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q-DIFFERENCE EQUATION FOR GENERALIZED TRIVARIATE q-HAHN POLYNOMIALS

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Abstract. In this paper, we introduce a family of trivariate q-Hahn polynomials $\Psi_n^{(a)}(x,y,z|q)$ as a general form of Hahn polynomials $\psi_n^{(a)}(x|q)$, $\psi_n^{(a)}(x,y|q)$ and $F_n(x,y,z;q)$. We represent $\Psi_n^{(a)}(x,y,z|q)$ by the homogeneous q-difference operator $\widetilde{L}(a,b;\theta_{xy})$ introduced by Srivastava *et al* [H. M. Srivastava, S. Arjika and A. Sherif Kelil, *Some homogeneous q-difference operators and the associated generalized Hahn polynomials*, Appl. Set-Valued Anal. Optim. **1** (2019), pp. 187–201.] to derive: extended generating, Rogers formula, extended Rogers formula and Srivastava-Agarwal type generating functions involving $\Psi_n^{(a)}(x,y,z|q)$ by the q-difference equation.

1. Introduction

In this paper, we adopt the common conventions and notations on q-series. For the convenience of the reader, we provide a summary of the mathematical notations, basics properties and definitions to be used in the sequel. We refer to the general references (see [10]) for the definitions and notations. Throughout this paper, we assume that $|q|<1$.

For complex numbers a, the q-shifted factorials are defined by:

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k) \tag{1.1}$$

and $(a_1,a_2,\ldots,a_r;q)_m = (a_1;q)_m(a_2;q)_m\cdots(a_r;q)_m,\ m\in\{0,1,2\cdots\}$.
The q-binomial coefficient is defined as [6]

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} = \frac{(q^{-n};q)_k}{(q;q)_k}(-1)^k q^{nk-\binom{k}{2}}, \quad \text{for } 0\le k\le n.$$

The basic or q-hypergeometric function in the variable z (see Slater [13, Chap. 3], Srivastava and Karlsson [14, p. 347, Eq. (272)] for details) is defined as:

$${}_r\Phi_s\left[\begin{matrix} a_1,a_2,\ldots,a_r; \\ & q;z \\ b_1,b_2,\ldots,b_s; \end{matrix}\right] = \sum_{n=0}^{\infty}\Big[(-1)^n q^{\binom{n}{2}}\Big]^{1+s-r}\frac{(a_1,a_2,\ldots,a_r;q)_n}{(b_1,b_2,\ldots,b_s;q)_n}\frac{z^n}{(q;q)_n},$$

when $r>s+1$. Note that, for $r=s+1$, we have:

$${}_{r+1}\Phi_r\left[\begin{matrix} a_1,a_2,\ldots,a_{r+1}; \\ & q;z \\ b_1,b_2,\ldots,b_r; \end{matrix}\right] = \sum_{n=0}^{\infty}\frac{(a_1,a_2,\ldots,a_{r+1};q)_n}{(b_1,b_2,\ldots,b_r;q)_n}\frac{z^n}{(q;q)_n}.$$

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We will be mainly concerned with the Cauchy polynomials as given below [5]

$$p_n(x,y) := (x-y)(x-qy)\cdots(x-q^{n-1}y) = (y/x;q)_n\, x^n \tag{1.2}$$

with the Srivastava-Agarwal type generating function

$$\sum_{n=0}^{\infty} p_n(x,y)\frac{(\lambda;q)_n\, t^n}{(q;q)_n} = {}_2\Phi_1\left[\begin{matrix}\lambda, y/x; & \\ & q; xt\\ 0; & \end{matrix}\right]. \tag{1.3}$$

For $\lambda = 0$, we get the generating function [5]

$$\sum_{n=0}^{\infty} p_n(x,y)\frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(xt;q)_\infty}. \tag{1.4}$$

The generating function (1.4) is also the homogeneous version of the Cauchy identity or the q-binomial theorem given by [6]

$$\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}z^k = {}_1\Phi_0\left[\begin{matrix} a \\ - \end{matrix}; q, z\right] = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z|<1. \tag{1.5}$$

Putting $a = 0$, the relation (1.5) becomes Euler's identity [6]

$$\sum_{k=0}^{\infty}\frac{z^k}{(q;q)_k} = \frac{1}{(z;q)_\infty} \quad |z|<1 \tag{1.6}$$

and its inverse relation [6]

$$\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}\, z^k}{(q;q)_k} = (z;q)_\infty. \tag{1.7}$$

Saad and Sukhi [12] defined the q-difference operator θ_{xy}

$$\theta_{xy}\{f(x,y)\} := \frac{f(q^{-1}x,y) - f(x,qy)}{q^{-1}x-y}, \tag{1.8}$$

which turns out to be suitable for dealing with the Cauchy polynomials. Their corresponding q-exponential operator is

$$\mathbb{E}(z\theta_{xy}) = \sum_{k=0}^{\infty}\frac{q^{\binom{k}{2}}}{(q;q)_k}\,(z\,\theta_{xy})^k. \tag{1.9}$$

Recently, Srivastava, Arjika and Kelil [15] (see [3]) have introduced the q-difference operator $\widetilde{L}(a,b;\theta_{xy})$

$$\widetilde{L}(a,b;\theta_{xy}) = \sum_{k=0}^{\infty}\frac{q^{\binom{k}{2}}\,(a;q)_k}{(q;q)_k}\,(b\,\theta_{xy})^k, \tag{1.10}$$

to study q-polynomials and related generating functions.

In this paper, our goal is to generalize the results of Srivastava, Arjika and Kelil [15], and Mohameed [1]. We first construct the following generalized trivariate q-Hahn polynomials as

$$\Psi_n^{(a)}(x,y,z|q) = (-1)^n q^{-\binom{n}{2}}\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q (-1)^k q^{\binom{k}{2}}(a;q)_k p_{n-k}(y,x) z^k. \tag{1.11}$$

Remark 1.1. For $a = 0$, the generalized trivariate q-Hahn polynomials $\Psi_n^{(a)}(x,y,z|q)$ are the well known trivariate q-polynomials $F_n(x,y,z;q)$ investigated by Mohameed (see [1] for more details), i.e.,

$$\Psi_n^{(0)}(x,y,z|q) = F_n(x,y,z;q). \tag{1.12}$$

If we let $a = 0$, $y = ax$ and $z = y$, the generalized trivariate q-Hahn polynomials $\Psi_n^{(a)}(x,y,z|q)$ reduce to the second Hahn polynomials $\psi_n^{(a)}(x,y|q)$ [4], i.e.,

$$\Psi_n^{(0)}(x,ax,y|q) = \psi_n^{(a)}(x,y|q). \tag{1.13}$$

Also, $a = 0$, $y = ax$ and $z = 1$, the generalized trivariate q-Hahn $\Psi_n^{(a)}(x,y,z|q)$ reduce to Hahn polynomials $\psi_n^{(a)}(x|q)$ [2], i.e.,

$$\Psi_n^{(0)}(x,ax,1|q) = \psi_n^{(a)}(x|q). \tag{1.14}$$

The polynomials (1.11) can be represented by the homogeneous q-difference operator (1.10) as follows.

Proposition 1.2.

$$\Psi_n^{(a)}(x,y,z|q) = \widetilde{L}(a,z;\theta_{xy}) \left\{(-1)^n q^{-\binom{n}{2}} p_n(y,x)\right\}. \tag{1.15}$$

Proof. By identity (1.10) and taking into account $\theta_{xy} p_n(y,x) = -(1-q^n)\, p_{n-1}(y,x)$, we get the result. □

In light of $\theta_{xy}^k[(xt;q)_\infty/(yt;q)_\infty] = (-t)^k[(xt;q)_\infty/(yt;q)_\infty]$, we have the following identity

$$\widetilde{L}(a,z;\theta_{xy}) \left\{\frac{(xt;q)_\infty}{(yt;q)_\infty}\right\} = \frac{(xt;q)_\infty}{(yt;q)_\infty}\, {}_1\Phi_1\left[\begin{matrix} a; \\ 0; \end{matrix}\; q; zt\right]. \tag{1.16}$$

The main object of this paper is to use the q-difference equation to derive some identities such as: extended generating function, Rogers formula, extended Rogers formula and Srivastava-Agarwal type generating functions.

The paper is organized as follows. In Section 2, we state two theorems and give the proofs. We derive an extended generating function for these q-polynomials. In Section 3, we state the Rogers formula and extended Rogers formula and give the proofs by the q-difference equation. In Section 4, we obtain Srivastava-Agarwal type generating functions involving the generalized trivariate q-Hahn polynomials by the method of q-difference equation.

2. Main results and proofs

In this section, we introduce another extension of q-Hahn polynomials. Then, we represent it by the homogeneous q-difference operator and derive an extended generating function.

Theorem 2.1. *Let $f(a,b,x,y,z)$ be an 5-variable analytic function at $(a,b,x,y,z) = (0,0,0,0,0) \in \mathbb{C}^5$. If $f(a,b,x,y,z)$ satisfies the q-difference equation*

$$(q^{-1}x - y)\Big[f(a,b,x,y,z) - f(a,b,x,y,qz)\Big] = z\Big[f(a,b,q^{-1}x,y,qz) - \tag{2.1}$$

$$f(a,b,x,qy,qz)\Big]+az\Big[f(a,b,x,qy,q^2z)-f(a,b,q^{-1}x,y,q^2z)\Big],$$

then we have:

$$f(a,b,x,y,z)=\widetilde{L}(a,z;\theta_{xy})\Big\{f(a,b,x,y,0)\Big\}. \tag{2.2}$$

Corollary 2.2. *Let $f(b,x,y,z)$ be an 4-variable analytic function at $(b,x,y,z)=(0,0,0,0)\in\mathbb{C}^4$. If $f(b,x,y,z)$ satisfies the q-difference equation*

$$(q^{-1}x-y)\Big[f(b,x,y,z)-f(b,x,y,qz)\Big] = z\Big[f(b,q^{-1}x,y,qz)-f(b,x,qy,qz)\Big], \tag{2.3}$$

then we have:

$$f(b,x,y,z)=\mathbb{E}(z\theta_{xy})\Big\{f(b,x,y,0)\Big\}. \tag{2.4}$$

Proof. From the theory of several complex variables [11], we begin to solve the q-difference equation (2.3). First we may assume that

$$f(a,b,x,y,z)=\sum_{n=0}^{\infty}A_n(a,b,x,y)z^n. \tag{2.5}$$

Substituting (2.5) into (2.3), we get:

$$(q^{-1}x-y)\sum_{n=0}^{\infty}(1-q^n)A_n(a,b,x,y)z^n=\sum_{n=0}^{\infty}q^n(1-aq^n)\Big[A_n(a,b,q^{-1}x,y) - A_n(a,b,x,qy)\Big]z^{n+1}.$$

Comparing coefficients of z^n, $n\geq 1$, we find that

$$(q^{-1}x-y)(1-q^n)A_n(a,b,x,y) = q^{n-1}(1-aq^{n-1})\Big[A_{n-1}(a,b,q^{-1}x,y)-A_{n-1}(a,b,x,qy)\Big].$$

After simplification, we get:

$$A_n(a,b,x,y)=q^{n-1}\frac{1-aq^{n-1}}{1-q^n}\theta_{xy}\Big\{A_{n-1}(a,b,x,y)\Big\}.$$

By iteration, we gain

$$A_n(a,b,x,y)=q^{\binom{n}{2}}\frac{(a;q)_n}{(q;q)_n}\theta_{xy}^n\Big[A_0(a,b,x,y)\Big]. \tag{2.6}$$

Just taking $z=0$ in (2.5), we immediately obtain $A_0(a,b,x,y)=f(a,b,x,y,0)$. Substituting (2.6) back into (2.5), we achieve (2.3). □

Theorem 2.3 (Extended generating function for $\Psi_n^{(a)}(x,y,z|q)$)**.** *For $|yt|<1$, we have:*

$$\sum_{n=0}^{\infty}\Psi_{n+k}^{(a)}(x,y,z|q)\frac{(-1)^{n+k}q^{\binom{n+k}{2}}\,t^n}{(q;q)_n} \tag{2.7}$$

$$= t^{-k}\frac{(xt;q)_\infty}{(yt;q)_\infty}\sum_{n=0}^{k}\frac{(q^{-k},yt;q)_n\,q^n}{(xt,q;q)_n}\,{}_1\Phi_1\left[\begin{matrix}a;\\ \\ 0;\end{matrix}\;q;ztq^n\right].$$

Corollary 2.4. *For $|yt|<1$, we have:*

$$\sum_{n=0}^{\infty}F_{n+k}(x,y,z;q)\frac{(-1)^{n+k}q^{\binom{n+k}{2}}\,t^n}{(q;q)_n} \tag{2.8}$$
$$= t^{-k}\frac{(xt,zt;q)_\infty}{(yt;q)_\infty}\,{}_3\Phi_2\left[\begin{matrix}q^{-k},yt,0;\\ \\ xt,zt;\end{matrix}\;q;q\right].$$

Remark 2.5. For $a=0$, (2.7) reduces (2.8). For $a=0$ and $k=0$, (2.7) and (2.8) reduce to the generating function for $\Psi_n^{(a)}(x,y,z|q)$

$$\sum_{n=0}^{\infty}\Psi_n^{(a)}(x,y,z|q)\frac{(-1)^nq^{\binom{n}{2}}\,t^n}{(q;q)_n}=\frac{(xt;q)_\infty}{(yt;q)_\infty}\,{}_1\Phi_1\left[\begin{matrix}a;\\ \\ 0;\end{matrix}\;q;zt\right],\quad |yt|<1 \tag{2.9}$$

and [1, Theorem 2.6].

To prove the Theorem 2.3, the following Lemma is necessary.

Lemma 2.6. *q-Chu-Vandermonde formula [6, Eq. (II.6)]*

$${}_2\Phi_1\left[\begin{matrix}q^{-n},a;\\ \\ c;\end{matrix}\;q;q\right]=\frac{(c/a;q)_n}{(c;q)_n}a^n. \tag{2.10}$$

Proof of Theorem 2.3. Denoting the right-hand side of equation (2.7) by $F(a,t,x,y,z)$, we have:

$$F(a,t,x,y,z)=t^{-k}\sum_{n=0}^{k}\frac{(q^{-k};q)_n\,q^n}{(q;q)_n}\frac{(xtq^n;q)_\infty}{(ytq^n;q)_\infty}\,{}_1\Phi_1\left[\begin{matrix}a;\\ \\ 0;\end{matrix}\;q;ztq^n\right]. \tag{2.11}$$

Because equation (2.11) satisfies (2.3), we have:

$$\begin{aligned}
F(a,t,x,y,z) &= \widetilde{L}(a,z;\theta_{xy})\left\{F(a,t,x,y,0)\right\}\\
&= \widetilde{L}(a,z;\theta_{xy})\left\{t^{-k}\sum_{n=0}^{k}\frac{(q^{-k};q)_n\,q^n}{(q;q)_n}\frac{(xtq^n;q)_\infty}{(ytq^n;q)_\infty}\right\}\\
&= \widetilde{L}(a,z;\theta_{xy})\left\{\frac{(xt;q)_\infty}{(yt;q)_\infty}t^{-k}\sum_{n=0}^{k}\frac{(q^{-k},yt;q)_n\,q^n}{(xt,q;q)_n}\right\}\\
&= \widetilde{L}(a,z;\theta_{xy})\left\{\frac{(xt;q)_\infty}{(yt;q)_\infty}t^{-k}{}_2\Phi_1\left[\begin{matrix}q^{-k},yt;\\ \\ xt;\end{matrix}\;q;q\right]\right\}\text{ by (2.10)}\\
&= \widetilde{L}(a,z;\theta_{xy})\left\{\frac{p_k(y,x)}{(xt;q)_k}\frac{(xt;q)_\infty}{(yt;q)_\infty}\right\}\text{ by (1.4)}\\
&= \widetilde{L}(a,z;\theta_{xy})\left\{p_k(y,x)\sum_{n=0}^{\infty}\frac{p_n(y,xq^k)\,t^n}{(q;q)_n}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \widetilde{L}(a,z;\theta_{xy})\left\{\sum_{n=0}^{\infty} p_{n+k}(y,x)\frac{t^n}{(q;q)_n}\right\}\\
&= \sum_{n=0}^{\infty}\widetilde{L}(a,z;\theta_{xy})\left\{(-1)^{n+k}q^{-\binom{n+k}{2}}p_{n+k}(y,x)\right\}\\
&\quad \times(-1)^{n+k}q^{\binom{n+k}{2}}\,\frac{t^n}{(q;q)_n} \text{ by (1.15)}\\
&= \sum_{n=0}^{\infty}\Psi_{n+k}^{(a)}(x,y,z|q)\frac{(-1)^{n+k}q^{\binom{n+k}{2}}\,t^n}{(q;q)_n},
\end{aligned}
$$

which is the left-hand side of (2.7). □

3. The Rogers formula for $\Psi_n^{(a)}(x,y,z|q)$

In this section, we give Rogers formula and extended Rogers formula for the generalized trivariate q-Hahn polynomials $\Psi_n^{(a)}(x,y,z|q)$ by using the homogeneous q-difference equations.

Theorem 3.1 (Roger's-type formula for $\Psi_n^{(a)}(x,y,z|q)$)**.** *We have:*

$$
\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\Psi_{n+m}^{(a)}(x,y,z|q)(-1)^{n+m}q^{\binom{n+m}{2}}\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m} = \frac{(xs;q)_\infty}{(t/s,ys;q)_\infty}\sum_{k=0}^{\infty}\frac{(ys;q)_k\,q^k}{(sq/t,xs,q;q)_k}\,{}_1\Phi_1\left[\begin{matrix} a; \\ 0; \end{matrix}\; q;zsq^k\right], \tag{3.1}
$$

where $max\{|t/s|,|ys|\}<1$.

Corollary 3.2. *We have:*

$$
\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}F_{n+m}(x,y,z;q)(-1)^{n+m}q^{\binom{n+m}{2}}\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m} = \frac{(xs,zs;q)_\infty}{(t/s,ys;q)_\infty}\,{}_4\Phi_3\left[\begin{matrix} yt,0,0,0; \\ sq/t,xs,zs; \end{matrix}\; q;q\right], \tag{3.2}
$$

where $max\{|t/s|,|ys|\}<1$.

Proof. Denoting the right-hand side of equation (3.1) by $G(a,s,x,y,z)$, we have:

$$
G(a,s,x,y,z) = \frac{1}{(t/s;q)_\infty}\sum_{k=0}^{\infty}\frac{q^k}{(sq/t,q;q)_k}\frac{(xsq^k;q)_\infty}{(ysq^k;q)_\infty}\,{}_1\Phi_1\left[\begin{matrix} a; \\ 0; \end{matrix}\; q;zsq^k\right]. \tag{3.3}
$$

Because equation (3.3) satisfies (2.3), by (2.4), we have:

$$
\begin{aligned}
G(a,s,x,y,z) &= \widetilde{L}(a,z;\theta_{xy})\{G(a,s,x,y,0)\}\\
&= \widetilde{L}(a,z;\theta_{xy})\left\{\frac{1}{(t/s;q)_\infty}\sum_{k=0}^{\infty}\frac{q^k}{(sq/t,q;q)_k}\frac{(xsq^k;q)_\infty}{(ysq^k;q)_\infty}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \widetilde{L}(a,z;\theta_{xy})\left\{\frac{(xs;q)_\infty}{(ys;q)_\infty}\sum_{k=0}^{\infty}\frac{(ys;q)_k q^k}{(xs,q;q)_k}\frac{1}{(t/s;q)_\infty(sq/t;q)_k}\right\}\\
&= \widetilde{L}(a,z;\theta_{xy})\left\{\frac{(xs;q)_\infty}{(ys;q)_\infty}\sum_{k=0}^{\infty}\frac{(ys;q)_k q^k}{(xs,q;q)_k}\frac{(-t/s)^k q^{-\binom{k}{2}-k}}{(tq^{-k}/s;q)_\infty}\right\}\\
&= \widetilde{L}(a,z;\theta_{xy})\left\{\frac{(xs;q)_\infty}{(ys;q)_\infty}\sum_{k=0}^{\infty}\frac{(ys;q)_k q^k(-t/s)^k q^{-\binom{k}{2}-k}}{(xs,q;q)_k}\sum_{n=0}^{\infty}\frac{(t/s)^n q^{-nk}}{(q;q)_n}\right\}\\
&= \widetilde{L}(a,z;\theta_{xy})\left\{\frac{(xs;q)_\infty}{(ys;q)_\infty}\sum_{k=0}^{\infty}\frac{(ys;q)_k q^k}{(xs,q;q)_k}\sum_{n=k}^{\infty}\frac{(-t/s)^n q^{\binom{k}{2}-nk}}{(q;q)_{n-k}}\right\}\\
&= \widetilde{L}(a,z;\theta_{xy})\left\{\frac{(xs;q)_\infty}{(ys;q)_\infty}\sum_{n=0}^{\infty}\frac{(t/s)^n}{(q;q)_n}\sum_{k=0}^{n}\frac{(q^{-n},ys;q)_k\, q^k}{(xs,q;q)_k}\right\}\\
&= \widetilde{L}(a,z;\theta_{xy})\left\{\frac{(xs;q)_\infty}{(ys;q)_\infty}\sum_{n=0}^{\infty}\frac{(t/s)^n}{(q;q)_n}\,{}_2\Phi_1\left[\begin{matrix} q^{-n},ys; \\ xs; \end{matrix}\; q;q\right]\right\}\\
&= \sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}\widetilde{L}(a,z;\theta_{xy})\left\{\frac{(xs;q)_\infty}{(ys;q)_\infty}s^{-n}\,{}_2\Phi_1\left[\begin{matrix} q^{-n},ys; \\ xs; \end{matrix}\; q;q\right]\right\} \quad \text{by (2.10)}\\
&= \sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}\widetilde{L}(a,z;\theta_{xy})\left\{\frac{p_n(y,x)\,(xs;q)_\infty}{(xs;q)_n(ys;q)_\infty}\right\} \quad \text{by (2.12)}\\
&= \sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}\sum_{m=0}^{\infty}\Psi_{n+m}^{(a)}(x,y,z|q)(-1)^{n+m}q^{\binom{n+m}{2}}\frac{s^m}{(q;q)_m},
\end{aligned}
$$

which is the left-hand side of (3.1). □

Theorem 3.3 (Extended Roger's-type formula for $\Psi_n^{(a)}(x,y,z|q)$)**.** *We have:*

$$
\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\Psi_{n+m+k}^{(a)}(x,y,z|q)\frac{(-1)^{n+m+k}q^{\binom{n+m+k}{2}}\,t^n\,s^m\omega^k}{(q;q)_{n+m}(q;q)_m(q;q)_k}
= \frac{(x\omega;q)_\infty}{(s/t,t/\omega,y\omega;q)_\infty}\sum_{j=0}^{\infty}\frac{(y\omega;q)_j\,q^j}{(x\omega,q\omega/t,q;q)_j}\,{}_1\Phi_1\left[\begin{matrix} a; \\ 0; \end{matrix}\; q;z\omega q^j\right], \tag{3.4}
$$

where $max\{|s/t|,|t/\omega|,|y\omega|\}<1$.

Remark 3.4. For $s=0$, (3.4) reduces to (3.1).

Proof of Theorem 3.3. Denoting the right-hand side of equation (3.4) by $H(a,\omega,x,y,z)$, we have:

$$
H(a,\omega,x,y,z) = \frac{1}{(s/t,t/\omega;q)_\infty}\sum_{k=0}^{\infty}\frac{q^k}{(q\omega/t,q;q)_k}\cdot\frac{(x\omega q^k;q)_\infty}{(y\omega q^k;q)_\infty}\,{}_1\Phi_1\left[\begin{matrix} a; \\ 0; \end{matrix}\; q;z\omega q^k\right]. \tag{3.5}
$$

Because equation (3.5) satisfies (2.3), by (2.4), we have:

$$
\begin{aligned}
H(a,\omega,x,y,z) &= \widetilde{L}(a,z;\theta_{xy})\left\{H(a,\omega,x,y,0)\right\}\\
&= \widetilde{L}(a,z;\theta_{xy})\left\{\frac{1}{(s/t,t/\omega;q)_\infty}\sum_{k=0}^{\infty}\frac{q^k}{(q\omega/t,q;q)_k}\frac{(x\omega q^k;q)_\infty}{(y\omega q^k;q)_\infty}\right\}\\
&= \frac{1}{(s/t;q)_\infty}\widetilde{L}(a,z;\theta_{xy})\left\{\frac{1}{(t/\omega;q)_\infty}\sum_{k=0}^{\infty}\frac{q^k}{(q\omega/t,q;q)_k}\frac{(x\omega q^k;q)_\infty}{(y\omega q^k;q)_\infty}\right\}\\
&= \frac{1}{(s/t;q)_\infty}\widetilde{L}(a,z;\theta_{xy})\left\{F(a,\omega,x,y,0)\right\}\quad\text{by (2.10)}\\
&= \frac{1}{(s/t;q)_\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\Psi_{m+k}^{(a)}(x,y,z|q)\frac{(-1)^{m+k}q^{\binom{m+k}{2}}t^m}{(q;q)_m}\frac{\omega^k}{(q;q)_k}\\
&= \sum_{n=0}^{\infty}\frac{(s/t)^n}{(q;q)_n}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\Psi_{m+k}^{(a)}(x,y,z|q)\frac{(-1)^{m+k}q^{\binom{m+k}{2}}t^m}{(q;q)_m}\frac{\omega^k}{(q;q)_k}\\
&= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{m=n}^{\infty}\Psi_{n+m}^{(a)}(x,y,z|q)\frac{(-1)^{m+k}q^{\binom{m+k}{2}}t^{m-n}}{(q;q)_m}\frac{s^n}{(q;q)_n}\frac{\omega^k}{(q;q)_k}. \qquad (3.6)
\end{aligned}
$$

Replacing m by $m+n$ in (3.6), we obtain:

$$H(a,\omega,x,y,z)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\Psi_{n+m+k}^{(a)}(x,y,z|q)\,\frac{(-1)^{n+m+k}q^{\binom{n+m+k}{2}}t^n\,s^m\omega^k}{(q;q)_m(q;q)_{n+m}(q;q)_k},$$

which is the left-hand side of (3.4). □

4. Srivastava-Agarwal type generating functions for generalized trivariate q-Hahn polynomials

The Hahn polynomials [8,9] (or Al-Salam and Carlitz polynomials [2]) are defined as

$$\Phi_n^{(a)}(x|q)=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q (a;q)_k x^k. \qquad (4.1)$$

Srivastava and Agarwal gave the following generating function.

Lemma 4.1 ([16, Eq. (3.20)]). *Suppose that* $max\{|t|,|xt|\}<1$*, we have:*

$$\sum_{n=0}^{\infty}\Phi_n^{(\alpha)}(x|q)\frac{(\lambda;q)_n t^n}{(q;q)_n}=\frac{(\lambda t;q)_\infty}{(t;q)_\infty}{}_2\Phi_1\left[\begin{matrix}\lambda,\alpha;\\ \lambda t;\end{matrix}\; q;xt\right]. \qquad (4.2)$$

The generating function (4.2) is called Srivastava-Agarwal type generating functions for the Al-Salam-Carlitz polynomials [16].

In this section, we give Srivastava-Agarwal type generating function for the generalized trivariate q-Hahn polynomials $\Psi_n^{(a)}(x,y,z|q)$ by the homogeneous q-difference equation.

Theorem 4.2. *For* $|y\nu t|<1$*, we have:*

$$\sum_{n=0}^{\infty}\Psi_n^{(a)}(x,y,z|q)p_n(\nu,\mu)\frac{(-1)^n q^{\binom{n}{2}}\,t^n}{(q;q)_n} = \frac{(\mu/\nu,x\nu t;q)_\infty}{(y\nu t;q)_\infty}\sum_{n=0}^{\infty}\frac{(y\nu t;q)_n\,(\mu/\nu)^n}{(x\nu t,q;q)_n}\ {}_1\Phi_1\left[\begin{matrix} a; \\ 0; \end{matrix}\ q; z\nu t q^n\right]. \tag{4.3}$$

Corollary 4.3. *For $|y\nu t|<1$, we have:*

$$\sum_{n=0}^{\infty}F_n(x,y,z;q)p_n(\nu,\mu)\frac{(-1)^n q^{\binom{n}{2}}\,t^n}{(q;q)_n} = \frac{(\mu/\nu,x\nu t,z\nu t;q)_\infty}{(y\nu t;q)_\infty}\ {}_3\Phi_2\left[\begin{matrix} y\nu t,0,0; \\ x\nu t,z\nu t; \end{matrix}\ q;\frac{\mu}{\nu}\right]. \tag{4.4}$$

Corollary 4.4. *For $|yt|<1$, we have:*

$$\sum_{n=0}^{\infty}\Psi_n^{(a)}(x,y,z|q)(\lambda;q)_n\frac{(-1)^n q^{\binom{n}{2}}\,t^n}{(q;q)_n} = \frac{(\lambda,xt;q)_\infty}{(yt;q)_\infty}\sum_{n=0}^{\infty}\frac{(yt;q)_n\,\lambda^n}{(xt,q;q)_n}\ {}_1\Phi_1\left[\begin{matrix} a; \\ 0; \end{matrix}\ q; ztq^n\right]. \tag{4.5}$$

Remark 4.5. For $\nu=1$, (4.4) reduces to (4.5).

Corollary 4.6. *For $|axt|<1$, we have:*

$$\sum_{n=0}^{\infty}\psi_n^{(a)}(x,y|q)(\lambda;q)_n\frac{(-1)^n q^{\binom{n}{2}}\,t^n}{(q;q)_n} = \frac{(\lambda,xt,yt;q)_\infty}{(axt;q)_\infty}\ {}_3\Phi_2\left[\begin{matrix} axt,0,0; \\ xt,yt; \end{matrix}\ q;\lambda\right] \tag{4.6}$$

and

$$\sum_{n=0}^{\infty}\psi_n^{(a)}(x|q)(\lambda;q)_n\frac{(-1)^n q^{\binom{n}{2}}\,t^n}{(q;q)_n} = \frac{(\lambda,xt,t;q)_\infty}{(axt;q)_\infty}\ {}_3\Phi_2\left[\begin{matrix} axt,0,0; \\ xt,t; \end{matrix}\ q;\lambda\right]. \tag{4.7}$$

Remark 4.7. Setting $\nu=1$, $a=0$, $y=ax$ and $z=y$ in (4.4), we get (4.6). For $\nu=1$, $a=0$, $y=ax$ and $z=1$, (4.4) reduces to (4.7). For $y=1$, (4.6) reduces to (4.7).

Before we prove the Theorem 4.3, the following Lemma is necessary.

Lemma 4.8 ([7, Eq. (III.1)])**.** *For $\{|c|,|z|,|b|\}<1$, we have:*

$$ {}_2\Phi_1\left[\begin{matrix} a,b; \\ c; \end{matrix}\ q;z\right] = \frac{(b,az;q)_\infty}{(c,z;q)_\infty}\ {}_2\Phi_1\left[\begin{matrix} c/b,z; \\ az; \end{matrix}\ q;b\right]. \tag{4.8}$$

Proof of Theorems 4.3. Denoting the right-hand side of equation (4.4) by $H'(a,t,x,y,z)$, we have:

$$H'(a,t,x,y,z)=(\mu/\nu;q)_\infty\sum_{n=0}^{\infty}\frac{(x\nu tq^n;q)_\infty\,(\mu/\nu)^n}{(y\nu tq^n;q)_\infty(q;q)_n}\;{}_1\Phi_1\left[\begin{matrix}a;\\ \\ 0;\end{matrix}\;q;z\nu tq^n\right]. \tag{4.9}$$

Because equation (4.9) satisfies (2.3), by (2.4), we have:

$$\begin{aligned}H'(a,t,x,y,z)&=\widetilde{L}(a,z;\theta_{xy})\left\{H'(a,t,x,y,0)\right\}\\&=\widetilde{L}(a,z;\theta_{xy})\left\{(\mu/\nu;q)_\infty\sum_{n=0}^{\infty}\frac{(x\nu tq^n;q)_\infty\,(\mu/\nu)^n}{(y\nu tq^n;q)_\infty(q;q)_n}\right\}\\&=\widetilde{L}(a,z;\theta_{xy})\left\{\frac{(\mu/\nu,x\nu t;q)_\infty}{(y\nu t;q)_\infty}\;{}_2\Phi_1\left[\begin{matrix}y\nu t,0;\\ \\ x\nu t;\end{matrix}\;q;\frac{\mu}{\nu}\right]\right\}.\end{aligned}$$

By using (4.8) and (1.3), the last relation becomes

$$\begin{aligned}H'(a,t,x,y,z)&=\widetilde{L}(a,z;\theta_{xy})\left\{\sum_{n=0}^{\infty}p_n(y,x)\frac{p_n(\nu,\mu)\,t^n}{(q;q)_n}\right\}\\&=\sum_{n=0}^{\infty}\widetilde{L}(a,z;\theta_{xy})\left\{(-1)^nq^{-\binom{n}{2}}p_n(y,x)\right\}\\&\quad\times p_n(\nu,\mu)\frac{(-1)^nq^{\binom{n}{2}}\,t^n}{(q;q)_n}\quad\text{by (1.15)}\\&=\sum_{n=0}^{\infty}\Psi_n^{(a)}(x,y,z|q)p_n(\nu,\mu)\frac{(-1)^nq^{\binom{n}{2}}\,t^n}{(q;q)_n},\end{aligned}$$

which is the left-hand side of (4.4). This achieves the proof. □

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