y)|\,ds \le \frac{E_1}{x^\alpha y^\beta} \int_0^{\varepsilon_1} \ln\left[\frac{r}{r_{12}}\right] ds \le \frac{E_2}{x^\alpha y^\beta}. \tag{4.9}$$

Thus, the obtained estimates (4.6) - (4.9) imply the validity of the Lemma 4.3. □

Theorem 4.4. *The following limit formulas hold true for a double-layer potential (4.2):*

$$w_i(s) = -\frac{1}{2}\mu(s) + \int_0^l \mu(t)K(s,t)dt, \tag{4.10}$$

$$w_e(s) = \frac{1}{2}\mu(s) + \int_0^l \mu(t)K(s,t)dt, \tag{4.11}$$

where

$$K(s,t) = A_\nu^{\alpha,\beta}\left[q\left(\xi(t),\eta(t);x(s),y(s)\right)\right],$$

$A_n^{\alpha,\beta}\left[w\left(x,y\right)\right]_i$ *and* $A_n^{\alpha,\beta}\left[w\left(x,y\right)\right]_e$ *are limiting values of the double-layer potential (4.2) at the point* $t \in \Gamma$ *from the inside and the outside, respectively.*

Proof. Theorem 4.4 follows from the Lemmas 4.1 and 4.3. □

5. The simple-layer potential

In this section, we consider the following integral:

$$v\left(x,y\right)=\int_{0}^{l}\rho(t)q(\xi,\eta;x,y)dt, \tag{5.1}$$

where the density $\rho(t)\in C\left(\overline{\Gamma}\right)$ and $q\left(\xi,\eta;x,y\right)$ is given in (2.8). We call the integral (5.1) a simple-layer potential with density $\rho(t)$.

The simple-layer potential (5.1) is defined throughout the quarter-plane $x>0,\ y>0$ and is a continuous function when passing through the curve Γ. Obviously, a simple-layer potential is a regular solution of equation (1.1) in any domain lying in the quarter-plane $x>0,\ y>0$. It is easy to see that, as the point (x,y) tends to ∞ , a simple-layer potential $v\left(x,y\right)$ tends to 0. Indeed, we let the point (x,y) be on the quarter-circle given by C_R: $x^2+y^2=R^2$ $(x>0,\ y>0)$. Then, by virtue of (2.8), we have

$$|v\left(x,y\right)|\leq\int_{0}^{l}|\rho\left(t\right)||q\left(\xi,\eta;x,y\right)|dt\leq PR^{-2\alpha-2\beta},\ \ (R\geq R_0)$$

where P is a constant.

We take an arbitrary point $N\left(x(x),y(s)\right)$ on the curve Γ and draw a normal at this point. By considering on this normal any point $M(x,y)$, not lying on the curve Γ, we find the conormal derivative of the simple-layer potential (5.1):

$$A_n^{\alpha,\beta}\left[v\left(x,y\right)\right]=\int_{0}^{l}\rho(t)A_n^{\alpha\beta}\left[q\left(\xi,\eta;x,y\right)\right]dt, \tag{5.2}$$

where

$$A_n^{\alpha,\beta}[\ \ ]=x^{2\alpha}y^{2\beta}\left(cos(n,x)\cdot\frac{\partial}{\partial x}+cos(n,y)\cdot\frac{\partial}{\partial y}\right).$$

The integral in (5.2) exists also in the case when the point $M(x,y)$ coincides with the point N, which we mentioned above.

Theorem 5.1. *The following limit formulas hold true for a simple-layer potential (5.1):*

$$A_n^{\alpha,\beta}\left[v\left(x,y\right)\right]_i=\frac{1}{2}\rho(s)+\int_{0}^{l}\rho(t)K(t,s)dt, \tag{5.3}$$

$$A_n^{\alpha,\beta}\left[v\left(x,y\right)\right]_e=-\frac{1}{2}\rho(s)+\int_{0}^{l}\rho(t)K(t,s)dt, \tag{5.4}$$

where

$$K(t,s)=A_n^{\alpha,\beta}\left[q\left(\xi(t),\eta(t);x(s),y(s)\right)\right],$$

$A_n^{\alpha,\beta}\left[v\left(x,y\right)\right]_i$ *and* $A_n^{\alpha,\beta}\left[v\left(x,y\right)\right]_e$ *are limiting values of the normal derivative of simple-layer potential (5.1) at the point* $t\in\Gamma$ *from the inside and the outside, respectively.*

Making use of these formulas, the jump in the normal derivative of the simple-layer potential follows immediately:

$$A_n^{\alpha,\beta}\left[v\left(x,y\right)\right]_i - A_n^{\alpha\beta}\left[v\left(x,y\right)\right]_e = \rho(x,y). \tag{5.5}$$

For future researches on the subject of the present investigation, it will be useful to note that when the point (x,y) tends to ∞, the following inequality

$$\left|A_n^{\alpha,\beta}\left[v\left(x,y\right)\right]\right| \leq QR^{-2\alpha-2\beta-1}, \ \ (R \geq R_0),$$

is valid, Q is a constant.

In exactly the same way as in the derivation of (3.4), it is not difficult to show that Green's formulas are applicable to the simple-layer potential (5.1) as follows:

$$\int\int_D x^{2\alpha}y^{2\beta}\left[\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]dxdy = \int_\Gamma vA_n^{\alpha,\beta}\left[v\right]_i ds. \tag{5.6}$$

$$\int\int_{D'} x^{2\alpha}y^{2\beta}\left[\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]dxdy = -\int_\Gamma vA_n^{\alpha,\beta}\left[v\right]_e ds. \tag{5.7}$$

Hereinafter $D' = R_2^{2+} \setminus \bar{D}$ is the unbounded domain at $x > 0$, $y > 0$.

6. Integral Equations For Denseness

Formulas (4.10), (4.11), (5.3) and (5.4) can be written as the following integral equations for densities:

$$\mu(s) - \lambda\int_0^l K(s,t)\mu(t)dt = f(s), \tag{6.1}$$

$$\rho(s) - \lambda\int_0^l K(t,s)\rho(t)dt = g(s), \tag{6.2}$$

where

$$\lambda = 2, \ \ f(s) = -2w_i(s), \ g(s) = -2A_n^{\alpha,\beta}\left[v\right]_e,$$

$$\lambda = -2, \ \ f(s) = 2w_e(s), \ g(s) = 2A_n^{\alpha,\beta}\left[v\right]_i.$$

Equations (6.1) and (6.2) are mutually conjugated and, by Lemma 4.2, Fredholm theory is applicable to them. We show that $\lambda = 2$ is not an eigenvalue of the kernel $K\left(s,t\right)$. This assertion is equivalent to the fact that the homogeneous integral equation

$$\rho(s) - 2\int_0^l K(t,s)\rho(t)dt = 0, \tag{6.3}$$

has no non-trivial solutions.

Let $\tilde{\rho}\left(t\right)$ be a continuous non-trivial solution of the equation (6.3). The simple-layer potential with density $\tilde{\rho}\left(t\right)$ gives us a function $\tilde{v}\left(x,y\right)$, which is a solution of the equation (1.1) in the domains D and D'. By virtue of the equation (6.3), the limiting values of the normal derivative of $A_n^{\alpha,\beta}[\tilde{v}]_e$ are zero. The formula (5.7) is applicable to the simple-layer potential $\tilde{v}(x,y)$, from which it follows that $\tilde{v}(x,y) = const$ in domain D'. At infinity, a simple layer potential is zero, and consequently $\tilde{v}(x,y) \equiv 0$ in D', and also on the curve Γ. Applying now (5.6), we find that $\tilde{v}(x,y) \equiv 0$ is valid also inside the domain D. But then $A_n^{\alpha,\beta}[\tilde{v}]_i = 0$, and by virtue of formula (5.5) we obtain $\tilde{\rho}(t) \equiv 0$. Thus, clearly, the homogeneous equation (6.3) has only the trivial solution; consequently, $\lambda = 2$ is not an eigenvalue of the kernel $K(s;t)$.

7. The Uniqueness of the Solution of Holmgren's Problem

We apply the obtained results of potential theory to the solving the boundary value problem for the equation (1.1) in the domain D.

We consider the Holmgren problem for equation (1.1) in the domain D defined in Section 4. We assume that the curve Γ satisfies conditions (i) and (ii) in Section 4.

Holmgren problem. Find a regular solution $u(x,y)$ of equation (1.1) in the domain D that is continuous in the closed domain $\overline{D}$ and satisfies the following boundary conditions:

$$u|_\Gamma = \varphi(s) \ (0 \le s \le l), \tag{7.1}$$

$$\lim_{x\to 0} x^{2\alpha}\frac{\partial u(x,y)}{\partial x} = \nu_1(y), \quad \lim_{y\to 0} y^{2\beta}\frac{\partial u(x,y)}{\partial y} = \nu_2(x) \ (0 < x, y < a), \tag{7.2}$$

where $\varphi(s) \in C[0,l]$, $\nu_1(y) \in C(0,a)$ and $\nu_2(x) \in C(0,a)$ are given continuous functions, and the function $\nu_1(y)$ can tend to ∞ of order less $1-2\alpha+2\beta$ and $1-2\alpha$ as $y \to 0$ and $y \to a$; the function $\nu_2(x)$ can tend to ∞ of order less $1+2\alpha-2\beta$ and $1-2\beta$ as $x \to 0$ and $x \to a$, respectively.

The uniqueness of the solution. Consider the domain $D_{\varepsilon,\delta_1,\delta_2} \subset D$, bounded by the curve Γ_ε, parallel to the curve Γ, and line segments $x = \delta_1 > \varepsilon$ and $y = \delta_2 > \varepsilon$.

Integrating both sides of the identity (3.1) along the domain D_ε and using the Ostrogradsky formula, we obtain

$$\int\!\!\int_{D_{\varepsilon,\delta_1,\delta_2}} x^{2\alpha}y^{2\beta}\left[uE(v) - vE(u)\right]dxdy$$
$$= \int_{S_{\varepsilon,\delta_1,\delta_2}} \left(uA_n^{\alpha,\beta}[v] - vA_n^{\alpha,\beta}[u]\right)dS_{\varepsilon,\delta_1,\delta_2},$$

where $S_{\varepsilon,\delta_1,\delta_2}$ is a contour of the domain $D_{\varepsilon,\delta_1,\delta_2}$.

One can easily check that the following equality holds:

$$\int\!\!\int_{D_{\varepsilon,\delta_1,\delta_2}} x^{2\alpha}y^{2\beta}uE(u)dxdy = \int\!\!\int_{D_{\varepsilon,\delta_1,\delta_2}} x^{2\alpha}y^{2\beta}\left[u_x^2 + u_y^2\right]dxdy$$
$$-\int\!\!\int_{D_{\varepsilon,\delta_1,\delta_2}} \left[y^{2\beta}\frac{\partial}{\partial x}\left(x^{2\alpha}uu_x\right) + x^{2\alpha}\frac{\partial}{\partial y}\left(y^{2\beta}uu_y\right)\right]dxdy.$$

Application of the Gauss-Ostrogradsky formula to this equality after $\delta_1 \to 0$, $\delta_2 \to 0$ and $\varepsilon \to 0$ yields

$$\int\int_D x^{2\alpha} y^{2\beta} \left[u_x^2 + u_y^2\right] dxdy = \int_0^a y^{2\beta} u(0,y)\nu_1(y)dy + \int_0^a x^{2\alpha} u(x,0)\nu_2(x)dx - \int_\Gamma \varphi(s) A_n^{\alpha,\beta}[u]ds. \tag{7.3}$$

If we consider the homogeneous Holmgren problem, then we find from (7.3) that

$$\int\int_D x^{2\alpha} y^{2\beta} \left[u_x^2 + u_y^2\right] dxdy = 0.$$

Hence, it follows that $u(x,y) = 0$ in $\overline{D}$.

Therefore, the following uniqueness theorem holds true.

Theorem 7.1. *If the Holmgren problem has a regular solution, then it is unique.*

8. Green's Function Revisited

To solve this problem, we use the Green's function method. First, we construct the Green's function for solving the Holmgren problem for an equation in a domain which is bounded by an arbitrary curve and two mutually perpendicular line segments. We then show that, in view of the Green's function, the solution of the Holmgren problem in a quadrant takes a simpler form as described below.

Definition 8.1. We refer to $G(x,y;x_0,y_0)$ as Green's function of the Holmgren problem, if it satisfies following conditions:

1) The function $G(x,y;x_0,y_0)$ is a regular solution of equation (1.1) in the domain D, expect at the point (x_0,y_0), which is any fixed point of D.

2) The function $G(x,y;x_0,y_0)$ satisfies the boundary conditions given by

$$G(x,y;x_0,y_0)|_\Gamma = 0, \quad x^{2\alpha}\frac{\partial G}{\partial x}\bigg|_{x=0} = 0, \quad y^{2\beta}\frac{\partial G}{\partial y}\bigg|_{y=0} = 0. \tag{8.1}$$

3) The function $G(x,y;x_0,y_0)$ can be represented as follows:

$$G(x,y;x_0,y_0) = q(x,y;x_0,y_0) + v(x,y;x_0,y_0) \tag{8.2}$$

where $q\,(x,y;x_0,y_0)$ is a fundamental solution of the equation (1.1), defined in the domain D, and the function $v(x,y;x_0,y_0)$ is a regular solution of the equation (1.1) in the domain D.

The construction of the Green's function $G(x,y;x_0,y_0)$ reduces to finding its regular part $v(x,y;x_0,y_0)$ which, by virtue of (2.9), (8.1) and (8.2), must satisfy the following boundary conditions:

$$v(x,y;x_0,y_0)|_\Gamma = -q(x,y;x_0,y_0)|_\Gamma\,, \tag{8.3}$$

$$x^{2\alpha}\frac{\partial v(x,y;x_0,y_0)}{\partial x}\bigg|_{x=0} = 0, \quad y^{2\beta}\frac{\partial v(x,y;x_0,y_0)}{\partial y}\bigg|_{y=0} = 0.$$

We now look for the function $v(x,y;x_0,y_0)$ in the form of a double-layer potential given by

$$v\left(x,y;x_0,y_0\right)=\int_0^l \mu\left(t;x_0,y_0\right)A_\nu^{\alpha,\beta}[q\left(\xi,\eta;x,y\right)]dt. \tag{8.4}$$

By taking into account the equality (4.10) and the boundary condition (8.3), we obtain the integral equation for the density $\mu\left(t;x_0,y_0\right)$ as follows:

$$\mu\left(s;x_0,y_0\right)-2\int_0^l K(s,t)\mu\left(t;x_0,y_0\right)dt=2q\left(x(s),y(s);x_0,y_0\right). \tag{8.5}$$

The right-hand side of (8.5) is a continuous function of s (the point (x_0,y_0) lies inside D). In Section 6, it was proved that $\lambda=2$ is not an eigenvalue of the kernel $K(s,t)$ and, consequently, the Equation (8.5) is solvable and its continuous solution can be written in the following form:

$$\mu\left(s;x_0,y_0\right)=2q\left(x(s),y(s);x_0,y_0\right)+4\int_0^l R(s,t;2)q\left(\xi,\eta;x_0,y_0\right)dt, \tag{8.6}$$

where $R(s,t;2)$ is the resolvent of kernel $K(s,t)$, $(x(s),y(s))\in\Gamma$. Thus, upon substituting from (8.6) into (8.4), we obtain

$$\begin{aligned}v\left(x,y;x_0,y_0\right)&=2\int_0^l q\left(\xi,\eta;x_0,y_0\right)A_\nu^{\alpha,\beta}[q\left(\xi,\eta;x,y\right)]dt\\&+4\int_0^l\int_0^l A_\nu^{\alpha,\beta}[q(\xi,\eta;x,y)]R_0(t,s;2)q\left(x(s),y(s);x_0,y_0\right)dtds.\end{aligned}$$

We now define the function $g(x,y)$ as follows:

$$g(x,y)=\begin{cases} v(x,y;x_0,y_0), & (x,y)\in D,\\ -q(x,y;x_0,y_0), & (x,y)\in D'.\end{cases} \tag{8.7}$$

The function $g(x,y)$ is a regular solution of equation (1.1) both inside the domain D and inside D' and equal to zero at infinity. Because the point (x_0,y_0) lies inside D, therefore, in D', the function $g(x,y)$ has derivatives of any order in all variables that are continuous up to Γ. We can consider $g(x,y)$ in D' as a solution of Equation (1.1) satisfying the boundary conditions given by

$$\begin{gathered}A_n^{\alpha,\beta}\left[g(x,y)\right]\Big|_\Gamma=-A_n^{\alpha,\beta}\left[q(x(s),y(s);x_0,y_0)\right],\\ x^{2\alpha}\frac{\partial g(x,y)}{\partial x}\bigg|_{x=0}=0,\ \ y^{2\beta}\frac{\partial g(x,y)}{\partial y}\bigg|_{y=0}=0.\end{gathered} \tag{8.8}$$

We represent this solution in the form of a simple-layer potential as follows:

$$g(x,y)=\int_0^l \rho(t;x_0,y_0)q(\xi,\eta;x,y)dt,\ \ (x,y)\in D' \tag{8.9}$$

with an unknown density $\rho(t;x_0,y_0)$.

Using the formula (5.3), by virtue of condition (8.8), we obtain the following integral equation for the density $\rho(s;x_0,y_0)$:

$$\rho(s;x_0,y_0)-2\int_0^l K(t,s)\rho(t;x_0,y_0)dt=2A_n^{\alpha,\beta}\left[q(x(s),y(s);x_0,y_0)\right]. \tag{8.10}$$

Equation (8.10) is conjugated with the equation (8.5). Its right-hand side is a continuous function of s. Thus, clearly, the equation (8.10) has the following continuous solution:

$$\begin{aligned}\rho(s;x_0,y_0) &= 2A_n^{\alpha,\beta}\left[q(x(s),y(s);x_0,y_0)\right]\\ &\quad + 4\int_0^l R(t,s;2)A_\nu^{\alpha,\beta}\left[q(\xi,\eta;x_0,y_0)\right]dt.\end{aligned} \tag{8.11}$$

The values of a simple-layer potential $g(x,y)$ on the curve Γ are equal to $-q(x,y;x_0,y_0)$, that is, just as the functions $v(x,y;x_0,y_0)$ and on the axes x and y their partial derivatives with respect to y and x multiplied, respectively, by $y^{2\beta}$ and $x^{2\alpha}$ are equal to zero. Hence, by virtue of the uniqueness theorem for the Holmgren problem, it follows that the formula (8.9) for the function $g(x,y)$ defined by (8.7) holds throughout in the quarter-plane $x \geq 0$, $y \geq 0$, that is,

$$v(x,y;x_0,y_0) = \int_0^l \rho(t;x_0,y_0)q(\xi,\eta;x,y)dt, \quad (x,y)\in D. \tag{8.12}$$

Thus, the regular part $v(x,y;x_0,y_0)$ of Green's function is representable in the form of a simple-layer potential.

Applying the formula (5.3) to (8.12), we obtain

$$2A_n^{\alpha,\beta}\left[v\left(x(s),y(s);x_0,y_0\right)\right]_i = \rho(s;x_0,y_0) + 2\int_0^l K(t,s)\rho(t;x_0,y_0)dt.$$

But, according to (8.10), we have

$$2A_n^{\alpha,\beta}\left[q\left(x(s),y(s);x_0,y_0\right)\right]_i = \rho(s;x_0,y_0) - 2\int_0^l K(t,s)\rho(t;x_0,y_0)dt.$$

Summing the last two equalities by term-by-term and taking equation (8.2) into account, we find that

$$A_n^{\alpha,\beta}\left[G\left(x(s),y(s);x_0,y_0\right)\right] = \rho(s;x_0,y_0). \tag{8.13}$$

Consequently, formula (8.12) can be written in the following form:

$$v(x,y;x_0,y_0) = \int_0^l A_\nu^{\alpha,\beta}\left[G\left(\xi,\eta;x_0,y_0\right)\right]q(\xi,\eta;x,y)dt.$$

Multiplying both sides of (8.11) by $q\left(x(s),y(s);x,y\right)$, integrating by s over the curve Γ from 0 to l and, by virtue of (8.6) and (8.4), we obtain

$$v(x_0,y_0;x,y) = \int_0^l \rho(t;x_0,y_0)q(\xi,\eta;x,y)dt.$$

Comparing this last equation with the formula (8.12), we have

$$v(x,y;x_0,y_0) = v(x_0,y_0;x,y). \tag{8.14}$$

if the points (x,y) and (x_0,y_0) are inside the domain D.

Lemma 8.2. *If points (x,y) and (x_0,y_0) are inside domain D, then Green's function $G(x,y;x_0,y_0)$ is symmetric about those points.*

The proof of Lemma 8.2 follows from the representation (8.2) of Green's function and the equality (8.14).

For a quarter circle D_0 bounded by two segments $[0, a]$ of the axes x and y and a quarter circle given by $x^2 + y^2 = a^2$ $(x \geq 0,\, y \geq 0)$, the Green's function of the Holmgren problem has the following form

$$G_0(x, y; x_0, y_0) = q(x, y; x_0, y_0) - \left(\frac{a}{R}\right)^{2\alpha+2\beta} q(x, y; \bar{x}_0, \bar{y}_0), \tag{8.15}$$

where

$$R^2 = x_0^2 + y_0^2,\;\; \bar{x}_0 = \frac{a^2}{R^2}x_0,\;\; \bar{y}_0 = \frac{a^2}{R^2}y_0.$$

We now show that the function given by

$$v_0(x, y; x_0, y_0) = -\left(\frac{a}{R}\right)^{2\alpha+2\beta} q(x, y; \bar{x}_0, \bar{y}_0)$$

can be represented in the following form:

$$v_0(x, y; x_0, y_0) = -\int_0^l \rho(s; x, y) v_0(x(s), y(s); x_0, y_0) ds, \tag{8.16}$$

where $\rho(s; x, y)$ is a solution of equation (8.12).

Indeed, by letting an arbitrary point (x_0, y_0) be inside the domain D, we consider the function given by

$$u(x, y; x_0, y_0) = -\int_0^l \rho(s; x, y) v_0(x(s), y(s); x_0, y_0) ds.$$

As a function of (x, y), the function $u\,(x, y; x_0, y_0)$ satisfies equation (1.1), because this equation is satisfied by the function $\rho(s; x, y)$. Substituting the expression (8.11) for $\rho(s; x, y)$, we obtain

$$u(x, y; x_0, y_0) = -\int_0^l \psi(s; x_0, y_0) A_n^{\alpha,\beta} \left[q(x(s), y(s); x, y)\right] ds, \tag{8.17}$$

where

$$\psi(s; x_0, y_0) = 2v_0\left(x(s), y(s); x_0, y_0\right) + 4\int_0^l R(s, t; 2) v_0\left(\xi, \eta; x_0, y_0\right) dt,$$

that is, $\psi(s; x_0, y_0)$ is a solution of the integral equation

$$\psi(s; x_0, y_0) - 2\int_0^l K(s, t)\psi(t; x_0, y_0) dt = 2v_0\left(x(s), y(s); x_0, y_0\right). \tag{8.18}$$

Applying formula (4.10) to the double-layer potential (8.17), we obtain

$$u_i\left(x(s), y(s); x_0, y_0\right) = \frac{1}{2}\psi(s; x_0, y_0) - \int_0^l K(s, t)\psi(t; x_0, y_0) dt,$$

whence, by virtue of (8.18) we get

$$u_i\left(x(s), y(s); x_0, y_0\right) = v_0\left(x(s), y(s); x_0, y_0\right),\; (x(s), y(s)) \in \Gamma.$$

It is easy to see that

$$x^{2\alpha}\frac{\partial u\left(x, y; x_0, y_0\right)}{\partial x}\bigg|_{x=0} = 0, \quad x^{2\alpha}\frac{\partial v_0\left(x, y; x_0, y_0\right)}{\partial x}\bigg|_{x=0} = 0,$$

$$y^{2\beta}\frac{\partial u\left(x,y;x_0,y_0\right)}{\partial y}\bigg|_{y=0}=0,\quad y^{2\beta}\frac{\partial v_0\left(x,y;x_0,y_0\right)}{\partial y}\bigg|_{y=0}=0.$$

Thus, clearly, the functions $u\left(x,y;x_0,y_0\right)$ and $v_0\left(x,y;x_0,y_0\right)$ satisfy the same equation (1.1) and the same boundary conditions. Also, by virtue of the uniqueness of the solution of the Holmgren problem, the equality $u\left(x,y;x_0,y_0\right)\equiv v_0\left(x,y;x_0,y_0\right)$ is satisfied.

Now, subtracting the expression (8.15) from (8.2), we obtain

$$\begin{aligned}H\left(x,y;x_0,y_0\right)&=G\left(x,y;x_0,y_0\right)-G_0\left(x,y;x_0,y_0\right)=\\&=v\left(x,y;x_0,y_0\right)-v_0\left(x,y;x_0,y_0\right)\end{aligned}$$

or, by virtue of (8.12), (8.14), (8.15) and (8.16), we obtain

$$H(x,y;x_0,y_0)=\int_0^l\rho(t;x,y)G_0(\xi,\eta;x_0,y_0)dt. \tag{8.19}$$

9. Solving the Holmgren Problem for Equation (1.1)

Let (x_0,y_0) be a point inside the domain D. Consider the domain $D_{\varepsilon,\delta_1,\delta_2}\subset D$ bounded by the curve Γ_ε, which is parallel to the curve Γ, and the line segments $x=\delta_1>\varepsilon$ и $y=\delta_2>\varepsilon$.

We choose ε, δ_1 and δ_2 to be so small that the point (x_0,y_0) is inside $D_{\varepsilon,\delta_1,\delta_2}$. We cut out from the domain $D_{\varepsilon,\delta_1,\delta_2}$ a circle of small radius ρ with center at the point (x_0,y_0), and we denote the remainder part of $D_{\varepsilon,\delta_1,\delta_2}$ by $D^{\rho}_{\varepsilon,\delta}$, in which the Green's function $G(x,y;x_0,y_0)$ is a regular solution of equation (1.1).

Let $u(x,y)$ be a regular solution of the equation (1.1) in the domain D that satisfies the boundary conditions (7.1) and (7.2). Applying the formula (3.3), we obtain

$$\begin{aligned}&\int_{\Gamma_\varepsilon}\left(GA_n^{\alpha,\beta}[u]-uA_n^{\alpha,\beta}[G]\right)ds+\int_{\delta_1}^{x_1}x^{2\alpha}y^{2\beta}\left(u\frac{\partial G}{\partial y}-G\frac{\partial u}{\partial y}\right)\bigg|_{y=\delta_2}dx\\&+\int_{\delta_2}^{y_1}x^{2\alpha}y^{2\beta}\left(u\frac{\partial G}{\partial x}-G\frac{\partial u}{\partial x}\right)\bigg|_{x=\delta_1}dy=\int_{C_\rho}\left(GA_n^{\alpha,\beta}[u]-uA_n^{\alpha,\beta}[G]\right)ds,\end{aligned}$$

x_1 and y_1 are an abscissa and ordinate of the intersection points of the curve Γ_ε with the straight lines $y=\delta_2$ and $x=\delta_1$, respectively, and C_ρ is a circumference of the cut circle.

Proceeding to the limit as $\rho\to0$ and then as $\varepsilon\to0$, $\delta_1\to0$ and $\delta_2\to0$, we obtain

$$\begin{aligned}u\left(x_0,y_0\right)=&-\int_0^a y^{2\beta}G\left(0,y;x_0,y_0\right)\nu_1(y)dy\\&-\int_0^a x^{2\alpha}G\left(x,0;x_0,y_0\right)\nu_2(x)dx-\int_0^l A_\nu^{\alpha,\beta}[G\left(\xi,\eta;x_0,y_0\right)]\varphi(s)ds\\&=I_1(x_0,y_0)+I_2(x_0,y_0)+I_3(x_0,y_0).\end{aligned} \tag{9.1}$$

We show that formula (9.1) gives a solution of the Holmgren problem.

It is easy to see that the first integral $I_1\left(x_0,y_0\right)$ in the formula (9.1) is a solution of the equation (1.1) and is regular in the domain D, continuous in $\overline{D}$.

We use the following notation:

$$\begin{aligned}\vartheta(x_0,y_0) &= \int_0^a y^{2\beta} q\left(0,y;x_0,y_0\right)\nu_1(y)dy \\ &= \kappa\int_0^a \frac{y^{2\beta}F\left(\alpha+\beta,\beta;2\beta;-\dfrac{4yy_0}{x_0^2+(y-y_0)^2}\right)}{\left[x_0^2+(y-y_0)^2\right]^{\alpha+\beta}}\nu_1(y)dy.\end{aligned} \tag{9.2}$$

Here, $\vartheta(x_0,y_0)$ is a continuous function in $\overline{D}$. In view of (9.2) and (8) and the symmetry of the function $v(x,y;x_0,y_0)$, the integral $I_1(x_0,y_0)$ can be represented in the following form:

$$\begin{aligned}I_1\left(x_0,y_0\right) &= -\vartheta\left(x_0,y_0\right) - 2\int_0^l \vartheta\left(\xi,\eta\right) A_\nu^{\alpha,\beta}[q\left(\xi,\eta;x_0,y_0\right)]dt- \\ &- 4\int_0^l\int_0^l R\left(t,s;2\right)\vartheta\left(x(s),y(s)\right)A_\nu^{\alpha,\beta}[q\left(\xi,\eta;x_0,y_0\right)]dtds.\end{aligned} \tag{9.3}$$

The last two integrals in the formula (9.3) are double-layer potentials. Taking into account the formula (4.10) and the integral equation for the resolvent $R(s,t;2)$ from formula (9.3), we obtain

$$I_1\left(x_0,y_0\right)|_\Gamma = 0,$$

It is easy to see that

$$\lim_{x_0\to 0} x_0^{2\alpha}\frac{\partial I_1\left(x_0,y_0\right)}{\partial x_0} = \nu_1\left(y_0\right) \quad (0<y_0<a).$$

In fact, by virtue of (8.12) and the symmetry of the function $v\left(x,y;x_0,y_0\right)$, the above integral can also be written in the following form:

$$\begin{aligned}I_1(x_0,y_0) = &-\int_0^a \nu_1(y)q(0,y;x_0,y_0)dy \\ &-\int_0^a \nu_1(y)dy\int_0^l \rho(t;0,y)q(\xi,\eta;x_0,y_0)dt.\end{aligned}$$

Following the work [21], it is easy to show that

$$\lim_{x_0\to 0} x_0^{2\alpha}\int_0^a \nu_1(y)q(0,y;x_0,y_0)dy = -\nu_1\left(y_0\right) \quad (0<y_0<a)$$

and

$$\lim_{x_0\to 0} x_0^{2\alpha}\int_0^a \nu_1(y)dy\int_0^l \rho(t;0,y)q(\xi,\eta;x_0,y_0)dt \quad (0<y_0<a),$$

because

$$x_0^{2\alpha}\frac{\partial q}{\partial x_0} = 0$$

when $x_0=0,\ 0<y_0<a$.

By virtue of the last from the conditions (8.1), we have

$$\lim_{y_0\to 0} y_0^{2\beta}\frac{\partial I_1\left(x_0,y_0\right)}{\partial y_0} = 0 \quad (0<x_0<a).$$

Similarly, we get

$$I_2(x_0,y_0)|_\Gamma=0;\ \lim_{x_0\to 0}x_0^{2\alpha}\frac{\partial I_2(x_0,y_0)}{\partial x_0}=0,\ \lim_{y_0\to 0}y_0^{2\beta}\frac{\partial I_2(x_0,y_0)}{\partial y_0}=\nu_2\left(x_0\right).$$

We consider the third integral $I_3(x_0,y_0)$ in the formula (9.1), which, by virtue of (8.13) and (8.11), can be written in the following form:

$$I_3(x_0,y_0)=-\int_0^l\varphi(s)\rho(s;x_0,y_0)ds=-\int_0^l\theta(t)A_\nu^{\alpha,\beta}\left[q(\xi,\eta;x_0,y_0)\right]dt,$$

where

$$\theta(t)=2\varphi(t)+4\int_0^l R(t,s;2)\varphi(s)ds,$$

that is, the function $\theta(s)$ is a solution of the integral equation

$$\theta(s)-2\int_0^l K(s,t)\theta(t)dt=2\varphi(s). \tag{9.4}$$

Because $\theta(s)$ is a continuous function, $I_3(x_0,y_0)$ is a solution of Equation (1.1), regular in the domain D, that is continuous in $\overline{D}$, which, by virtue of (4.10) and (9.4), satisfies following condition:

$$I_3\left(x_0,y_0\right)|_\Gamma=\varphi(s).$$

It is now easy to see that

$$\lim_{x_0\to 0}x_0^{2\alpha}\frac{\partial I_3\left(x_0,y_0\right)}{\partial x_0}=0\ \ (0<y_0<a),$$
$$\lim_{y_0\to 0}y_0^{2\beta}\frac{\partial I_3\left(x_0,y_0\right)}{\partial y_0}=0\ \ (0<x_0<a).$$

By using formulas (8.19) and (8.15), solution (9.1) of the Holmgren problem given by (7.1) and (7.2) for Equation (1.1) can be written in the following form:

$$\begin{aligned}
&u\left(x_0,y_0\right)=\\
&-\int_0^a\nu_1(y)y^{2\beta}\left[G_0\left(0,y;x_0,y_0\right)+H\left(0,y;x_0,y_0\right)\right]dy\\
&-\int_0^a\nu_2(x)x^{2\alpha}\left[G_0\left(x,0;x_0,y_0\right)+H\left(x,0;x_0,y_0\right)\right]dx\\
&-\int_0^l\varphi(s)\left\{A_\nu^{\alpha,\beta}\left[G_0\left(\xi,\eta;x_0,y_0\right)\right]+A_\nu^{\alpha,\beta}\left[H\left(\xi,\eta;x_0,y_0\right)\right]\right\}ds,
\end{aligned}\tag{9.5}$$

where

$$H\left(x,y;x_0,y_0\right)=\int_0^l\rho\left(t;x_0,y_0\right)G_0\left(\xi,\eta;x,y\right)dt;$$

ρ and G_0 are defined by (8.11) and (8.15), respectively.

We remark that solution (9.5) of the Holmgren problem is more convenient for further investigations. The resulting explicit integral representation (9.5) plays an important role in the study of problems for equation of the mixed type (that is, elliptic-hyperbolic or elliptic-parabolic types): it makes it easy to derive the basic functional relationship between the traces of the sought solution and of its derivative on the line of degeneration from the elliptic part of the mixed domain.

In the case of a quarter circle D_0, the function $H\left(x,y;x_0,y_0\right)\equiv 0$ and solution (9.5) assumes a simpler form as follows:

$$
\begin{aligned}
u\left(x_0,y_0\right) =& -\kappa\int_0^a \nu_1(y)y^{2\beta} \\
&\times\left[\frac{F\left(\alpha+\beta,\beta;2\beta;-4yy_0/X_1^2\right)}{X_1^{2\alpha+2\beta}}-\frac{F\left(\alpha+\beta,\beta;2\beta;-4yy_0/Y_1^2\right)}{Y_1^{2\alpha+2\beta}}\right]dy \\
&-\kappa\int_0^a \nu_2(x)x^{2\alpha} \\
&\times\left[\frac{F\left(\alpha+\beta,\alpha;2\alpha;-4xx_0/X_2^2\right)}{X_2^{2\alpha+2\beta}}-\frac{F\left(\alpha+\beta,\alpha;2\alpha;-4xx_0/Y_2^2\right)}{Y_2^{2\alpha+2\beta}}\right]dx \\
&+2(\alpha+\beta)\kappa\int_0^l \varphi(s)\xi^{2\alpha}\eta^{2\beta} \\
&\times F_2\left(1+\alpha+\beta,\alpha,\beta;2\alpha,2\beta;\frac{r_1^2-r^2}{r_{12}^2},\frac{r_2^2-r^2}{r_{12}^2}\right)\frac{R^2-a^2}{r_{12}^{2+2\alpha+2\beta}}ds,
\end{aligned}
\tag{9.6}
$$

where

$$r^2=\left(\xi-x_0\right)^2+\left(\eta-y_0\right)^2,\quad r_{12}^2=\left(\xi+x_0\right)^2+\left(\eta+y_0\right)^2,\quad R^2=x_0^2+y_0^2,$$

$$r_1^2=\left(\xi+x_0\right)^2+\left(\eta-y_0\right)^2,\ r_2^2=\left(\xi-x_0\right)^2+\left(\eta+y_0\right)^2,\ a^2=\xi^2+\eta^2,$$

$$X_1^2=x_0^2+\left(y-y_0\right)^2,\ Y_1^2=\left(a-\frac{yy_0}{a}\right)^2+\frac{y^2}{a^2}x_0^2,$$

$$X_2^2=\left(x-x_0\right)^2+y_0^2,\ Y_2^2=\left(a-\frac{xx_0}{a}\right)^2+\frac{x^2}{a^2}y_0^2.$$

Thus, clearly, we obtain the solution of the Holmgren problem for Equation (1.1) in the quarter of the circle.

10. Concluding Remarks and Observations

In our present investigation of the two dimensional singular elliptic equation (1.1), we use potential theory results in order to represent boundary value problems in integral equation form. In fact, in problems with known Green's functions, an integral equation formulation leads to powerful numerical approximation schemes. Thus, by seeking the representation of the solution of the boundary value problem as a double-layer potential with unknown density, we are eventually led to a Fredholm equation of the second kind for the explicit determination of the solution in terms of Appell hypergeometric function. Appell hypergeometric function $F_2(a,b_1,b_2;c_1,c_2;x,y)$ possesses easily-accessible numerical algorithms for computational purposes, can indeed be used to numerically compute the solution presented here for many different special values of the parameters a, b_1, b_2, c_1 and c_2 and of the arguments x and y.

Numerical applications of several suitably specialized versions of the solutions presented in this paper can be found in solid mechanics, fluid mechanics, elastic dynamics, electro-magnetics, and acoustics (see, for details, some of the citations [4, 8] handling special situations which were motivated by such widespread applications).

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