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PROPERTIES OF BILATERAL MOCK THETA FUNCTIONS OF GENERAL ORDER

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Dedicated to Prof H M Srivastava on his 80th birthday Anniversary

Abstract. Bilateral mock theta functions were obtained and studied in [22]. We express them in terms of Lerch's transcendental function $f(x, \xi; q, p)$. We also express some bilateral mock theta functions as sum of other mock theta functions. We generalize these functions and show that these generalizations are F_q functions. We give an integral representation for these generalized functions.

1. Introduction

The mock theta functions were first introduced by Ramanujan [17] in his last letter to G. H. Hardy in January 1920. He provided a list of seventeen mock theta functions and labelled them as of third, fifth and seventh order without mentioning the reason for his labelling. Watson [28] added to this set three more third order mock theta functions.

His general definition of a mock theta function is a function $f(q)$ defined by a q-series convergent when $|q| < 1$ which satisfies the following two conditions.

1. For every root ξ of unity, there exists a theta function[1] $\theta_\xi(q)$ such that the difference between $f(q)$ and $\theta_\xi(q)$ is bounded as $q \to \xi$ radially.
2. There is no single theta function which works for all ξ i.e. for every theta function $\theta_\xi(q)$ there is some root of unity ξ for which $f(q)$ minus the theta function $\theta_\xi(q)$ is unbounded as $q \to \xi$ radially.

Andrews and Hickerson [6] announced the existence of eleven more identities given in the ' Lost ' note book of Ramanujan involving seven new functions which they labelled as mock theta functions of order six. Y. S. Choi [8] has discovered four functions which he called the mock theta function of order ten. B. Gordon and R. J. McIntosh [12] have announced the existence of eight mock theta functions of order eight and R. J. McIntosh [15] has announced the existence of three mock theta functions of order two.

Hikami [9], [10] has introduced a mock theta function of order two, another of order four and two of order eight. Very recently Andrews [5] while studying q-orthogonal polynomials found four new mock theta functions and Bringmann et al [7] have also found two more new mock theta functions but they did not mention the order of their mock theta functions.

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[1] When Ramanujan refers to theta functions, he means sums, products, and quotients of series of the form $\sum_{n\in z} \epsilon^n q^{an^2+bn}$ with $a, b \in Q$ and $\epsilon = -1, 1$.

Watson [29] has defined four bilateral series, which he has called the 'Complete' or Bilateral forms for four of the ten mock theta functions of order five. Further he has expressed them in terms of the transcendental function $f(x,\xi;q,p)$ studied by M. Lerch [14]. S. D. Prasad [16] in 1970 has defined the 'Complete' or 'Bilateral' forms of the five generalized third order mock theta functions. The 'Complete' sixth order mock theta functions were studied by A. Gupta [13]. Bhaskar Srivastava [23–26] have studied bilateral mock theta functions of order five, eight, two and new mock theta functions by Andrews [5] and Bringmann et al [7].

Truesdell [27] calls the functions which satisfy the equation $\frac{\partial}{\partial z}F(z,\alpha) = F(z,\alpha+1)$ as F-functions. He has tried to unify the study of these F-functions. The function which satisfy the q-analogue of the equation $D_{q,z}F(z,\alpha) = F(z,\alpha+1)$ where $zD_{q,z}F(z,\alpha) = F(z,\alpha) - F(zq,\alpha)$ are called F_q-functions.

Shukla and Ahmad [18] to [22] and M Ahmad [1] to [3] have obtained and studied bilateral mock theta functions of differen orders.

The following eight bilateral mock theta functions of general order were studied in [22]

$$f_{0,c_r}(q) = \sum_{-\infty}^{\infty}(-1)^{rn}\frac{q^{\frac{r(n^2-n)}{2}}q^n}{(-q;q)_n} \tag{1.1}$$

$$f_{1,c_r}(q) = \sum_{-\infty}^{\infty}(-1)^{rn}\frac{q^{\frac{r(n^2-n)}{2}}q^{2n}}{(-q;q)_n} \tag{1.2}$$

$$F_{0,c_r}(q^2) = \sum_{-\infty}^{\infty}(-1)^{rn}\frac{q^{r(n^2-n)}q^{2n}}{(q;q^2)_n} \tag{1.3}$$

$$F_{1,c_r}(q^4) = \sum_{-\infty}^{\infty}(-1)^{rn}\frac{q^{r(2n^2-2n)}q^{8n}}{(q^6;q^4)_n} \tag{1.4}$$

$$\Psi_{0,c_r}(q) = \sum_{-\infty}^{\infty}(-1)^{rn}q^{(r-1)}\frac{n^2+3n}{2}(-q;q)_n \tag{1.5}$$

$$\Phi_{1,c_r}(q^2) = \sum_{-\infty}^{\infty}(-1)^{rn}q^{(r-1)(n^2+2n)}(-q;q^2)_n \tag{1.6}$$

$$\Phi_{0,c_r}(q^2) = \sum_{-\infty}^{\infty}(-1)^{rn}\frac{q^{rn^2}}{(-q;q^2)_n} \tag{1.7}$$

$$\Psi_{1,c_r}(q) = \sum_{-\infty}^{\infty}(-1)^{r(n+1)}\frac{q^{\frac{rn(n+1)}{2}}}{2(-q;q)_n} \tag{1.8}$$

The paper is divided as follows: In section 2 we list few important definitions. In section 3 we develop certain identities of these functions by expressing some of them as sums of other mock theta functions. In section 4 we express these functions in terms of the Lerch transcendental function $f(x, \xi; q, p)$. In section 5 we generalize these functions which are then proved to be F_q functions. We further give an integral representation of these functions in section 6.

2. Notation and Definitions

We use the following q-notation. Suppose q and z are complex numbers and n is an integer. If $n \geq 0$ we define

$$(z)_n = (z;q)_n = \prod_{i=0}^{n-1}(1-q^i z) \ if \ n \ \leq 0 \text{ and } (z)_{-n} = (z;q)_{-n} = \frac{(-z)^{-n} q^{\frac{n(n+1)}{2}}}{\left(\frac{q}{z};q\right)_n} \text{ and}$$

more generally $(z_1, z_2, \ldots, z_r; q)_n = (z_1)_n (z_2)_n \ldots (z_r)_n$.

For $\left|q^k\right| < 1$ let us define $(z;q^k)_n = (1-z)(1-zq^k)\ldots(1-zq^{k(n-1)})$, $n \geq 1$ $(z;q^k)_0 = 1$ and $(z;q^k)_\infty = \lim_{n\to\infty}(z;q^k)_n = \prod_{i\geq 0}(1-q^{ki}z)$ and even more generally,

$$(z_1, z_2 \ldots z_r; q^k)_\infty = (z_1;q^k)_\infty \ldots (z_r;q^k)_\infty$$

A basic hypergeometric series ${}_{r+1}\Phi_r$ on base q^k is defined as

(2.1)
$$ {}_{r+1}\Phi_r \left[\begin{matrix} a_1, a_2 & \ldots & a_{r+1} \\ b_1, b_2 & \ldots & b_r \end{matrix} \ ; \ q^k; z \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q^k)_n z^n}{(q^k;q^k)_n (b_1, b_2, \ldots b_r; q^k)_n}, (|z| < 1)$$

and a bilateral basic hypergeometric series ${}_r\Psi_r$ is defined as

$$ {}_r\Psi_r \left[\begin{matrix} a_1, & \ldots & a_r \\ b_1, & \ldots & b_r \end{matrix} \ ; q, z \right] = \sum_{n=-\infty}^{\infty} \frac{(a_1, \ldots, a_r; q)_n \ z^n}{(b_1 \ldots b_r; q)_n}, \left(\left| \frac{b_1 \ldots b_r}{a_1 \ldots a_r} \right| < |z| < 1 \right) \tag{2.2}$$

The Lerch transcendental function $f(x, \xi; q, p)$ is defined by:

$$f(x, \xi; q, p) = \sum_{-\infty}^{\infty} \frac{(pq)^{n^2} (x\xi)^{-2n}}{(-p\xi^{-2}; p^2)_n} \tag{2.3}$$

and by

$$f(x, \xi; q, p) = \sum_{-\infty}^{\infty} (-\xi^2 p; p^2)_n q^{n^2} x^{2n} \tag{2.4}$$

3. Certain identities

The following identities between the bilateral mock theta functions given in Equations 1.1,1.2,1.5,1.6,1.7 and the corresponding mock theta functions may be verified by hypergeometric transformations:

$$f_{0,c_r}(q) = f_{0,r}(q) + 2(-1)^r q^{(r-1)} \Psi_{0,r}(q) \tag{3.1}$$

$$f_{1,c_r}(q) = f_{1,r}(q) + 2(-1)^r q^{(r-2)} \Phi_{1,r}(q) \sum_{0}^{\infty} q^{\frac{(r-3)n}{2}} \tag{3.2}$$

$$\Phi_{0,c_r}(q^2) = \Phi_{0,r}(q^2) + (-1)^r q^{(r-1)} \Phi_{1,r}(q^2) \sum_{0}^{\infty} (1 + q^{2n+1}) \tag{3.3}$$

Here $f_{0,r}(q), f_{1,r}(q), \Psi_{0,r}(q), \Phi_{0,r}(q^2), \Phi_{1,r}(q^2)$ are the corresponding mock theta functions.

4. REPERSENTATION IN TERMS OF LERCH TRANSCENDENTAL FUNCTION

The bilateral mock theta functions defined in Section 1 can be expressed in terms of the Lerch transcendant by means of the following lemma.

Lemma 4.1. *For* $\epsilon = \pm 1$,

$$\sum_{n=-\infty}^{\infty} (-1)^{rn} \frac{q^{\alpha n^2} q^{\beta n}}{(\epsilon q^{\gamma}; q^{\delta})_n} = f(i^r (-\epsilon)^{-1/2} q^{\frac{2\gamma - 2\beta - \delta}{4}}, (-\epsilon)^{1/2} q^{\frac{\delta - 2\gamma}{4}}; q^{\frac{2\alpha - \delta}{2}}, q^{\frac{\delta}{2}}). \tag{4.1}$$

and

$$\sum_{n=-\infty}^{\infty} (-1)^{rn} (-q; q^{\gamma})_n q^{\alpha n^2} q^{\beta n} = f(i^r q^{\frac{\beta}{2}}, q^{\frac{2-\gamma}{4}}; q^{\alpha}, q^{\frac{\gamma}{2}}). \tag{4.2}$$

Proof. The proof follows from direct substitution and use of basic hypergeometric transformations. ☐

As an example we note that
$f_{0,c_r}(q) = \sum_{-\infty}^{\infty} (-1)^{rn} \frac{q^{\frac{rn^2}{2}} q^{(1-\frac{r}{2})n}}{(-q;q)_n} = f(i^r q^{(r-1)/4}, q^{-1/4}; q^{(r-1)/2}, q^{1/2})$ by taking $\alpha = r/2, \beta = 1 - r/2$, $\epsilon = -1$, $\gamma = \delta = 1$
and $\Psi_{0,c_r}(q) = \sum_{-\infty}^{\infty} (-1)^{rn} q^{(r-1)\frac{n^2+3n}{2}} (-q;q)_n = f(i^r q^{3(r-1)/4}, q^{1/4}; q^{(r-1)/2}, q^{1/2})$ by taking $\alpha = (r-1)/2, \beta = 3(r-1)/2$, $\gamma = 1$ in the above lemma. In this way all other bilateral mock theta functions defined by Equations 1.1 to 1.8 can be expressed in terms of the Lerch Transcendental function defined by equations (2.3) and (2.4).

5. GENERALIZATION OF THESE BILATERAL MOCK THETA FUNCTIONS

We generalize the functions given by Equations 1.1 to 1.8 by introducing two parameters α, z. For $\alpha = 1, z = 0$ these are reduced to the original functions.

$$f_{0,c_r}(z, \alpha; q) = \frac{1}{(z)_{\infty}} \sum_{-\infty}^{\infty} (-1)^{rn} \frac{(z)_n q^{\frac{rn^2}{2} + n\alpha - \frac{rn}{2}}}{(-q;q)_n} \tag{5.1}$$

$$f_{1,c_r}(z, \alpha; q) = \frac{1}{(z)_{\infty}} \sum_{-\infty}^{\infty} (-1)^{rn} \frac{(z)_n q^{\frac{rn^2}{2} + n\alpha + (1-r/2)n}}{(-q;q)_n} \tag{5.2}$$

$$F_{0,c_r}(z,\alpha;q^2) = \frac{1}{(z)_\infty}\sum_{-\infty}^{\infty}(-1)^{rn}\frac{(z)_n q^{rn^2+n\alpha+(1-r)n}}{(q;q^2)_n} \tag{5.3}$$

$$F_{1,c_r}(z,\alpha;q^4) = \frac{1}{(z)_\infty}\sum_{-\infty}^{\infty}(-1)^{rn}\frac{(z)_n q^{2rn^2+n\alpha+(7-2r)n}}{(q^6;q^4)_n} \tag{5.4}$$

$$\Psi_{0,c_r}(z,\alpha;q) = \frac{1}{(z)_\infty}\sum_{-\infty}^{\infty}(-1)^{rn}(z)_n q^{(r-1)\frac{n^2}{2}+n\alpha+(3r-5)n/2}(-q;q)_n \tag{5.5}$$

$$\Phi_{1,c_r}(z,\alpha;q^2) = \frac{1}{(z)_\infty}\sum_{-\infty}^{\infty}(-1)^{rn}(z)_n q^{(r-1)n^2+n\alpha+(2r-3)n}(-q;q^2)_n \tag{5.6}$$

$$\Phi_{0,c_r}(z,\alpha;q^2) = \frac{1}{(z)_\infty}\sum_{-\infty}^{\infty}(-1)^{rn}\frac{(z)_n q^{rn^2+n\alpha-n}}{(-q;q^2)_n} \tag{5.7}$$

$$\Psi_{1,c_r}(z,\alpha;q) = \frac{1}{(z)_\infty}\sum_{-\infty}^{\infty}(-1)^{r(n+1)}\frac{(z)_n q^{\frac{rn^2}{2}+n\alpha+(r/2-1)n}}{2(-q;q)_n} \tag{5.8}$$

We now show that these generalized functions are F_q functions.

Theorem 5.1. *The functions defined by the Equations 5.1-5.8 are F_q functions.*

Proof. We give the proof only for $f_{0,c_r}(z,\alpha;q)$. The remaining cases are similar. For $f_{0,c_r}(z,\alpha;q)$ note that

$$\begin{aligned}
zD_{q,z}f_{0,c_r}(z,\alpha;q) &= f_{0,c_r}(z,\alpha;q) - f_{0,c_r}(zq,\alpha;q)\\
&= \frac{1}{(z)_\infty}\sum_{-\infty}^{\infty}(-1)^{rn}\frac{(z)_n q^{\frac{rn^2}{2}+n\alpha-\frac{rn}{2}}}{(-q;q)_n}\\
&- \frac{1}{(zq)_\infty}\sum_{-\infty}^{\infty}(-1)^{rn}\frac{(zq)_n q^{\frac{rn^2}{2}+n\alpha-\frac{rn}{2}}}{(-q;q)_n}\\
&= \frac{1}{(z)_\infty}\sum_{-\infty}^{\infty}(-1)^{rn}\frac{(z)_n q^{\frac{rn^2}{2}+n\alpha-\frac{rn}{2}}}{(-q;q)_n}(1-(1-zq^n))\\
&= \frac{z}{(z)_\infty}\sum_{-\infty}^{\infty}(-1)^{rn}\frac{(z)_n q^{\frac{rn^2}{2}+n(\alpha+1)-\frac{rn}{2}}}{(-q;q)_n}\\
&= zf_{0,c_r}(z,\alpha+1;q)
\end{aligned}$$

and hence $f_{0,c_r}(z,\alpha;q)$ is a F_q function. □

6. INTEGRAL REPERSENTATION

We now give integral representations of these generalized functions. Jackson (on Page 23 of [11]) defined the q - integral on $(0,\infty)$ by

$$\int_0^\infty f(t)d_qt = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n$$

Now let $f(t) = t^{x-1}(tq;q)_\infty$ for some fixed x. We have

$$\begin{aligned}\int_0^\infty t^{x-1}(tq;q)_\infty d_qt &= (1-q)\sum_{n=-\infty}^{\infty}(q^{n+1};q)_\infty q^{nx}\\ &= (1-q)\frac{(q;q)_\infty}{(q^x;q)_\infty}\end{aligned}$$

and so

$$\frac{1}{(q^x;q)_\infty} = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^\infty t^{x-1}(tq;q)_\infty d_qt \tag{6.1}$$

We now use Equation 6.1 to give integral representations of the F_q functions 5.1 to 5.8. We let $a = q^\alpha$ for convenience.

$$f_{0,c_r}(q^z,\alpha;q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^\infty t^{z-1}(tq;q)_\infty f_{0,c_r}(0,at;q)d_qt \tag{6.2}$$

$$f_{1,c_r}(q^z,\alpha;q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^\infty t^{z-1}(tq;q)_\infty f_{1,c_r}(0,at;q)d_qt \tag{6.3}$$

$$F_{0,c_r}(q^z,\alpha;q^2) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^\infty t^{z-1}(tq;q)_\infty F_{0,c_r}(0,at;q^2)d_qt \tag{6.4}$$

$$F_{1,c_r}(q^z,\alpha;q^4) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^\infty t^{z-1}(tq;q)_\infty F_{1,c_r}(0,at;q^4)d_qt \tag{6.5}$$

$$\Psi_{0,c_r}(q^z,\alpha;q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^\infty t^{z-1}(tq;q)_\infty \Psi_{0,c_r}(0,at;q)d_qt \tag{6.6}$$

$$\Phi_{1,c_r}(q^z,\alpha;q^2) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^\infty t^{z-1}(tq;q)_\infty \Phi_{1,c_r}(0,at;q^2)d_qt \tag{6.7}$$

$$\Phi_{0,c_r}(q^z,\alpha;q^2) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^\infty t^{z-1}(tq;q)_\infty \Phi_{0,c_r}(0,at;q^2)d_qt \tag{6.8}$$

$$\Psi_{1,c_r}(q^z,\alpha;q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^\infty t^{z-1}(tq;q)_\infty \Psi_{1,c_r}(0,at;q)d_qt \tag{6.9}$$

Theorem 6.1. *Equations 6.2 to 6.9 hold.*

Proof. We prove only 6.2. The remaining cases are similar. We have,

$$\begin{aligned}
f_{0,c_r}(q^z,\alpha;q) &= \frac{1}{(q^z;q)_\infty}\sum_{-\infty}^{\infty}(-1)^n\frac{a^n(q^z;q)_n q^{\frac{rn^2}{2}-\frac{rn}{2}}}{(-q;q)_n}\\
&= \sum_{-\infty}^{\infty}(-1)^{rn}\frac{a^n q^{\frac{rn^2}{2}-\frac{rn}{2}}}{(-q;q)_n(q^{n+z};q)_\infty}\\
&= \frac{(1-q)^{-1}}{(q;q)_\infty}\sum_{-\infty}^{\infty}(-1)^{rn}\frac{a^n q^{\frac{rn^2}{2}-\frac{rn}{2}}}{(-q;q)_n}\int_0^\infty t^{n+z-1}(tq;q)_\infty d_qt\\
&= \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^\infty t^{z-1}(tq;q)_\infty\sum_{-\infty}^{\infty}(-1)^{rn}\frac{(at)^n q^{\frac{rn^2}{2}-\frac{rn}{2}}}{(-q;q)_n}d_qt\\
&= \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^\infty t^{z-1}(tq;q)_\infty f_{0,c_r}(0,at;q)d_qt
\end{aligned}$$

which completes the proof. ☐

We remark that for $at = q$ the function $f_{0,c_r}(0, at; q)$ reduces to the bilateral mock theta function $f_{0,c_r}(q)$ defined previously.

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