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A NOTE ON MULTIVARIATE PSEUDO-CHEBYSHEV FUNCTIONS OF FRACTIONAL DEGREE

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Abstract. In a recent article, the second-kind multivariate pseudo-Chebyshev functions of fractional degree have been introduced, in order to find an explicit representation of the square roots of non-singular complex matrices. In this paper, by using preceding results about the connection of the first kind multivariate Chebyshev and Lucas polynomials, and the relevant integral repesentations, even the first kind pseudo-Chebyshev functions are studied. A possible application of these new functions is shown in Section 7.

1. Introduction

It is not necessary to underline the importance of Special functions which appear as the main tools for finding explicit solutions of all problems in mathematical physics, engineering, and applied mathematics. Among them the hypergeometric functions constitute an important class that unifies the most (if not all) parts of special functions (see, for example, [25–27]).

Many multivariate generalizations of hypergeometric functions have been studied in the literature, even through an extension of the Pochhammer symbol [24, 28–31]. Connections with several extensions of the celebrated Zeta function and the relevant applications to Number Theory and Statistics have been recently highlighted in [23].

In this framework the classical Chebyshev polynomials have been generalized to the multivariate case in earlier works [2, 7, 12, 13, 18]. Furthermore, in [4] a link between the multivariate Lucas [14, 16] and the Chebyshev polynomials both of the first and second kind has been shown.

In the above mentioned article [4] integral representations and generating functions [5] of the multivariate Lucas and Chebyshev polynomials were derived.

As the two last cited articles were written in Italian, and are not available on the web, we devote a wide part of this article to expose the results which are necessary to understand our investigation.

In recent works a special set of univariate hypergeometric functions, called the pseudo-Chebyshev functions, were introduced in [19] and studied in [20].

The second kind multivariate pseudo-Chebyshev functions have been also introduced in [21] showing their connection with the problem of computing matrix roots. As a matter of fact, it seems that these polynomials are naturally connected with

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this problem, just as the multivariate Chebyshev polynomials are with the integer matrix powers, as it has been proven previously in [17].

In this article, starting from the generating functions and the connections between the multivariate Lucas and Chebyshev polynomials, and using the Dunford-Taylor (or Riesz-Fantappiè) formula, an integral representation of the first kind multivariate Chebyshev polynomials is derived, further properties of these polynomials are recalled and an extension of these polynomials to the case of rational indexes is achieved, introducing the multivariate first kind pseudo-Chebyshev functions.

A possible application of these pseudo-Chebyshev functions to the fractional moments of the density of zeros of a polynomial, extending the results in [10] is touched on in the 7th Section.

2. Basic definitions

Definition 2.1. Given the $r \times r$ matrix $\mathcal{A} = (a_{ij})$, whose invariants are

$$\left\{ \begin{array}{lcl} u_1 & := & tr\ \mathcal{A} = a_{11} + a_{22} + \cdots + a_{rr} \\ \\ u_2 & := & \sum_{i<j}^{1,r} \left| \begin{array}{cc} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{array} \right| \\ \cdots & & \\ u_r & := & det\ \mathcal{A}\,, \end{array} \right. \tag{2.1}$$

putting for shortness $\mathbf{u} := (u_1, u_2, \ldots, u_r)$, its *characteristic polynomial* is given by

$$\begin{array}{c} P(\mathbf{u};\lambda) \ := det(\lambda\mathcal{I} - \mathcal{A}) \ = \\ = \lambda^r - u_1\lambda^{r-1} + u_2\lambda^{r-2} + \ldots + (-1)^r u_r\,, \end{array} \tag{2.2}$$

and the relative characteristic equation writes:

$$P(\lambda) = P(\mathbf{u};\lambda) \ = 0\,. \tag{2.3}$$

2.1. Recalling the $F_{k,n}$ functions and the second kind multivariate Lucas polynomials.

It is well known [4, 16] that a basis for the r-dimensional vectorial space of solutions of the homogeneous linear bilateral recurrence relation with constant coefficients u_k $(k = 1, 2, \ldots, r)$, with $u_r \neq 0$,

$$X_n = u_1 X_{n-1} - u_2 X_{n-2} + \cdots + (-1)^{r-1} u_r X_{n-r}\,,\ (n \in \mathbf{Z})\,, \tag{2.4}$$

is given by the functions $F_{k,n} = F_{k,n}(\mathbf{u})\,,\ (k = 1, 2, \ldots, r\,,\ n \geq -1)\,,$ defined by the initial conditions:

$$F_{r-k+1,h-2}(\mathbf{u}) = \delta_{k,h}\,, \qquad (k, h = 1, 2, \ldots, r)\,, \tag{2.5}$$

where δ is the Kronecker symbol.

Since $u_r \neq 0$, the $F_{k,n}$ functions can be defined even if $n < -1$, by means of the so called *reflection properties*:

$$\begin{array}{c} F_{k,n}(\mathbf{u}) = F_{r-k+1,-n+r-3}\left(\dfrac{u_{r-1}}{u_r}, \ldots, \dfrac{u_1}{u_r}, \dfrac{1}{u_r}\right)\,, \\ (k = 1, 2, \ldots, r\,;\ n < -1)\,. \end{array} \tag{2.6}$$

It has been show by É Lucas [4, 14] that all $\{F_{k,n}\}_{n\in\mathbf{Z}}$ functions are expressed through the only bilateral sequence $\{F_{1,n}\}_{n\in\mathbf{Z}}$, corresponding to the initial conditions in (2.5). More precisely, the following equations hold

$$(2.7)\quad \begin{cases} F_{k,n} = (-1)^{k-1} u_k F_{1,n-1} + F_{k+1,n-1}\,, & (k = 1, 2, \ldots, r-1)\,, \\ F_{r,n} = (-1)^{r-1} u_r F_{1,n-1}\,. \end{cases}$$

Therefore, the bilateral sequence $\{F_{1,n}\}_{n\in\mathbf{Z}}$ is called the *fundamental solution* of the recursion (2.4) (the *"fonction fondamentale"* by É. Lucas [14]).

Remark 2.2. - The $F_{k,n}$ functions are the standard basis for the recursion (2.4) which is different from the usual one, which makes use of the roots of the characteristic equation [3]. This basis does not imply the knowledge of roots and does not depend on their multiplicity, so that it is sometimes more convenient in applications.

The functions $F_{1,n}(\mathbf{u})$ are called in literature [16] *generalized Lucas polynomials of the second kind.*
In what follows, we will use the notation:

$$(2.8)\qquad \Phi_n(\mathbf{u}) := F_{1,n}(\mathbf{u}) = F_{1,n}(u_1, u_2, \ldots, u_r)\,,\ (n \in \mathbf{Z})\,,$$

for denoting these multivariate second kind Lucas polynomials, (shortly MSK Lucas polynomials).

3. The multivariate first kind Lucas polynomials

Among the solutions of the recursion (2.4) the most used one is the sum of integer powers $n\,(n \in \mathbf{Z})$ of the roots of the characteristic equation (2.3). This solution is called the *primordial solution* (the *"fonction primordiale"* by É. Lucas [14]), and will be denoted by $\{\Psi(\mathbf{u})\}_{n\in\mathbf{Z}}$, and its terms, for $n \geq r-2$ costitute a polynomial sequence of the r variables $(u_1, u_2, \ldots, u_r)$, called *generalized Lucas polynomials of the first kind.* In what follows we denote these multivariate polynomials by the symbol

$$(3.1)\qquad \Psi_n(\mathbf{u}) := \Psi_n(u_1, u_2, \ldots, u_r)\,,\ (n \in \mathbf{Z})\,,$$

and we use for them the acronym MFK Lucas polynomias.

In order to get these polynomials it is suffcient to assume the initial conditions borrowed by the classic Newton formulas.

Denoting by $\lambda_1, \lambda_2, \dots, \lambda_r$ the roots of the characteristic equation (2.3), we put:

$$
(3.2)\qquad \begin{cases}
\Psi_{r-1} = u_1 = \sum_{h=1}^{r} \lambda_h \\
\Psi_r = u_1 \Psi_{r-1} - 2u_2 = u_1^2 - 2u_2 = \sum_{h=1}^{r} \lambda_h^2 \\
\dots \\
\Psi_{2r-3} = u_1 \Psi_{2r-4} - u_2 \Psi_{2r-5} + \cdots + (-1)^{r-3} u_{r-2} \Psi_{r-1} + \\
\qquad\qquad (-1)^{r-2}(r-1)u_{r-1} = \sum_{h=1}^{r} \lambda_h^{r-1} \\
\Psi_{2r-2} = u_1 \Psi_{2r-3} - u_2 \Psi_{2r-4} + \cdots + (-1)^{r-2} u_{r-1} \Psi_{r-1} + \\
\qquad\qquad (-1)^{r-1} r u_r = \sum_{h=1}^{r} \lambda_h^{r}
\end{cases}
$$

For $n > 2r - 2$ we get:

$$
(3.3)\qquad \Psi_n = u_1 \Psi_{n-1} - u_2 \Psi_{n-2} + \cdots + (-1)^{r-1} u_r \Psi_{n-r} = \sum_{h=1}^{r} \lambda_h^{n-r+2} .
$$

Note that:

$$
\Psi_{r-2} = r = \sum_{h=1}^{r} \lambda_h^0 .
$$

The Ψ_n, as all particular solutions of the recurrence (2.4), can be expressed in terms of the fundamental solution Φ_n, according to the equations:

$$
(3.4)\qquad \Psi_n = u_1 \Phi_{n-1} - 2u_2 \Phi_{n-2} + \cdots + (-1)^{r-1} r u_r \Phi_{n-r} , \quad (\forall n \in \mathbf{Z}) .
$$

or by means of:

$$
(3.5)\qquad \begin{aligned} \Psi_n = r\Phi_n - (r-1)u_1 \Phi_{n-1} + (r-2)u_2 \Phi_{n-2} + \cdots \\ + (-1)^{r-1} u_{r-1} \Phi_{n-r+1} , \quad (\forall n \in \mathbf{Z}) . \end{aligned}
$$

In the last equation (3.5) the variable u_r does not appear explicitly.

3.1. **Links with the multivariate Chebyshev polynomials.** It is worth to note, as it was remarked in [4], that the first and second kind multivariate Lucas polynomials are related, in a very simple way, to an extension of the Chebyshev polynomials to the multivariate case, introduced by R. Lidl - C. Wells [13], R. Lidl [12], M. Bruschi - P.E. Ricci [5], and studied by K.B. Dunn - R. Lidl [7], R.J. Beerends [2].

In fact, putting for shortness $\tilde{\mathbf{u}} := (u_1, u_2, \dots, u_{r-1})$, and denoting respectively by $T_n^{(r-1)}(\tilde{\mathbf{u}})$ [by $U_n^{(r-1)}(\tilde{\mathbf{u}})$] the multivariate Chebyshev polynomials of the first [of the second] kind in $r-1$ variables, it results:

$$
(3.6)\qquad T_n^{(r-1)}(\tilde{\mathbf{u}}) = \Psi_n(u_1, u_2, \dots, u_{r-1}, 1) = \Psi_n(\tilde{\mathbf{u}}, 1)
$$

and

$$
(3.7)\qquad U_n^{(r-1)}(\tilde{\mathbf{u}}) = \Phi_n(u_1, u_2, \cdots, u_{r-1}, 1) = \Phi_n(\tilde{\mathbf{u}}, 1) .
$$

Remark 3.1. In the above equations the choice of indexes has been performed in such a way that, in the particular case $r = 2$, putting $u_1 = 2x$, and $u_2 = 1$, we recover the classical Chebyshev polynomials with the same indexes.
We have, in fact:

$$\Psi_n(u_1, 1) = \Psi_n(2x, 1) \equiv T_n(x)\,,$$
$$\Phi_n(u_1, 1) = \Phi_n(2x, 1) \equiv U_n(x)\,, \quad (n \in \mathbf{N})\,.$$

where $\{T_n(x)\}_{n\in\mathbf{N}}$ and $\{U_n(x)\}_{n\in\mathbf{N}}$ are respectively the first and second kind Chebyshev polynomials.

From the above consideration, it follows that the second kind multivariate Chebyshev polynomials are defined by the recursion:

$$\begin{aligned} U_n^{(r-1)}(\tilde{\mathbf{u}}) &= u_1\, U_{n-1}^{(r-1)}(\tilde{\mathbf{u}}) - u_2\, U_{n-2}^{(r-1)}(\tilde{\mathbf{u}}) + \cdots + \\ &+ (-1)^{r-2} u_{r-1}\, U_{n-r+1}^{(r-1)}(\tilde{\mathbf{u}}) + (-1)^{r-1}\, U_{n-r}^{(r-1)}(\tilde{\mathbf{u}})\,, \end{aligned} \tag{3.8}$$

with initial conditions:

$$\begin{aligned} U_0^{(r-1)}(\tilde{\mathbf{u}}) = U_1^{(r-1)}(\tilde{\mathbf{u}}) = \cdots = U_{r-3}^{(r-1)}(\tilde{\mathbf{u}}) = 0\,, \\ U_{r-2}^{(r-1)}(\tilde{\mathbf{u}}) = 1\,, \quad U_{r-1}^{(r-1)}(\tilde{\mathbf{u}}) = u_1\,, \end{aligned} \tag{3.9}$$

while the first kind mutivariate Chebyshev polynomials are defined by the same recursion:

$$\begin{aligned} T_n^{(r-1)}(\tilde{\mathbf{u}}) &= u_1\, T_{n-1}^{(r-1)}(\tilde{\mathbf{u}}) - u_2\, T_{n-2}^{(r-1)}(\tilde{\mathbf{u}}) + \cdots \\ &+ (-1)^{r-2} u_{r-1}\, T_{n-r+1}^{(r-1)}(\tilde{\mathbf{u}}) + (-1)^{r-1}\, T_{n-r}^{(r-1)}(\tilde{\mathbf{u}})\,, \end{aligned} \tag{3.10}$$

with initial conditions given by equations (3.2).

4. Matrix powers representation

In preceding articles [4], [17], the following result has been proven:

Theorem 4.1. - *Given an $r \times r$ matrix $\mathcal{A}$, with characteristic polynomial $P(\lambda) \equiv P(\mathbf{u};\lambda)$ in equation* (2.2)*, the matrix powers, with integer exponent n, can be represented by the equation:*

$$\mathcal{A}^n = F_{1,n-1}(\mathbf{u})\,\mathcal{A}^{r-1} + F_{2,n-1}(\mathbf{u})\,\mathcal{A}^{r-2} + \cdots + F_{r,n-1}(\mathbf{u})\,\mathcal{I}\,, \tag{4.1}$$

where the functions $F_{k,n}(u_1, \ldots, u_r)$ are defined in Section 2.1.
Moreover, if $\mathcal{A}$ is non-singular, i.e. $u_r \neq 0$, equation (4.1) *still works for negative integers n, assuming the definition* (2.6) *for the $F_{k,n}$ functions.*

By using equations (2.7), only the multivariate second kind Lucas polynomials can be used in (4.1), so that we find

$$\begin{aligned} \mathcal{A}^n &= F_{1,n-1}(\mathbf{u})\mathcal{A}^{r-1} + \left[-u_2\, F_{1,n-2}(\mathbf{u}) + u_3\, F_{1,n-3}(\mathbf{u}) + \cdots \right. \\ &\left. + (-1)^{r-2} u_{r-1}\, F_{1,n-r+1}(\mathbf{u}) + (-1)^{r-1}\, F_{1,n-r}(\mathbf{u})\right] \mathcal{A}^{r-2} + \cdots \\ &+ \left[(-1)^{r-2} u_{r-1}\, F_{1,n-2}(\mathbf{u}) + (-1)^{r-1}\, F_{1,n-3}(\mathbf{u})\right] \mathcal{A} + (-1)^{r-1}\, F_{1,n-2}(\mathbf{u})\,\mathcal{I}\,. \end{aligned} \tag{4.2}$$

Let $\mathcal{A} = [a_{h,k}]_{r\times r}$ be a non-singular complex matrix of order r, put:

$$v_1 = u_1 u_r^{-1/r}\,,\ v_2 = u_2 u_r^{-2/r}\,,\ \dots\,,\ v_{r-1} = u_{r-1} u_r^{-(r-1)/r}\,, \tag{4.3}$$

and, for shortness, $\mathbf{v} = (v_1, v_2, \dots, v_{r-1})$.
Consider the $(r-1)$variable Chebyshev polynomials [5],

$$U_n^{(r-1)}(\mathbf{v}) \equiv U_n^{(r-1)}(v_1, v_2, \dots, v_{r-1})\,, \tag{4.4}$$

defined by the recursion (3.8)-(3.9), where ve have assumed the notation: $v_k \equiv u_k$, $(k = 1, 2, \dots, r-1)$, that is: $\mathbf{v} \equiv \tilde{\mathbf{u}}$.

Then, from Theorem 1, we can derive the result [17]:

Theorem 4.2. *The integer powers of the matrix $\mathcal{A}$ are given by the equation:*

$$\begin{aligned} \mathcal{A}^n = U_{n-1}^{(r-1)}(\mathbf{v})\, u_r^{\frac{n-r+1}{r}}\, \mathcal{A}^{r-1} + \Big[-u_2\, U_{n-2}^{(r-1)}(\mathbf{v}) + u_3\, U_{n-3}^{(r-1)}(\mathbf{v}) + \cdots \\ + (-1)^{r-2} u_{r-1}\, U_{n-r+1}^{(r-1)}(\mathbf{v}) + (-1)^{r-1}\, U_{n-r}^{(r-1)}(\mathbf{v})\Big]\, u_r^{\frac{n-r+2}{r}}\, \mathcal{A}^{r-2} + \cdots \\ + \Big[(-1)^{r-2} u_{r-1}\, U_{n-2}^{(r-1)}(\mathbf{v}) + (-1)^{r-1}\, U_{n-3}^{(r-1)}(\mathbf{v})\Big]\, u_r^{\frac{n-1}{r}}\, \mathcal{A} + \\ (-1)^{r-1}\, U_{n-2}^{(r-1)}(\mathbf{v})\, u_r^{\frac{n}{r}}\, \mathcal{I}\,. \end{aligned} \tag{4.5}$$

Remark 4.3. Note that equation (4.5) can be simplified assuming the condition $\det \mathcal{A} = u_r = 1$, which is not a restriction. In fact, letting:

$$\mathcal{A} = u_r\, \tilde{\mathcal{A}}\,, \qquad \text{with} \qquad \det \tilde{\mathcal{A}} = 1\,,$$

it results: $\mathcal{A}^n = (u_r)^n\, \tilde{\mathcal{A}}^n$, and we have again $\tilde{\mathbf{u}} \equiv \mathbf{v}$.

By using a matrix $\tilde{\mathcal{A}}$, such that $\det \tilde{\mathcal{A}} = 1$, the equation (4.5) becomes:

$$\begin{aligned} \tilde{\mathcal{A}}^n = U_{n-1}^{(r-1)}(\mathbf{v})\, \tilde{\mathcal{A}}^{r-1} + \Big[-u_2\, U_{n-2}^{(r-1)}(\mathbf{v}) + u_3\, U_{n-3}^{(r-1)}(\mathbf{v}) + \cdots \\ + (-1)^{r-2} u_{r-1}\, U_{n-r+1}^{(r-1)}(\mathbf{v}) + (-1)^{r-1}\, U_{n-r}^{(r-1)}(\mathbf{v})\Big]\, \tilde{\mathcal{A}}^{r-2} + \cdots \\ + \Big[(-1)^{r-2} u_{r-1}\, U_{n-2}^{(r-1)}(\mathbf{v}) + (-1)^{r-1}\, U_{n-3}^{(r-1)}(\mathbf{v})\Big]\, \tilde{\mathcal{A}} + \\ (-1)^{r-1}\, U_{n-2}^{(r-1)}(\mathbf{v})\, \mathcal{I}\,. \end{aligned} \tag{4.6}$$

5. The Dunford-Taylor integral

Theorem 5.1. - *Consider an $r \times r$ matrix $\mathcal{A} = \{a_{h,k}\}$, with invariants* (2.1) *and characteristic polynomial given by equation* (2.2). *Let f be a function holomorphic in an open set $\mathcal{O}$, containing all the eigenvaues of $\mathcal{A}$. Then, the matrix functions $f(\mathcal{A})$ are given by the Dunford-Taylor* (*also called Riesz-Fantappiè*) *integral* [11]*:*

$$f(\mathcal{A}) = \frac{1}{2\,\pi\,\mathrm{i}} \left[\sum_{k=1}^{r} \oint_\gamma \frac{f(\lambda)\, \sum_{h=0}^{k-1} (-1)^h u_h\, \lambda^{k-h-1}}{P(\lambda)}\, d\lambda\ \mathcal{A}^{r-k} \right]\,, \tag{5.1}$$

where γ denotes a simple contour encircling all the zeros of $P(\lambda)$.

In particular the integer powers of $\mathcal{A}$ are given by the equation:

$$\mathcal{A}^n = \frac{1}{2\,\pi\,\mathrm{i}} \left[\sum_{k=1}^{r} \oint_\gamma \frac{\lambda^n \sum_{h=0}^{k-1} (-1)^h u_h \, \lambda^{k-h-1}}{P(\lambda)} \, d\lambda \; \mathcal{A}^{r-k} \right] . \tag{5.2}$$

Remark 5.2. Note that for computing the integrals in equation (5.1) it is sufficient the knowledge of a circular domain D, ($\gamma := \partial D$), encircling the spectrum of $\mathcal{A}$. By the Gerschgorin theorem, this can be done without computing its eigenvalues. Therefore, this method is computationally more convenient that using the Lagrange-Sylvester formula.

5.1. **Integral representations.**

Theorem 5.3. *The SKM Lucas polynomials are represented by the integral formula:*

$$\Phi_n(\mathbf{u}) = \frac{1}{2\,\pi\,\mathrm{i}} \oint_\gamma \frac{\lambda^{n+1}}{P(\lambda)} \, d\lambda \, . \tag{5.3}$$

Proof. - It is sufficient to compare equation (4.1) with equation (5.2), assuming $k = 1$ and increasing the index n by one unit.

Theorem 5.4. *The FKM Lucas polynomials are represented by the integral formula:*

$$\Psi_n(\mathbf{u}) = \frac{1}{2\,\pi\,\mathrm{i}} \oint_\gamma \frac{\lambda^{n-r+2}\, P'(\lambda)}{P(\lambda)} \, d\lambda \, . \tag{5.4}$$

Proof. - It is a consequence of the identity [1]:

$$\frac{P'(\lambda)}{P(\lambda)} = \frac{1}{\lambda - \lambda_1} + \frac{1}{\lambda - \lambda_2} + \cdots \frac{1}{\lambda - \lambda_r} \, ,$$

so that, by using the residue theorem and recalling equation (3.3), we find:

$$\frac{1}{2\,\pi\,\mathrm{i}} \oint_\gamma \frac{\lambda^{n-r+2}\, P'(\lambda)}{P(\lambda)} \, d\lambda = \lambda_1^{n-r+2} + \lambda_2^{n-r+2} + \cdots + \lambda_r^{n-r+2} = \Psi_n(\mathbf{u}) \, .$$

In what follows, when $u_r = 1$, in order to simplify the notation it is convenient to put, by definition:

$$\Delta(\lambda) = \Delta(\mathbf{v}; \lambda) := \lambda^r - u_1\, \lambda^{r-1} + \cdots + (-1)^{r-1} u_{r-1}\, \lambda + (-1)^r \, . \tag{5.5}$$

Therefore, the integral representations (5.3)-(5.4), for the multivariate Chebyshev polynomials, become:

Theorem 5.5. *The SKM Chebyshev polynomials are represented by the integral formula:*

$$U_n^{(r-1)}(\mathbf{v}) = \frac{1}{2\,\pi\,\mathrm{i}} \oint_\gamma \frac{\lambda^{n+1}}{\Delta(\lambda)} \, d\lambda \, . \tag{5.6}$$

Theorem 5.6. *The FKM Chebyshev polynomials are represented by the integral formula:*

$$T_n^{(r-1)}(\mathbf{v}) = \frac{1}{2\,\pi\,\mathrm{i}} \oint_\gamma \frac{\lambda^{n-r+2}\, \Delta'(\lambda)}{\Delta(\lambda)} \, d\lambda \, . \tag{5.7}$$

6. Further elementary properties

6.1. **Generating functions.** In a former article [4] the generting functions of the $F_{k,n}(\mathbf{u})$ functions have been recalled. In particular, for the FKM and SKM Lucas polynomials the following results follow.

Theorem 6.1. *The generating function of th SKM Lucas polynomials is given by*

$$\sum_{n=0}^{\infty} \Phi_{n+r-2}(\mathbf{u})\lambda^n = \frac{1}{P(\mathbf{u};\lambda)}\,.$$

The generating function of th FKM Lucas polynomials is given by

$$\sum_{n=0}^{\infty} \Psi_{n+r-2}(\mathbf{u})\lambda^n = \frac{P'(\mathbf{u};\lambda)}{P(\mathbf{u};\lambda)}\,,$$

where $P(\mathbf{u};\lambda)$ denotes the characteristic polynomial defined in (2.2).

As a consequence, the generating funtions of the second and first kind Multivariate Chebyshev polynomials follow.

Theorem 6.2. *The generating function of th SKM Chebyshev polynomials is given by*

$$\sum_{n=0}^{\infty} U^{(r-1)}_{n+r-2}(\mathbf{v})\lambda^n = \frac{1}{\Delta(\mathbf{v};\lambda)}\,.$$

The generating function of th FKM Chebyshev polynomials is given by

$$\sum_{n=0}^{\infty} T^{(r-1)}_{n+r-2}(\mathbf{v})\lambda^n = \frac{\Delta'(\mathbf{v};\lambda)}{\Delta(\mathbf{v};\lambda)}\,,$$

where $\Delta(\mathbf{u};\lambda)$ is defined in equation (5.5).

6.2. **Isobaric property and PDE of the $F_{k,n}(\mathbf{u})$ functions.** In the above mentioned article [4] it has been shown that the $F_{k,n}$ functions satisfy the isobaric property:

$$F_{k,n}(tu_1, t^2u_2, \dots, t^r u_r) = t^{n+k-r+1}\, F_{k,n}(\mathbf{u})\,, \quad (k = 1, 2, \dots, r),$$

and therefore we have the result:

Theorem 6.3. *The $F_{k,n}(\mathbf{u})$ functions satisfy the PDE*

$$u_1\frac{\partial F_{k,n}}{\partial u_1} + 2u_2\frac{\partial F_{k,n}}{\partial u_2} + \dots + ru_r\frac{\partial F_{k,n}}{\partial u_r} = (n+k-r+1)\, F_{k,n}\,, \quad (k = 1, 2, \dots, r).$$

Proof. - It is a trivial consequence of Euler's theorem on homogeneous functions.

7. The Multivariate pseudo-Chebyshev functions and their applications

The integral representations (5.6)-(5.7) make possible to define the second (first) kind multivariate Chebyshev polynomials even for rational values of their indexes. Of course, in this case the functions so found are no longer polynomials, but functions called with the adjective *multivariate pseudo-Chebyshev*, and will be indicated, for shortness, with the acronyms SKMP-C functions and FKMP-C functions. Therefore, by using the above notations, we put the definition:

Definition 7.1. The SKMP-C functions are defined by the integral representation:

$$U^{(r-1)}_{\frac{p}{q}}(\mathbf{v}) = \frac{1}{2\,\pi\,\mathrm{i}} \oint_\gamma \frac{\lambda^{\frac{p}{q}+1}}{\Delta(\lambda)}\, d\lambda\,. \tag{7.1}$$

The FKMP-C functions are defined by the integral representation:

$$T^{(r-1)}_{\frac{p}{q}}(\mathbf{v}) = \frac{1}{2\,\pi\,\mathrm{i}} \oint_\gamma \frac{\lambda^{\frac{p}{q}-r+2}\,\Delta'(\lambda)}{\Delta(\lambda)}\, d\lambda\,. \tag{7.2}$$

7.1. **Applications of the SKMP-C functions.** In a recent article [21], the SKMP-C functions have been used in order to compute matrix powers with rational indexes, as we have found the equation:

$$\begin{aligned}\mathcal{A}^{p/q} = {} & U^{(r-1)}_{\frac{p}{q}-1}(\mathbf{v})\, u_r^{\frac{p-q(r-1)}{qr}}\mathcal{A}^{r-1} + \Big[-u_2\, U^{(r-1)}_{\frac{p}{q}-2}(\mathbf{v}) + u_3\, U^{(r-1)}_{\frac{p}{q}-3}(\mathbf{v}) + \cdots \\ & + (-1)^{r-2}u_{r-1}\, U^{(r-1)}_{\frac{p}{q}-r+1}(\mathbf{v}) + (-1)^{r-1}\, U^{(r-1)}_{\frac{p}{q}-r}(\mathbf{v})\Big]\, u_r^{\frac{p-q(r-2)}{qr}}\mathcal{A}^{r-2} + \cdots \\ & + \Big[(-1)^{r-2}u_{r-1}\, U^{(r-1)}_{\frac{p}{q}-2}(\mathbf{v}) + (-1)^{r-1}\, U^{(r-1)}_{\frac{p}{q}-3}(\mathbf{v})\Big]\, u_r^{\frac{p-q}{qr}}\mathcal{A} \\ & + (-1)^{r-1}\, U^{(r-1)}_{\frac{p}{q}-2}(\mathbf{v})\, u_r^{\frac{p}{qr}}\mathcal{I}\,,\end{aligned} \tag{7.3}$$

where the 2nd kind pseudo-Chebyshev functions with rational indexes are defined by equation (7.1), (see [21] for details).

In particular the SKMP-C functions $U^{(r-1)}_{\frac{1}{2}}(\mathbf{v})$ have been used to find the square roots of a non-singular complex $r \times r$ matrix. Assuming $u_r = 1$, and $n = 1/2$, equation (4.6) becomes:

$$\begin{aligned}\tilde{\mathcal{A}}^{1/2} = {} & U^{(r-1)}_{-\frac{1}{2}}(\mathbf{v})\, \tilde{\mathcal{A}}^{r-1} + \Big[-u_2\, U^{(r-1)}_{-\frac{3}{2}}(\mathbf{v}) + u_3\, U^{(r-1)}_{-\frac{5}{2}}(\mathbf{v}) + \cdots \\ & + (-1)^{r-2}u_{r-1}\, U^{(r-1)}_{\frac{3}{2}-r}(\mathbf{v}) + (-1)^{r-1}\, U^{(r-1)}_{\frac{1}{2}-r}(\mathbf{v})\Big]\, \tilde{\mathcal{A}}^{r-2} + \cdots \\ & + \Big[(-1)^{r-2}u_{r-1}\, U^{(r-1)}_{-\frac{3}{2}}(\mathbf{v}) + (-1)^{r-1}\, U^{(r-1)}_{-\frac{5}{2}}(\mathbf{v})\Big]\, \tilde{\mathcal{A}} + (-1)^{r-1}\, U^{(r-1)}_{-\frac{3}{2}}(\mathbf{v})\,\mathcal{I}\,.\end{aligned} \tag{7.4}$$

7.2. **Applications of the FKMP-C functions.** We want now to consider a possible applications of the FKMP-C functions. As a matter of fact, in an old article (see [10] and the references therein), the FKM Lucas polynomials, according to equation (3.3), have been used in order to represent the moments of the density of zeros of an orthogonal polynomial set, defined by a three term recurrence relation.

In the case of the FKMP-C functions, denoting by $\xi_1, \xi_2, \dots, \xi_r$ the roots of the algebraic equation

$$\Delta(\lambda) = \lambda^r - u_1\,\lambda^{r-1} + \cdots + (-1)^{r-1} u_{r-1}\,\lambda + (-1)^r = 0\,, \tag{7.5}$$

we can conclude with the result

Theorem 7.2. *The sum of powers with rational exponent* $p/q - r + 2$, $(q \neq 0)$ *of the roots of the equation* (7.5) *can be represented, in terms of the FKMP-C functions as follows:*

$$\begin{aligned} T^{(r-1)}_{\frac{p}{q}}(\mathbf{v}) &= \frac{1}{2\,\pi\,\mathrm{i}} \oint_\gamma \frac{\lambda^{\frac{p}{q}-r+2}\,\Delta'(\lambda)}{\Delta(\lambda)}\,d\lambda = \\ &= \xi_1^{\frac{p}{q}-r+2} + \xi_2^{\frac{p}{q}-r+2} + \cdots + \xi_r^{\frac{p}{q}-r+2}\,. \end{aligned} \tag{7.6}$$

Note that, as it was recalled in Remark 5.2, the computation of the integral in equation (7.6) can be obtained without knowing the roots $\xi_1, \xi_2, \dots, \xi_r$, so that the second member of this equation is calculated by means of the integral at the first member.

We end this article by noting that the definition of the nth moment of the density of zeros of a polynomial of degree r

$$\mu_n^{(r)} := \frac{1}{r}\sum_{k=1}^r \xi_k^n\,.$$

could be extende to the fractional degree by the following one, which could be useful in applied mathematics and statistics:

Definition 7.3. The fractional moments of the density of zeros of the polynomial $\Delta(\lambda)$, whose zeros are given by $\xi_1, \xi_2, \dots, \xi_r$, can be defined as

$$\mu^{(r)}_{\frac{p}{q}} := \frac{1}{r}\sum_{k=1}^r \xi_k^{\frac{p}{q}}\,, \tag{7.7}$$

and are represented by

$$\begin{aligned} \mu^{(r)}_{\frac{p}{q}} = \frac{1}{r}\,T^{(r-1)}_{\frac{p}{q}+r-2}(\mathbf{v}) &= \frac{1}{2\,\pi\,\mathrm{i}} \oint_\gamma \frac{\lambda^{\frac{p}{q}}\,\Delta'(\lambda)}{r\,\Delta(\lambda)}\,d\lambda = \\ &= \tfrac{1}{r}\left[\xi_1^{\frac{p}{q}} + \xi_2^{\frac{p}{q}} + \cdots + \xi_r^{\frac{p}{q}}\right]. \end{aligned} \tag{7.8}$$

8. Conclusion

By using the integral representation formulas for the multivariate second (first) kind Lucas polynomials, which in a particular case (putting $u_r = 1$) reduce to the integral representations for the second (first) kind multivariate Chebyshev polynomials, we have introduced the multivariate second (first) kind pseudo-Chebyshev functions, that is the relative functions of fractional degree. Then we have highlighted their use in two different frameworks, as the computation of matrix roots

for the SKMPC functions, and the representation of sums of fractional powers of the characteristic equation's roots for the FKMPC functions.
A survey about the main properties of the multivariate second (first) kind Lucas polynomials has been premitted in order to make more readible the article.

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