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A VARIATIONAL PRINCIPLE TO A FRACTAL HUNTER-SAXTON EQUATION

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ABSTRACT. Hunter-Saxton equation models the evolution of crystal liquid, but it can not take into account the crystal geometry. A fractal modification is given using the two-scale fractal calculus, and two variational principles are established for the fractal Hunter-Saxton equation by the semi-inverse method, and its analytical solution is obtained by Ritz method.

1. Introduction

The Hunter-Saxton equation can be written in the form [2]:

$$u_{xt} + uu_{xx} + \frac{1}{2}u_x^2 = 0, \tag{1.1}$$

Eq.(1.1) was introduced in Ref. [13] as a simplified model to describe the evolution of crystal liquid, which shares some similar mechanical properties of a fluid, so the differential model given in Eq.(1.1) was widely used[1,2,3]. Arbabi et al. adopted the Haar wavelet quasilinearization approach to obtain a numerical solution of Eq.(1.1). Though a crystal liquid can be considered as an approximate continuum, Eq.(1.1) cannot model the effect of the crystal geometry on its evolution, the crystal liquid is discontinuous on a moelecule scale, so a fractal modification of Eq.(1.1) has to be considered, and its solution process will be elucidated by a variational approach.

2. Fractal liquid crystal and fractal Hunter-Saxton equation

The two-scale fractal calculus uses two scales to study the same problem, the large scale leads to a differential model, while a smaller scale, saying a molecule scale, results in a fractal model [1, 5, 12, 14].

To elucidate the fractal properties of the Hunter-Saxton equation, we give a simple example. We consider just the growth of a tree, and assume that the tree grows up only on the daytime, the zigzag growth can not be modelled by a differential model, and a fractal model has to be considered.

To consider the effect of the crystal structure on the evolution property, we have to use a molecule scale, and its fractal modification is,

$$\frac{\partial^2 u}{\partial x^\alpha \partial t^\beta} + u\frac{\partial^2 u}{\partial x^{2\alpha}} + \frac{1}{2}\left(\frac{\partial u}{\partial x^\alpha}\right)^2 = 0, \tag{2.1}$$

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The fractal derivatives are defined as [1, 5, 12, 14].

$$(2.2)\qquad \begin{cases} \frac{du}{dx^\alpha}(x_0,t) = \tau(1+\alpha)\lim_{x-x_0\to\Delta x,\Delta x\neq 0}\frac{x(x,t)-x(x_0,t)}{(x-x_0)^\alpha} \\ \frac{du}{dt^\beta}(x,t_0) = \tau(1+\beta)\lim_{t-t_0\to\Delta t,\Delta t\neq 0}\frac{u(x,t)-u(x,t_0)}{(t-t_0)^\beta}. \end{cases}$$

Where α, β are fractal dimensions in space and time, respectively. Δx is the crystal size beyond which all phenomena are ignored, Δt is the time response through Δx.

The two-scale fractal calculus can be widely applied to various porous problems or problems with unsmooth boundaries [4, 6, 11, 15–17, 21, 23, 24].

3. Fractal variational principle

Recently the fractal variational theory becomes a hot topic in both mathematics and engineering, much progress was made thanks to Yan Wang and colleagues [22], Kang-Le Wang and his colleagues [19, 20], Ji-Huan He and colleagues [7, 8, 18]. By the semi-inverse method [9, 10], we obtain the following two variational principles,

$$(3.1)\qquad J(u) = \int_{0^\beta}^{\infty^\beta}\left[\int_{0^\alpha}^{1^\alpha}\left(-\frac{1}{2}\frac{\partial u}{\partial x^\alpha}\frac{\partial u}{\partial t^\beta} + \frac{1}{4}\frac{\partial^2 u}{\partial x^{2\alpha}}u^2\right)dx^\alpha\right]dt^\beta$$

$$(3.2)\qquad J(u) = \int_{0^\beta}^{\infty^\beta}\left[\int_{0^\alpha}^{1^\alpha}\left(-\frac{1}{2}\frac{\partial u}{\partial x^\alpha}\frac{\partial u}{\partial t^\beta} - \frac{1}{2}\left(\frac{\partial u}{\partial x^\alpha}\right)^2 u\right)dx^\alpha\right]dt^\beta$$

Proof. The Euler-Lagrange equation can be expressed as,

$$(3.3)\qquad \frac{\partial L}{\partial u} - \frac{\partial}{\partial x^\alpha}\left(\frac{\partial L}{\partial u_{x^\alpha}}\right) - \frac{\partial}{\partial t^\beta}\left(\frac{\partial L}{\partial u_{t^\beta}}\right) + \frac{\partial}{\partial x^{2\alpha}}\left(\frac{\partial L}{\partial u_{x^{2\alpha}}}\right) + \frac{\partial}{\partial x^\alpha \partial t^\beta}\left(\frac{\partial L}{\partial u_{x^\alpha t^\beta}}\right) = 0.$$

where $u_{x^\alpha} = \frac{\partial u}{\partial x^\alpha}$, $u_{t^\beta} = \frac{\partial u}{\partial t^\beta}$, $u_{x^{2\alpha}} = \frac{\partial u}{\partial x^{2\alpha}}$, $u_{x^\alpha t^\beta} = \frac{\partial u}{\partial x^\alpha \partial t^\beta}$. For Eq.(3.1), the Lagrange function is,

$$(3.4)\qquad L = -\frac{1}{2}\frac{\partial u}{\partial x^\alpha}\frac{\partial u}{\partial t^\beta} + \frac{1}{4}\frac{\partial^2 u}{\partial x^{2\alpha}}u^2.$$

By Eq.(3.3) we have,

$$(3.5)\qquad \frac{1}{2}\frac{\partial^2 u}{\partial x^{2\alpha}}u + \frac{1}{2}\frac{\partial^2}{\partial x^\alpha}\left(\frac{\partial u}{\partial t^\beta}\right) + \frac{\partial}{\partial t^\beta}\left(\frac{\partial u}{\partial x^\alpha}\right) + \frac{1}{4}\frac{\partial^2}{\partial x^{2\alpha}}(u^2) = 0,$$

or

$$(3.6)\qquad \frac{1}{2}\frac{\partial^2 u}{\partial x^{2\alpha}}u + \frac{\partial^2 u}{\partial x^\alpha \partial t^\beta} + \frac{1}{2}\frac{\partial}{\partial x^\alpha}\left(u\frac{\partial u}{\partial x^\alpha}\right) = 0.$$

It is obvious that Eq.(3.6) is equivalent to Eq.(2.1). For Eq.(3.2), the Lagrange function is,

$$(3.7)\qquad L = -\frac{1}{2}\frac{\partial u}{\partial x^\alpha}\frac{\partial u}{\partial t^\beta} - \frac{1}{2}\left(\frac{\partial u}{\partial x^\alpha}\right)^2 u.$$

By Eq.(3.3) we have,

$$-\frac{1}{2}\left(\frac{\partial u}{\partial x^{\alpha}}\right)^{2}+\frac{1}{2}\frac{\partial}{\partial x^{\alpha}}\left(\frac{\partial u}{\partial t^{\beta}}\right)+\frac{1}{2}\frac{\partial}{\partial t^{\beta}}\left(\frac{\partial u}{\partial x^{\alpha}}\right)+\frac{\partial}{\partial x^{\alpha}}\left(\frac{\partial u}{\partial x^{\alpha}}u\right)=0, \tag{3.8}$$

Eq.(3.8) leads to Eq.(2.1) after simple calculation.

When $\alpha=\beta=1$ we have the following variational principles for Eq.(1.1):

$$J(u)=\int_0^{\infty}\left[\int_0^1\left(-\frac{1}{2}u_x u_t+\frac{1}{4}u_{xx}u^2\right)dx\right]dt, \tag{3.9}$$

$$J(u)=\int_0^{\infty}\left[\int_0^1\left(-\frac{1}{2}u_x u_t-\frac{1}{2}u_x^2 u\right)dx\right]dt. \tag{3.10}$$

Proof. Making Eq.(2.1) and Eq.(2.2) stationary, we have, respectively, the following Euler-Lagrange equations,

$$u_{xt}+\frac{1}{2}u_{xx}u+\frac{1}{4}(u^2)_{xx}=0. \tag{3.11}$$

$$u_{xt}-\frac{1}{2}u_x^2 u+(u_x u)_x=0. \tag{3.12}$$

It is obvious that Eq.(3.11) and Eq.(3.12) are equivalent to Eq.(1.1).

4. Ritz method

We can use the Ritz method to obtain an approximate solution for Eq.(2.1) based on either Eq.(3.1) or Eq.(3.2). To elucidate the solution process, we consider the following intial and boundary conditions[1]:

$$u(x^{\alpha},0^{\beta})=2x^{\alpha}. \tag{4.1}$$

$$\frac{\partial}{\partial t^{\beta}}u(x^{\alpha},0^{\beta})=-2x^{\alpha}. \tag{4.2}$$

$$u(0^{\alpha},t^{\beta})=0^{\alpha}. \tag{4.3}$$

$$u(1^{\alpha},t^{\beta})=\frac{2}{1+t^{\beta}} \tag{4.4}$$

We assume that the approximate solution is,

$$u(x^{\alpha},t^{\beta})=\sum_{i=0,j=0}^{i=M,j=N}a_{ij}x^{\alpha i}t^{\beta j}. \tag{4.5}$$

where a_{ij} are unknown constants to be further determined. Substituting Eq.(4.5) in Eq. (3.1) or Eq.(3.2), the stationary condition of Eq.(3.1) or Eq.(3.2) becomes,

$$\frac{\partial J}{\partial a_{ij}}=0. \tag{4.6}$$

Solving Eqs.(4.1)-(4.4) and (4.6) simultaneously, we can easily obtain the values of a_{ij}, For $M=5,N=7$, the obtained result is,

$$u(x^{\alpha},t^{\beta})=2x^{\alpha}(1-t^{\beta}+t^{2\phi}-t^{3\beta}+t^{4\beta}-t^{5\beta}+t^{6\beta}-t^{7\beta}). \tag{4.7}$$

When N tends to infinite, the obtained solution converges to the exact solution:

$$u(x^\alpha, t^\beta) = \frac{2x^\alpha}{1+t^\beta}. \tag{4.8}$$

5. Conclusion

The fractal Hunter-Saxton equation assumes that the evolution of crystal liquid is a function of x^α and t^β, instead of x and t. This paper gives a simple variational approach to the fractal model, and an approximate solution is obtained.

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