Applied Analysis and Optimization
Volume 5, Number 1, 2021, 29–44

Yokohama Publishers
ISSN 2189-1664 Online Journal

SOLUTION OF NONLOCAL FRACTIONAL-ORDER BOUNDARY VALUE PROBLEMS BY AN EFFECTIVE ACCURATE APPROXIMATION METHOD

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Abstract. The generalized Bessel polynomials are applied as basis functions in a collocation scheme to find an approximate solutions of a class of fractional differential equations subject to nonlocal boundary conditions. Our solution strategy is based on the truncated Bessel series expressed in the matrix representation form along with a collocation approach, which reduces the fractional boundary value problems (FBVPs) into a matrix equation corresponds to a set of linear algebraic equations consist of polynomial coefficients. To illustrate the effectiveness of this proposed strategy, some test cases are carried out. Comparisons with some existing well-established numerical models indicate that the presented technique is very accurate and convenient for solving nonlocal FBVPs.

1. Introduction

This study presents the generalized Bessel collocation method for solving a class of fractional-order boundary value problems (BVPs) with nonlocal boundary conditions. To be more precise, an approximate solution of the following fractional differential equation is considered

$$\mathcal{D}_t^{(\sigma)} u(t) + a(t)\, u'(t) + b(t)\, u(t) = f(t), \quad 0 \le t \le 1, \quad 1 < \sigma \le 2, \tag{1.1}$$

subjected to the boundary conditions

$$u(0) = u_0, \quad u(1) = \sum_{j=1}^{\ell} c_j\, u(\zeta_j), \tag{1.2}$$

where u_0, $\ell > 0$, and c_j, $0 < \zeta_j < 1$ are real constants. To ensure the existence and uniqueness of the solutions of boundary value problem (1.1)-(1.2), we assume that $a(t)$ and $b(t)$ are continuous on $[0, 1]$ and $f(t)$ is a given appropriate function. Here, $\mathcal{D}_t^{(\sigma)}$ is the standard Liouville-Caputo fractional derivative operator. Some results and discussions about existence and uniquness of fractional boundary value problems with nonlocal boundary conditions can be found cf. in [1], [31], [23].

Since the appearance of fractional calculus and consequently fractional differential equations (FDEs), various developments and applications have been achieved in the modeling numerous real world phenomena in different branches of science and engineering [16,21]. On the other hand, the solutions of most FDEs are not possible to be obtained in exact analytical form. As an alternative, numerical as well as approximate procedures are preferred to obtain a solution with an accebtable accuray.

2010 *Mathematics Subject Classification.* 34A08, 26A33, 41A10.

Key words and phrases. Bessel polynomials, Boundary value problems, Liouville-Caputo derivative, Collocation method, Nonlocal boundary conditions.

The following numerical tehniques have been proposed for the solution of FDEs. Among many available methods to solve FDES, we may mention the reproduing kernel schemes [5, 19], the homotopy analysis method [27], the operational matrix approaches [17, 32], the wavelet based techniqes [24, 30, 33], the numerial inverse Laplace tranform method [22], the spectral based collocation methods [7, 10–15], the LDG method [8, 9], to name a few.

For the model problem (1.1)-(1.2), according to the best of our knowledge, the following approximative and numerical procedures have been proposed. These include the reproducing kernel method (RKM) [19], the Haar wavelet method [24], the improved RKM [5], the shifted Jacobi operational matrix metod [17], and the Legendre wavelet approach [33]. In this study, we aimed at developing an approximation technique based on the generalized Bessel collocation approach, which was previously considered in [14]. The Bessel polynomials was first systemically introduced in [18]. However, most of the important subsequent developments on the subject of the Bessel polynomials and the generalized Bessel polynomials, which were published until early 1980s, can be found in [2–4, 20, 25, 26, 28, 29, 34]. The main characteristic of the considered methodology is that the governing FBVPs and the corresponding nonlocal boundary conditions can be easily handled. Thus, the present technique in not only easy to implement approach, but capable of giving a more accurate results than the available aforementioned numerical model results.

The outline of this paper is structured as follows. In the next section 2, some preliminary facts about fractional calculus and relevant properties are introduced. Next, the definitions of Bessel polynomials of fractional order are given. Section 3 is devoted to the presentation of the proposed collocation scheme applied to nonlocal FBVPs. In Section 4, we perform some experiments to illustrate the high accuracy and efficiency of the generalized Bessel collocation technique. Finally, Section 5 provides a conclusion.

2. Some basic preliminaries

We first state some definitions and fundamental facts of fractional calculus. Next, some basic definitions of generalized Bessel polynomials and theorems, which are useful for our subsequent sections have been introduced, see also [21], [16].

2.1. **Fractional calculus.**

Definition 2.1. Let us assume that $h(t)$ is n-times continuously differentiable. The Liouville-Caputo fractional derivative $\mathcal{D}_t^{(\sigma)}$ of $h(t)$ of order $\sigma > 0$ is

$$\mathcal{D}_t^{(\sigma)} h(t) = \begin{cases} I^{n-\sigma} h^{(n)}(t) & \text{if } n-1 < \sigma < n, \\ h^{(n)}(t), & \text{if } \sigma = n, \quad n \in \mathbb{N}, \end{cases} \tag{2.1}$$

where

$$I^{\sigma} h(t) = \frac{1}{\Gamma(\sigma)} \int_0^t h(s)(t-s)^{\sigma-1}\, ds, \quad t > 0.$$

The following properties of the operator $\mathcal{D}_t^{(\sigma)}$ will be used below:

$$\mathcal{D}_t^{(\sigma)}(C) = 0 \quad (C \text{ is a constant}), \tag{2.2}$$

(2.3)
$$\mathcal{D}_t^{(\sigma)} x^\beta = \begin{cases} \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\sigma)} x^{\beta-\sigma}, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta \geq \lceil \sigma \rceil, \text{ or } \beta \notin \mathbb{N}_0 \text{ and } \beta > \lfloor \sigma \rfloor, \\ 0, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta < \lceil \sigma \rceil. \end{cases}$$

2.2. **Generalized Bessel functions.** The Bessel polynomials naturally found in the study of classical wave equation when dealing in spherial coordinate [18] and are closely related to the Bessel functions of order equal to half integer, see also [6,14] and references therein. These polynomials are obtained recursively as

$$\mathrm{B}_{k+1}(z) = \mathrm{B}_{k-1}(z) + (2k+1)z\, \mathrm{B}_k(z), \quad k = 1, 2, \dots,$$

where $\mathrm{B}_0(z) = 1$ and $\mathrm{B}_1(z) = z + 1$. One can easily check that all coefficients of these polynomials are positive. With the help of change of variable $t = z^\alpha$, $\alpha > 0$, the fractional version of the polynomials are defined. Let us denote them by $\mathrm{B}_k^\alpha(t) = \mathrm{B}_k(z)$. This conversion was first introduced in [14]. In the explicit form, each $\mathrm{B}_k^\alpha(t)$ of degree $(k\alpha)$ is expressed as

$$\mathrm{B}_k^\alpha(t) = \sum_{j=0}^{k} b_{k,j}\, t^{\alpha j}, \quad b_{k,j} = \frac{(j+k)!}{(k-j)!} \frac{1}{j!} \left(\frac{t}{2}\right)^j, \quad j = 0, 1, \dots, k. \tag{2.4}$$

It can be shown that the set of fractional polynomial functions $\{\mathrm{B}_0^\alpha, \mathrm{B}_1^\alpha, \dots\}$ forms an orthogonal system on unit circle C with respect to the weight function, $w_\alpha(t) = t^{\alpha-1} \exp(-2/t^\alpha)$; i.e.

$$\frac{1}{2\pi i} \int_C w_\alpha(t) \mathrm{B}_k^\alpha(t)\, \mathrm{B}_{k'}^\alpha(t) dt = \frac{2(-1)^{k+1}\, \delta_{kk'}}{\alpha(2k+1)}.$$

Here, $\delta_{kk'}$ denotes the Kronecker delta function.

2.2.1. *Bessel function approximation.* Let assume that a square integrable function $u(t)$ in $(0,1)$ is given. We may expand $u(t)$ by a linear combination of fractional Bessel polynomials as

$$u(t) = \sum_{j=0}^{\infty} \alpha_j\, \mathrm{B}_j^\alpha(t),$$

being α_j, $j = 0, 1, \dots$ the unknown coefficients can be found by the aid of orthogonality properties of these Bessel polynomials. Practically, the first $(J+1)$-terms Bessel polynomials are considered to obtain an approximate solution of model problem (1.1) as follows

$$u_{J,\alpha}(t) = \sum_{j=0}^{J} \alpha_j\, \mathrm{B}_j^\alpha(t), \quad 0 \leq t \leq 1, \tag{2.5}$$

where the unknown coefficients α_j, $j = 0, 1, \dots, J$ have to be computed. The vector of Bessel polynomials $\mathrm{B}_j^\alpha(t), j = 0, 1, \dots, J$ can be represented as

$$\boldsymbol{\mathcal{B}}_\alpha(t) = \boldsymbol{\mathcal{T}}_\alpha(t)\, \boldsymbol{\mathcal{D}}. \tag{2.6}$$

Here, $\boldsymbol{\mathcal{B}}_\alpha(t) = [\mathrm{B}_0^\alpha(t) \quad \mathrm{B}_1^\alpha(t) \quad \dots \quad \mathrm{B}_J^\alpha(t)]$ and

$$\boldsymbol{\mathcal{T}}_\alpha(t) = \begin{bmatrix} 1 & t^\alpha & t^{2\alpha} & \dots & t^{J\alpha} \end{bmatrix}.$$

Moreover, the matrix $\boldsymbol{\mathcal{D}}$ can be expressed as

$$\boldsymbol{\mathcal{D}} = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & b_{1,1} & b_{2,1} & b_{3,1} & \dots & b_{J-1,1} & b_{J,1} \\ 0 & 0 & b_{2,2} & b_{3,2} & \dots & b_{J-1,2} & b_{J,2} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & b_{J-1,J-1} & b_{J,J-1} \\ 0 & 0 & 0 & \dots & 0 & 0 & b_{J,J} \end{pmatrix}_{(J+1)\times(J+1)}.$$

Next, by defining the vector of unknown $\boldsymbol{\mathcal{A}}_J = [\alpha_0 \quad \alpha_1 \quad \dots \quad \alpha_J]^t$ and utilizing the relation (2.6) we may rewrite (2.5) in the matrix representation form as

$$u_{J,\alpha}(t) = \boldsymbol{\mathcal{B}}_\alpha(t)\,\boldsymbol{\mathcal{A}}_J = \boldsymbol{\mathcal{T}}_\alpha(t)\,\boldsymbol{\mathcal{D}}\,\boldsymbol{\mathcal{A}}_J, \tag{2.7}$$

We finally state a result about the generalized Bessel polynomials by considering their convergence properties. Let us by $\mathcal{V}_J^\alpha$ we denote the space spanned by $\{\mathrm{B}_0^\alpha(t), \mathrm{B}_1^\alpha(t), \dots, \mathrm{B}_{J-1}^\alpha(t)\}$. The next theorem shows the exponential convergent of the approximation solution $u_{J,\alpha}(t) \in \mathcal{V}_J^\alpha$ to $u(t)$ when the number of basis function J will be increasd, see [14].

Theorem 2.2. *Suppose that for $j = 0, 1, \dots, J$ we have $\mathcal{D}_t^{(j\alpha)} u(t) \in C[0,1]$. If $u_{J-1,\alpha} = \boldsymbol{\mathcal{B}}_\alpha(t)\,\boldsymbol{\mathcal{A}}_{J-1}$ denote the best approximation to $u \in \mathcal{V}_J^\alpha$, then an upper bound of the error is obtained as*

$$\|u(t) - u_{J-1,\alpha}(t)\|_{w_\alpha} \le \frac{1}{\sqrt{(2J+1)\alpha}} \frac{exp(-1)\, M_\alpha}{\Gamma(J\alpha+1)},$$

being $M_\alpha \ge |\mathcal{D}_t^{(J\alpha)} u(t)|$, $t \in [0,1]$.

3. Generalized Bessel collocation approach

Let us denote by $u_{J,\alpha}(x)$ the $(J+1)$-terms generalized Bessel polynomials series as an approximation to the solution $u(t)$ of the linear BVPs (1.1) on $[0,1]$. In the vectorized representation form, we may write

$$u(t) \approx u_{J,\alpha}(t) = \boldsymbol{\mathcal{T}}_\alpha(t)\,\boldsymbol{\mathcal{D}}\,\boldsymbol{\mathcal{A}}_J. \tag{3.1}$$

To proceed, the following collocation points are utilized

$$t_s = s/J, \quad s = 0, 1, \dots, J. \tag{3.2}$$

We place the foregoing collocation points into (3.1) to obtain the following system of matrix equations

$$u_{J,\alpha}(t_s) = \boldsymbol{\mathcal{T}}_\alpha(t_s)\,\boldsymbol{\mathcal{D}}\,\boldsymbol{\mathcal{A}}_J, \quad s = 0, 1, \dots, J.$$

In a compact form, we may represent the preceding equations as

$$\boldsymbol{U} = \boldsymbol{T}\,\boldsymbol{\mathcal{D}}\,\boldsymbol{\mathcal{A}}_J, \quad \boldsymbol{U} = \begin{pmatrix} u_{J,\alpha}(t_0) \\ u_{J,\alpha}(t_1) \\ \vdots \\ u_{J,\alpha}(t_J) \end{pmatrix}, \quad \boldsymbol{T} = \begin{pmatrix} \boldsymbol{\mathcal{T}}_\alpha(t_0) \\ \boldsymbol{\mathcal{T}}_\alpha(t_1) \\ \vdots \\ \boldsymbol{\mathcal{T}}_\alpha(t_J) \end{pmatrix}. \tag{3.3}$$

According to (3.1) and using the properties (2.2) and (2.3), the fractional derivative of order σ will be determined

$$\mathcal{D}_t^{(\sigma)} u_{J,\alpha}(t) = \boldsymbol{\mathcal{T}}_\alpha^{(\sigma)}(t)\, \boldsymbol{\mathcal{D}}\, \boldsymbol{\mathcal{A}}_J, \tag{3.4}$$

where

$$\boldsymbol{\mathcal{T}}_\alpha^{(\sigma)}(t) := \left(\mathcal{D}_t^{(\sigma)} \boldsymbol{\mathcal{T}}_\alpha(t)\right) = [0 \quad \mathcal{D}_t^{(\sigma)} t^\alpha \quad \dots \quad \mathcal{D}_t^{(\sigma)} t^{\alpha J}].$$

Similarly, we replace the collocation points (3.2) into (3.4) to obtain the matrix representation

$$\boldsymbol{U}^{(\sigma)} = \boldsymbol{T}^{(\sigma)}\, \boldsymbol{\mathcal{D}}\, \boldsymbol{\mathcal{A}}_J, \quad \boldsymbol{U}^{(\sigma)} = \begin{pmatrix} \mathcal{D}_t^{(\sigma)} u_{J,\alpha}(t_0) \\ \mathcal{D}_t^{(\sigma)} u_{J,\alpha}(t_1) \\ \vdots \\ \mathcal{D}_t^{(\sigma)} u_{J,\alpha}(t_J) \end{pmatrix}, \quad \boldsymbol{T}^{(\sigma)} = \begin{pmatrix} \boldsymbol{\mathcal{T}}_\alpha^{(\sigma)}(t_0) \\ \boldsymbol{\mathcal{T}}_\alpha^{(\sigma)}(t_1) \\ \vdots \\ \boldsymbol{\mathcal{T}}_\alpha^{(\sigma)}(t_J) \end{pmatrix}. \tag{3.5}$$

Next aim is to establish a connection between $u_{J,\alpha}(t)$ and its first derivative in (3.1). Accordingly, we need to compute $\frac{d}{dt}\boldsymbol{\mathcal{T}}_\alpha(t)$. To this end, we use the properties (2.2)-(2.3) with $\sigma = 1$. For illustration, we pick $J = 8$ and $\alpha = 1/4$ to have

$$\boldsymbol{\mathcal{T}}_{\frac{1}{4}}(t) = \begin{bmatrix} 1 & t^{1/4} & t^{1/2} & t^{3/4} & t & t^{5/4} & t^{3/2} & t^{7/4} & t^2 \end{bmatrix}.$$

An straightforward differentiation with respect to t yields

$$\frac{d}{dt}\boldsymbol{\mathcal{T}}_{\frac{1}{4}}(t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & \frac{5}{4}t^{1/4} & \frac{3}{2}t^{1/2} & \frac{7}{4}t^{3/4} & 2t \end{bmatrix}.$$

Hence, it suffices to define

$$\boldsymbol{\mathcal{T}}_\alpha^{(1)}(t) := \frac{d}{dt}\boldsymbol{\mathcal{T}}_\alpha(t),$$

and applying the differentiation to the relation (3.1) to get

$$\frac{d}{dt} u_{J,\alpha}(t) = \boldsymbol{\mathcal{T}}_\alpha^{(1)}(t)\, \boldsymbol{\mathcal{D}}\, \boldsymbol{\mathcal{A}}_J. \tag{3.6}$$

If one substitutes the collocation points (3.2) into (3.6), we get the following matrix expression

$$\boldsymbol{U}^{(1)} = \boldsymbol{T}^{(1)}\, \boldsymbol{\mathcal{D}}\, \boldsymbol{\mathcal{A}}_J, \quad \boldsymbol{U}^{(1)} = \begin{bmatrix} u'_{J,\alpha}(t_0) \\ u'_{J,\alpha}(t_1) \\ \vdots \\ u'_{J,\alpha}(t_J) \end{bmatrix}, \quad \boldsymbol{T}^{(1)} = \begin{bmatrix} \boldsymbol{\mathcal{T}}_\alpha^{(1)}(t_0) \\ \boldsymbol{\mathcal{T}}_\alpha^{(1)}(t_1) \\ \vdots \\ \boldsymbol{\mathcal{T}}_\alpha^{(1)}(t_J) \end{bmatrix}. \tag{3.7}$$

We now are able to compute the generalized Bessel solutions of (1.1). In this respect, we insert the collocation points into the fractional BVPs (1.1) to get the equations

$$\mathcal{D}_t^{(\sigma)} u(t_s) + a(t_s)\, u'(t_s) + b(t_s)\, u(t_s) = f(t_s), \quad s = 0, 1, \dots, J.$$

Expressing the preceding equations in the matrix representation form, we have

$$\boldsymbol{U}^{(\sigma)} + \boldsymbol{A}\boldsymbol{U}^{(1)} + \boldsymbol{B}\boldsymbol{U} = \boldsymbol{F}. \tag{3.8}$$

Here, the matrices $\boldsymbol{A}$, $\boldsymbol{B}$, as well as the right-hand side vector $\boldsymbol{F}$ have the following representations

$$\boldsymbol{A} = \begin{pmatrix} a(t_0) & 0 & \dots & 0 \\ 0 & a(t_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a(t_J) \end{pmatrix}, \boldsymbol{B} = \begin{pmatrix} b(t_0) & 0 & \dots & 0 \\ 0 & b(t_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b(t_J) \end{pmatrix}, \boldsymbol{F} = \begin{pmatrix} f(t_0) \\ f(t_1) \\ \vdots \\ f(t_J) \end{pmatrix}.$$

By using the relations (3.3), (3.5), and (3.7) and putting them into (3.8), the following fundamental matrix equation will be obtained

$$\boldsymbol{X}\,\boldsymbol{\mathcal{A}}_J = \boldsymbol{F}, \quad \text{or,} \quad [\boldsymbol{X}; \boldsymbol{F}], \tag{3.9}$$

where

$$\boldsymbol{X} := \Big(\boldsymbol{T}^{(\sigma)} + \boldsymbol{A}\boldsymbol{T}^{(1)} + \boldsymbol{B}\boldsymbol{T}\Big)\boldsymbol{\mathcal{D}}.$$

Evidently, the relation (3.9) is a linear matrix equation to be solved for the vector of unknowns $\boldsymbol{\mathcal{A}}_J$ as the Bessel coefficients.

It remains to be considered the boundary conditions (1.2) into the former matrix equation. For the first condition $u(0) = u_0$, by approaching $t \to 0$ in (3.1) we arrive at the following matrix representation

$$\widehat{\boldsymbol{X}}_0\,\boldsymbol{\mathcal{A}}_J = u_0, \qquad \widehat{\boldsymbol{X}}_0 := \boldsymbol{\mathcal{T}}_\alpha(0)\,\boldsymbol{\mathcal{D}} = [1 \quad 1 \quad \dots \quad 1].$$

Analogously, for the end condition $u(1) = \sum_{j=1}^{\ell} c_j\, u(\zeta_j)$ we obtain the matrix expression

$$\widehat{\boldsymbol{X}}_1\,\boldsymbol{\mathcal{A}}_J = 0, \qquad \widehat{\boldsymbol{X}}_1 := \left(\boldsymbol{\mathcal{T}}_\alpha(1) - \sum_{j=1}^{\ell} c_j \boldsymbol{\mathcal{T}}_\alpha(\zeta_j)\right)\boldsymbol{\mathcal{D}} = [\hat{x}_{1,0} \quad \hat{x}_{1,1} \quad \dots \quad \hat{x}_{1,J}].$$

In consequence, we substitute the first row as well as the last row of the augmented matrix $[\boldsymbol{X}; \boldsymbol{F}]$ by the row matrices $[\widehat{\boldsymbol{X}}_0; u_0]$ and $[\widehat{\boldsymbol{X}}_1; 0]$. Finally, we arrive at the following modified augmented system

$$\Big[\widehat{\boldsymbol{X}}; \widehat{\boldsymbol{F}}\Big] = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 & ; & u_0 \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & \dots & x_{1,J} & ; & f(t_1) \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & \dots & x_{2,J} & ; & f(t_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & ; & \vdots \\ x_{J-1,0} & x_{J-1,1} & w_{J-1,2} & w_{J-1,3} & \dots & w_{J-1,J} & ; & f(t_{J-1}) \\ \hat{x}_{1,0} & \hat{x}_{1,1} & \hat{x}_{1,2} & \hat{x}_{1,3} & \dots & \hat{x}_{1,J} & ; & 0 \end{bmatrix}. \tag{3.10}$$

Therefore, the generalized Bessel coefficients in (3.1) will be determined via solving this linear system of equations. For this purpose, any classical linear solver can be used.

4. Illustrative Examples

Let us describe the efficiency of the presented generalized Bessel collocation approach. To this end, some numerical examples are given and comparisons are made with the results of other existing methods. MATLAB R2017a has been used for numerical simulations in this work.

Example 4.1. As the first example, we consider the nonlocal fractional boundary value problem [5, 17, 19, 33]

$$\mathcal{D}_t^{(1.3)} u(t) + \cos(t)\, u'(t) + 2\, u(t) = f(t), \quad 0 \leq t \leq 1,$$

with boundary conditions $u_0 = 0$ and $u(1) = u(\frac{1}{8}) + 2u(\frac{1}{2}) + \frac{31}{49} u(\frac{7}{8})$. The function $f(t)$ is given such that the exact solution of this BVPs is $u(t) = t^2$.

To find an approximate solution in the form $u_{J,\alpha}(t) = \sum_{j=0}^{J} \alpha_j \mathrm{B}_j^{\alpha}(t)$, we take $J = 2$ and $\alpha = 1$. We need to compute the unknown coefficients α_0, α_1, and α_2. The used collocation points are $\left\{x_0 = 0, x_1 = \frac{1}{2}, x_3 = 1\right\}$. Applying the proposed approach with $\sigma = 1.3$, the corresponding vectors and matrices in the fundamental matrix equation (3.10) are obtained as

$$\boldsymbol{\mathcal{D}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{bmatrix}, \boldsymbol{T}^{(1.3)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1401/1034 \\ 0 & 0 & 1609/731 \end{bmatrix}, \boldsymbol{T}^{(1)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix},$$

$$\boldsymbol{F} = \begin{bmatrix} 0 \\ 2227/815 \\ 8081/1530 \end{bmatrix}, \boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1699/1936 & 0 \\ 0 & 0 & 429/794 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

$$\boldsymbol{T} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1/2 & 1/4 \\ 1 & 1 & 1 \end{bmatrix}, \left[\widehat{\boldsymbol{X}}; \widehat{\boldsymbol{F}}\right] = \begin{bmatrix} 1 & 1 & 1 & ; & 0 \\ 2 & 2439/629 & 2612/165 & ; & \frac{2227}{815} \\ -129/49 & -649/190 & -915/196 & ; & 0 \end{bmatrix}.$$

Once we solve the linear system $\left[\widehat{\boldsymbol{X}}; \widehat{\boldsymbol{F}}\right]$, the unknown coefficients matrix will be found as

$$\boldsymbol{\mathcal{A}} = [2/3 \quad -1 \quad 1/3]^t.$$

Hence, we get the approximate solution as

$$u_{2,1}(t) = \begin{bmatrix} 1 & 1+t & 3t^2 + 3t + 1 \end{bmatrix} \boldsymbol{\mathcal{A}} = t^2,$$

which is evidently the true exact solution. The numerical solutions and the absolute errors

$$e_{J,\alpha}(t) := |u(t) - u_{J,\alpha}(t)|,$$

at some points $t \in (0, 1)$ are reported in Table 1. Note that this example was solved by a reproducing kernel method (RKM) [5], the improved RKM (IKRM) [19], the shited Jacobi operational matrices method (SJOMM) [17], and the Legendre wavelet method (LWM) [33]. Comparisons of absolute errors between our results and these approximation schemes are further shown in Table 1. It can be clearly seen that our approach with less computational efforts is considerably more accurate than the RKM, IRKM, SJOMM, and LWM.

Example 4.2. In this test example, let us consider the nonlocal fractional boundary value problem [5]

$$\mathcal{D}_t^{(1.6)} u(t) + \sinh(t)\, u(t) = f(t), \quad 0 \leq t \leq 1,$$

Table 1. Comparison of numerical solutions and absolute errors in Example 4.1 for $\sigma = 1.3, \alpha = 1$, and $J = 2$ for various $t \in [0, 1]$.

	Bessel		RKM [5]	IRKM [16]	JOMM [14]	LWM [30]
t	$u_{2,1}(t)$	$e_{2,1}(t)$	$M = 10$	$M = 10$	$M = 20$	$k = 2, M = 4$
0.1	0.01	0	1.11×10^{-7}	3.09×10^{-08}	3.09×10^{-08}	3.80×10^{-18}
0.2	0.04	0	2.87×10^{-7}	1.69×10^{-08}	1.69×10^{-08}	6.49×10^{-18}
0.3	0.09	0	5.51×10^{-7}	3.55×10^{-09}	3.55×10^{-09}	3.74×10^{-19}
0.4	0.16	0	9.23×10^{-7}	1.89×10^{-09}	1.89×10^{-09}	1.82×10^{-17}
0.5	0.25	0	1.41×10^{-6}	2.94×10^{-10}	2.94×10^{-10}	3.15×10^{-17}
0.6	0.36	0	2.03×10^{-6}	8.47×10^{-09}	8.47×10^{-09}	3.73×10^{-17}
0.7	0.49	0	2.77×10^{-6}	2.07×10^{-08}	2.07×10^{-08}	5.63×10^{-17}
0.8	0.64	0	3.64×10^{-6}	3.64×10^{-08}	3.64×10^{-08}	8.30×10^{-17}
0.9	0.81	0	4.62×10^{-6}	5.23×10^{-08}	5.23×10^{-08}	1.11×10^{-16}

where $f(t) = \frac{\Gamma(3)}{\Gamma(1.4)} t^{0.4} + \frac{\Gamma(4)}{\Gamma(2.4)} t^{1.4} + \sinh(t)(t^2 + t^3)$. In this case, the boundary conditions are

$$u(0) = 0, \quad u(1) = u\left(\frac{1}{10}\right) + u\left(\frac{1}{2}\right) + \frac{538}{513} u\left(\frac{9}{10}\right).$$

One can easily checked the exact solution of this model problem is $u(t) = t^3 + t^2$.

For this example we take $J = 3$, which is sufficient to get the desired approximation. Using $\alpha = 1$, the approximate solution obtained via Bessel collocation method on $0 \leq t \leq 1$ is as follows

$$u_{3,1}(t) = 1.0\, t^3 + 1.0\, t^2 - 7.648870818 \times 10^{-17}\, t + 4.230993657 \times 10^{-109}.$$

The graphs of the preceding approximation along with the exact solutions are plotted in Fig. (1). In addition, the corresponding absolute errors are also shown in this figure for $0 \leq t \leq 1$. The numerical results obtained by using the Bessel collocation

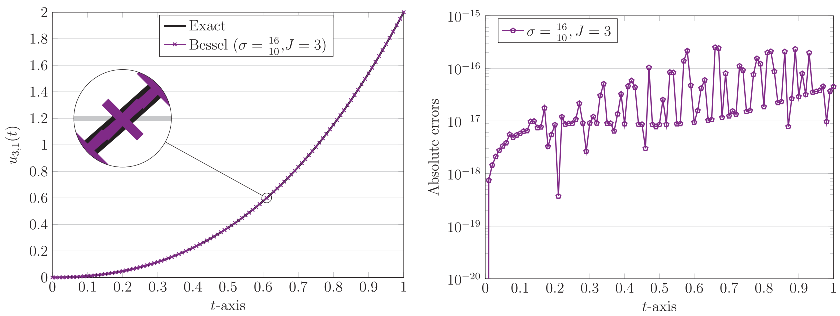

Figure 1. Comparison of approximated and exact solutions using Bessel functions (left) and the corresponding absolute errors (right) for Example 4.2 with $\sigma = 1.6$, $\alpha = 1$, and $J = 3$.

scheme at some points $t \in [0, 1]$ for Example (4.2) are presented in Table 2. The corresponding absolute errors are also reported in Table 2. Additionally, we emphasize that numerical approximated solutions for this model problem using RKM were proposed in [5] with achieved absolute errors larger than 1×10^{-6}, see Fig. 2 in this paper. In Table 2, we further present the numerical solutions and absolute errors for $= 2, 3$ evaluated at some points $t \in [0, 1]$. For $J = 2$, the approximate solution obtained via Bessel collocation method is

$$u_{2,1}(t) = 2.090573501\, t^2 - 0.4208122144\, t.$$

Table 2. Comparison of numerical solutions and absolute errors in Example 4.2 for $\sigma = 1.6, \alpha = 1$, and $J = 2, 3$ for various $t \in [0, 1]$.

t	$u_{2,1}(t)$	$e_{2,1}(t)$	$u_{3,1}(t)$	$e_{3,1}(t)$
0.1	−0.021175486429912	3.21755×10^{-2}	0.01100	5.7650×10^{-18}
0.2	−0.000539502838979	4.85395×10^{-2}	0.04800	8.4590×10^{-18}
0.3	0.061907950772799	5.50920×10^{-2}	0.11700	9.1272×10^{-18}
0.4	0.166166874405420	5.78331×10^{-2}	0.22400	8.8145×10^{-18}
0.5	0.312237268058887	6.27627×10^{-2}	0.37500	8.5663×10^{-18}
0.6	0.500119131733197	7.58809×10^{-2}	0.57600	9.4275×10^{-18}
0.7	0.729812465428353	1.03188×10^{-1}	0.83300	1.2443×10^{-17}
0.8	1.001317269144352	1.50683×10^{-1}	1.15200	1.8659×10^{-17}
0.9	1.314633542881196	2.24367×10^{-1}	1.53900	2.9119×10^{-17}

Example 4.3. Let us consider the following boundary value problem with three ponits boundary conditions [19, 24]

$$\mathcal{D}_t^{(1.5)} u(t) + \frac{\exp(-3\pi)}{\sqrt{\pi}}\, u(t) = f(t), \quad 0 \le t \le 1,$$

with the right-hand side

$$f(t) = \frac{\sqrt{t}}{\sqrt{\pi}} \left(\frac{128}{7} t^3 - \frac{74}{5} t + \frac{33}{10} \right) + \frac{exp(-3\pi)}{40\sqrt{\pi}} \left(40t^5 - 74t^3 + 33t^2 \right),$$

and also with boundary conditions

$$u(0) = 0, \quad u(1) + \frac{625}{596}\, u(\frac{2}{5}) = 0.$$

It can be verified that the exact solutions is $u(t) = t^5 - \frac{37}{20} t^3 + \frac{33}{40} t^2$.

In the third test case, we set $J = 5$ as the number of basis functions. The parameter $\alpha = 1$ is also sufficient to get the desired approximation. The approximate solution $u_{5,1}(t)$ of this model problem using Bessel basis functions in the interval $0 \le t \le 1$ is obtained as follows:

$$u_{5,1}(t) = 1.0\, t^5 - 5.393237855 \times 10^{-15}\, t^4 - 1.85\, t^3 + 0.825\, t^2$$

$$+ 3.122109065 \times 10^{-16}\, t,$$

which coincide with the true exact solution up to machine epsilon. Graphical representations of the above approximated solution and its related absolute error function are visualized in Fig. 2. To validate our results, we also plot the exact solution, which represented by a thick line. Clearly, our obtained approximated solution is very close to the exact one.

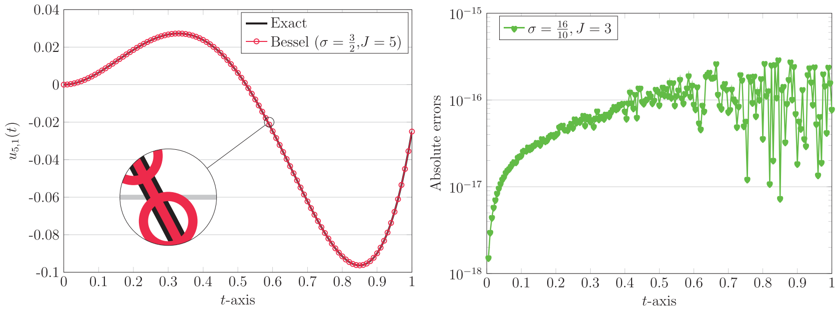

FIGURE 2. Comparison of approximated and exact solutions using Bessel functions (left) and the corresponding absolute errors (right) for Example 4.3 with $\sigma = 3/2$, $\alpha = 1$, and $J = 5$.

In Table 3, we report the numerical results as well as the absolute errors correspond to $J = 5$ obtained by the Bessel collocation procedure using $\sigma = 3/2$ and $\alpha = 1$ at some points $t \in [0, 1]$. A comparison in this table is also made with the IRKM [19] and the Haar wavelet method (HWM) from [24]. As one can see from Table 3 that the results obtained by our proposed scheme are superior in terms of accuracy compared to the IRKM and HWM.

Example 4.4. We consider an example with nonlocal intergral boundary conditions

$$\mathcal{D}_t^{(\sigma)} u(t) + \sinh(t)\, u'(t) + 2\, u(t) = 2t\, \sinh(t) + 2t^2 + \frac{\Gamma(3)}{\Gamma(3-\sigma)} t^{2-\sigma}, \quad 0 \le t \le 1,$$

with boundary conditions

$$u(\frac{1}{2}) = 6 \int_0^{\frac{1}{2}} u(s)ds, \quad u(1) = 4 \int_0^1 s u(s) ds.$$

One can check that the exact solutions is given by $u(t) = t^2$ for any $1 < \sigma \le 2$. This model problem with $\sigma = 1.3$ was considered in [19].

As for the previously solved test problems, we also consider $\alpha = 1$ here. For $\sigma = 1.3$ and using $J = 2$, we use the Bessel collocation to obtain the approximate solution $u_{2,1}(t)$. In this case, we get for $0 \le t \le 1$

$$u_{2,1}(t) = 1.0\, t^2 + 6.661338148 \times 10^{-17}\, t - 2.220446049 \times 10^{-17}.$$

As mentioned, this problem was previously solved via IRKM [19]. Referring to Fig. 1 in [19], the achieved absolute errors are larger than 1×10^{-8}. The same results for other values of σ will be obtained, which are similar up to machine epsilon. The effect of using different values of fractional orders $\sigma = 1.1, 1.3, 1.5, 1.7$, and $\sigma = 1.9$

TABLE 3. Comparison of numerical solutions and absolute errors in Example 4.3 for $\sigma = 1.5, \alpha = 1$, and $J = 5$ for various $t \in [0, 1]$.

	Bessel		IRKM [16]	HWM [21]
t	$u_{2,1}(t)$	$e_{2,1}(t)$	$M = 10$	$m = 256$
0.1	0.00641	2.31719×10^{-17}	8.74×10^{-7}	2.41789×10^{-10}
0.2	0.01852	3.98708×10^{-17}	7.12×10^{-7}	3.59522×10^{-10}
0.3	0.02673	5.72864×10^{-17}	9.52×10^{-7}	8.06354×10^{-10}
0.4	0.02384	7.51296×10^{-17}	1.31×10^{-6}	6.64016×10^{-10}
0.5	0.00625	8.83647×10^{-17}	1.61×10^{-6}	6.69882×10^{-10}
0.6	-0.02484	8.99422×10^{-17}	1.87×10^{-6}	1.41570×10^{-09}
0.7	-0.06223	7.35311×10^{-17}	2.13×10^{-6}	7.94582×10^{-10}
0.8	-0.09152	3.62512×10^{-17}	2.39×10^{-6}	8.23084×10^{-10}
0.9	-0.08991	1.85939×10^{-17}	2.15×10^{-6}	5.19118×10^{-10}

are depicted in Fig. 3. From Fig. 3 one infers that the same accuracies are achieved when using different σ in the range $[0, 1]$. The numerical solutions as well as the

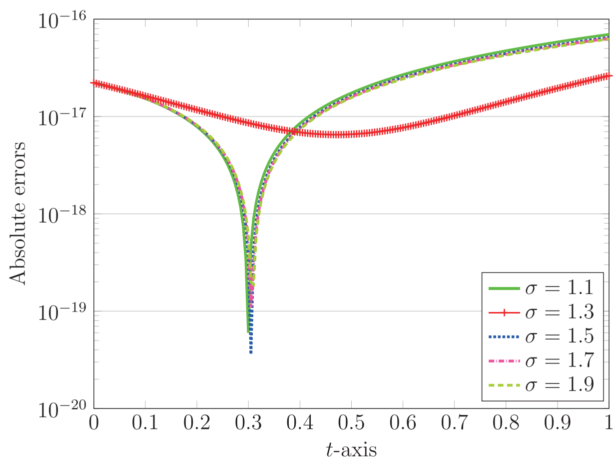

FIGURE 3. The absolute errors $e_{2,1}(t)$ for Example 4.4 using $J = 5, \alpha = 1$ for different $\sigma = 1.1, 1.3, 1.5, 1.7$, and $\sigma = 1.9$.

corresponding absolute errors for $J = 2$ and for $\sigma = 1.3$, $\sigma = 1.7$ at some points $t \in [0, 1]$ are tabulated in Table 3.

In the last experiment, we show that how using the fractional version of Bessel polynomials gains us. In this respect, we construct an nonlocal BVPs which has a fractional solution.

Table 4. Comparison of numerical solutions and absolute errors in Example 4.4 for $\sigma = 1.3, 1.7$, and $\alpha = 1$, and $J = 2, 3$ for various $t \in [0, 1]$.

	$\sigma = 1.3$		$\sigma = 1.7$		
t	$u_{2,1}(t)$	$e_{2,1}(t)$	$u_{3,1}(t)$	$e_{3,1}(t)$	Exact
0.1	0.01	1.6250×10^{-17}	0.01	1.5352×10^{-17}	0.01
0.2	0.04	1.1709×10^{-17}	0.04	8.1174×10^{-18}	0.04
0.3	0.09	8.5817×10^{-18}	0.09	5.0048×10^{-19}	0.09
0.4	0.16	6.8680×10^{-18}	0.16	7.4986×10^{-18}	0.16
0.5	0.25	6.5679×10^{-18}	0.25	1.5880×10^{-17}	0.25
0.6	0.36	7.6814×10^{-18}	0.36	2.4643×10^{-17}	0.36
0.7	0.49	1.0209×10^{-17}	0.49	3.3789×10^{-17}	0.49
0.8	0.64	1.4149×10^{-17}	0.64	4.3317×10^{-17}	0.64
0.9	0.81	1.9504×10^{-17}	0.81	5.3227×10^{-17}	0.81

Example 4.5. We finally consider the following nonlocal BVPs

$$\mathcal{D}_t^{(\sigma)} u(t) - \frac{2}{5}\, t\, u'(t) + u(t) = \frac{\Gamma(7/2)}{\Gamma(7/2-\sigma)} t^{5/2-\sigma}, \quad 0 \le t \le 1.$$

The boundary conditions are

$$u(0) = 0 \quad u(1) = u(\frac{1}{4}) + \frac{31\sqrt{3}}{27} u(\frac{3}{4}).$$

For any $1 < \sigma \le 2$, it can be shown that the exact solution is $u(t) = t^{\frac{5}{2}}$.

To start, we fix $J = 5$ and use two values of $\alpha = 1$ and $\alpha = 1/2$. By using these values, we will investigate the difference between fractional and non-fractional Bessel basis functions. By setting $\sigma = 3/2$, the approximate solutions $u_{5,1}(t)$ and $u_{5,\frac{1}{2}}(t)$ obtained via (3.10) of the model (1.1) in the interval $[0, 1]$ are as follows

$$\begin{aligned} u_{5,1}(t) &= 0.13817464199996248423\, t^5 - 0.48410906426886896909\, t^4 \\ &+ 1.0583665387779764119\, t^3 + 0.29168521645037729056\, t^2 \\ &- 0.0049927313066897211141\, t + 2.6612249000050941999 \times 10^{-109}, \end{aligned}$$

and

$$\begin{aligned} u_{5,\frac{1}{2}}(t) &= 5.2534708506020563582 \times 10^{-105}\, t + 1.70999667174727 \times 10^{-105}\, t^2 \\ &- 3.4157353836545 \times 10^{-105}\, t^{1/2} - 5.790825382411 \times 10^{-107}\, t^{3/2} + 1.0\, t^{5/2}. \end{aligned}$$

Clearly, utilizing the generalized Bessel basis functions yields to a considerable more accurate solution compared to the case $\alpha = 1$. This fact can be further confirmed in the next two Fig. 4 and Fig. 5, in which we plot these approximations with the corresponding absolute errors $e_{5,\alpha}(t)$ for $\alpha = 1, 1/2$. Moreover, the numerical solutions for $\alpha = 1, 1/2$ at some points $t \in [0, 1]$ are shown in Table 5. The corresponding exact solutions as well as the absolute errors are also reported in Table 5.

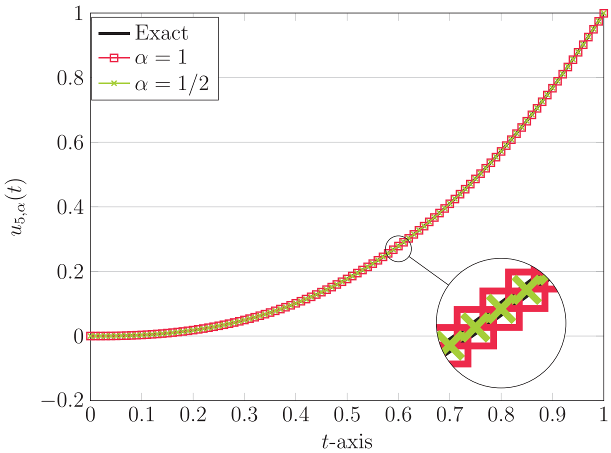

FIGURE 4. The approximated Bessel series solutions $u_{5,\alpha}(t)$ for Example 4.5 using $\sigma = 3/2$ for two different $\alpha = 1$ and $\alpha = 1/2$.

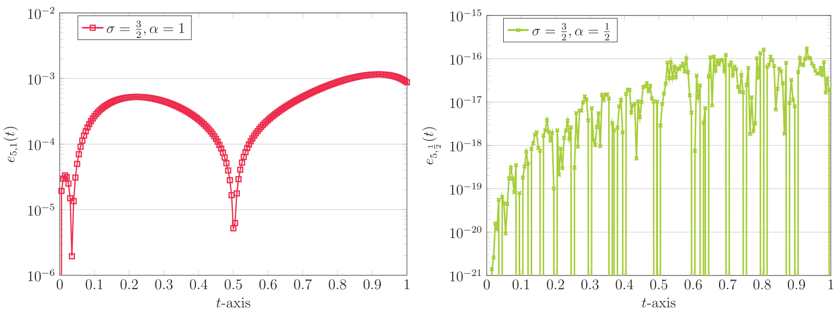

FIGURE 5. Comparison of absolute errors using $\alpha = 1$ (left) and $\alpha = 1/2$ (right) for Example 4.5 with $\sigma = 1/2$ and $J = 5$.

On the other hand, with lower number of basis functions is also possible to get a comparable accuracy as for $J = 5$ and $\alpha = 1/2$. For example, when using $J = 3$ and $\alpha = 5/6$, the following approximative solution is obtained

$$u_{3,\frac{5}{6}}(t) = 1.0\, t^{5/2} + 8.5159196800163014398 \times 10^{-108}\, t^{5/3} \\ + 1.7031839360032602 88 \times 10^{-108}\, t^{5/6},$$

or even with a lower $J = 2$ and $\alpha = 5/4$, we have the following approximation

$$u_{2,\frac{5}{4}}(t) = 1.0\, t^{5/2} + 6.8127357440130411519 \times 10^{-108}\, t^{5/4}.$$

As one see that both latter approximate solutions are in excellent alignment with the exact solution.

Table 5. Comparison of numerical solutions and absolute errors in Example 4.5 for $\sigma = 3/2$ and different $\alpha = 1, 1/2,$, and $J = 5$ for various $t \in [0, 1]$.

	$J=5, \alpha=1$		$J=5, \alpha=1/2$		
t	$u_{5,1}(t)$	$e_{5,1}(t)$	$u_{5,\frac{1}{2}}(t)$	$e_{5,\frac{1}{2}}(t)$	Exact
0.1	0.003428916413	2.67_{-4}	0.003162277660168	0	0.003162277660168
0.2	0.018405436090	5.17_{-4}	0.017888543819998	0	0.017888543819998
0.3	0.049744227595	4.49_{-4}	0.049295030175465	0	0.049295030175465
0.4	0.101429716880	2.37_{-4}	0.101192885125388	0	0.101192885125388
0.5	0.176781896852	5.20_{-6}	0.176776695296637	0	0.176776695296637
0.6	0.278622136947	2.33_{-4}	0.278854800926934	0	0.278854800926934
0.7	0.409438992697	5.24_{-4}	0.409963413001697	0	0.409963413001697
0.8	0.571554015303	8.79_{-4}	0.572433402239946	0	0.572433402239946
0.9	0.767287561206	1.15_{-3}	0.768433471420916	0	0.768433471420916

5. Conclusions

An accurate approximation algorithm based on generalized Bessel functions was developed for the solution of fractional-order differential equation under nonlocal boundary conditions. Utilizing the (fractional) Bessel functions with together the collocation points, the underlying differential equations is reduced into an algebraic system of linear equations. Illustrative examples were given to demonstrate the efficiency and accuracy of the proposed method and a comparison between the method and other existing schemes has been performed. Moreover, the performance of fractional and non-fractional basis functions has been assessed. From Figures and Tables, one can conclude that the present approximative technique is not only straightforward in implementation but also accurate and powerful tool for obtaining the approximate solutions of nonlocal FBVPs compared to the numerical results of other existing well-known numerical methods. The method can be easily extended to the solutions of higher-order nonlocal FBVPs and systems appearing in the modelling of many problems in science and engineering fields.

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Manuscript received September 20 2020
revised October 20 2020

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