Applied Analysis and Optimization
Volume 3, Number 3, 2019, 423–437

Yokohama Publishers
ISSN 2189-1664 Online Journal

COMPOSITE IMPLICIT VISCOSITY EXTRAGRADIENT ALGORITHMS FOR SYSTEMS OF VARIATIONAL INEQUALITIES WITH FIXED POINT CONSTRAINTS OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

LU-CHUAN CENG

Abstract. In this paper, a composite implicit viscosity extragradient method based on Korpelevich's extragradient method, implicit viscosity approximation method, and Mann's iteration method is studied and we consider a general system of variational inequalities and a common fixed point problem of an asymptotically nonexpansive mapping and countably many nonexpansive mappings in real Hilbert spaces.

1. Introduction

In a real Hilbert space $(H, \|\cdot\|)$, we denote by $\langle\cdot,\cdot\rangle$ its inner product. Given a nonempty closed convex subset $C \subset H$. Let P_C be the metric projection from H onto C. The notations $\mathbf{R}$, $\to$ and $\rightharpoonup$ are used to stand for the set of all real numbers, the strong convergence and the weak convergence, respectively. Given a mapping $T : C \to C$. We denote by $\mathrm{Fix}(T)$ the fixed point set of T, i.e., $\mathrm{Fix}(T) = \{u \in C : Tu = u\}$. Recall that T is called asymptotically nonexpansive if $\exists\{\theta_n\} \subset [0,\infty)$ s.t. $\lim_{n\to\infty} \theta_n = 0$ and

$$\|T^n u - T^n v\| \le (1+\theta_n)\|u - v\| \quad \forall u, v \in C, n \ge 1. \tag{1.1}$$

In particular, if $\theta_n = 0\ \forall n \ge 1$, then T is called nonexpansive. A mapping $f : C \to C$ is called a contractive map if $\exists\delta \in [0,1)$ s.t. $\|f(u) - f(v)\| \le \delta\|u - v\|\ \forall u, v \in C$. An operator $A : C \to H$ is called monotone if $\langle Au - Av, u - v\rangle \ge 0\ \forall u, v \in C$. It is called α-strongly monotone if $\exists\alpha > 0$ s.t. $\langle Au - Av, u - v\rangle \ge \alpha\|u - v\|^2\ \forall u, v \in C$. Also, it is called β-inverse-strongly monotone (or β-cocoercive) if $\exists\beta > 0$ s.t. $\langle Au - Av, u - v\rangle \ge \beta\|Au - Av\|^2\ \forall u, v \in C$. It is not hard to find that each inverse-strongly monotone operator is monotone and Lipschitzian and that each strongly monotone and Lipschitzian operator is inverse-strongly monotone but the converse is not true.

Given both nonlinear mappings $A_1, A_2 : C \to H$. Consider the following problem of finding $(u^*, v^*) \in C \times C$ s.t.

$$\begin{cases} \langle \mu_1 A_1 v^* + u^* - v^*, u - u^*\rangle \ge 0 & \forall u \in C, \\ \langle \mu_2 A_2 u^* + v^* - u^*, v - v^*\rangle \ge 0 & \forall v \in C, \end{cases} \tag{1.2}$$

2010 *Mathematics Subject Classification.* 47H05, 47H09.
Key words and phrases. asymptotically nonexpansive mapping, variational inequality, iterative algorithm, Hilbert space.

with constants $\mu_1, \mu_2 > 0$, which is called a general system of variational inequalities (GSVI). It is remarkable that GSVI (1.2) can be transformed into a fixed point problem in the following way.

Lemma 1.1 ([7]). *Given two points $u^*, v^* \in C$. Then (u^*, v^*) is a solution of GSVI (1.2) if and only if $x^* \in$GSVI(C, A_1, A_2), where GSVI(C, A_1, A_2) is the fixed point set of the operator $G := P_C(I - \mu_1 A_1)P_C(I - \mu_2 A_2)$, and $y^* = P_C(I - \mu_2 A_2)x^*$.*

The literature on the GSVI is vast and Korpelevich's extragradient method has received great attention given by many authors, who improved it in various ways and applied it for solving the GSVI (1.2) and other optimization problems; see e.g., [1–6, 10,12] and references therein, to name but a few. In the case when $A_1 = A_2 = A$ and $u^* = v^*$, the GSVI (1.2) reduces to the classical variational inequality problem (VIP) of finding $u^* \in C$ s.t. $\langle Au^*, v - u^*)\rangle \geq 0\ \forall v \in C$. In 2018, Cai et al. [2] designed a viscosity implicit rule for finding a common element of the solution set of GSVI (1.2) and the fixed point set of an asymptotically nonexpansive mapping T, and proved that the sequence constructed by the proposed rule converges strongly to a point in $\Omega = \mathrm{GSVI}(C, A_1, A_2) \cap \mathrm{Fix}(T)$, which solves a certain VIP. Very recently, Ceng and Wen [8] suggested a hybrid extragradient-like implicit rule for finding a common solution of the GSVI (1.2) and the CFPP of countably many uniformly Lipschitzian pseudocontractive mappings $\{S_n\}_{n=0}^{\infty}$ and an asymptotically nonexpansive mapping T, i.e., for any given $x_0 \in C$, the sequence $\{x_n\}$ is constructed by

$$(1.3)\qquad \begin{cases} z_n = \beta_n x_n + (1 - \beta_n) S_n z_n, \\ q_n = P_C(z_n - \mu_2 A_2 z_n), \\ p_n = P_C(q_n - \mu_1 A_1 q_n), \\ x_{n+1} = P_C[\alpha_n f(x_n) + (I - \alpha_n \rho F) T^n p_n] \quad \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0, 1]$ are such that

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(ii) $\lim_{n\to\infty} \theta_n / \alpha_n = 0$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ and $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;

(iv) $\sum_{n=0}^{\infty} \|T^{n+1} p_n - T^n p_n\| < \infty$.

They proved that the sequence $\{x_n\}$ generated by (1.3) converges strongly to a point $x^* \in \Omega = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, A_1, A_2) \cap \mathrm{Fix}(T)$, which also solves the VIP: $\langle (f - \rho F)x^*, x - x^* \rangle \leq 0\ \forall x \in \Omega$.

On the other hand, the implicit midpoint rule has become one of the most effective numerical methods for solving ordinary differential equations. In 2015, Xu et al. [15] considered the following viscosity implicit midpoint rule:

$$(1.4)\qquad x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T\left(\frac{x_n + x_{n+1}}{2}\right) \quad \forall n \geq 0.$$

They proved that the sequence $\{x_n\}$ constructed by (1.4) converges strongly to a point $x^* \in \mathrm{Fix}(T)$, which solves the VIP: $\langle (I - f)x^*, x - x^* \rangle \geq 0\ \forall x \in \mathrm{Fix}(T)$. In

2018, Yan and Cai [16] suggested a modified viscosity implicit rule for an asymptotically nonexpansive mapping T with a sequence $\{\theta_n\}$:

$$x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \gamma_n T^n(\frac{x_n + x_{n+1}}{2}) \quad \forall n \geq 0, \tag{1.5}$$

where $f : C \to C$ is a contractive map with constant $\delta \in [0, 1)$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [0, 1]$ are such that

(i) $\alpha_n + \beta_n + \gamma_n = 1$ and $\limsup_{n\to\infty} \alpha_n < 1$;
(ii) $\lim_{n\to\infty} \theta_n/\beta_n = 0$;
(iii) $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=0}^{\infty} \beta_n = \infty$;
(iv) $\lim_{n\to\infty} \|T^{n+1}x_n - T^n x_n\| = 0$.

They proved that if $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, then the sequence $\{x_n\}$ constructed by (1.5) converges strongly to a point $x^* \in \mathrm{Fix}(T)$, which solves the VIP: $\langle (I - f)x^*, x - x^*\rangle \geq 0\ \forall x \in \mathrm{Fix}(T)$.

In this paper, we introduce a composite implicit viscosity extragradient method for solving the GSVI (1.2) and the CFPP of an asymptotically nonexpansive mapping T and countably many nonexpansive mappings $\{S_n\}_{n=0}^{\infty}$ in a real Hilbert space H. Here the composite implicit viscosity extragradient method is based on Korpelevich's extragradient method, implicit viscosity approximation method, Mann's iteration method and the W-mappings constructed by $\{S_n\}_{n=0}^{\infty}$. Under suitable assumptions imposed on the parameters, we prove some strong convergence theorems for finding an element $x^* \in \Omega = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, A_1, A_2) \cap \mathrm{Fix}(T)$. As an application, we apply our main results to find a common solution of fixed point problems of nonexpansive mappings, variational inequality problems and general system of variational inequalities in H.

2. Preliminaries

Given a nonempty closed convex subset $C \subset H$ and a sequence $\{x_n\} \subset H$. The notation $x_n \to x$ (resp., $x_n \rightharpoonup x$) stands for the strong (resp., weak) convergence of $\{x_n\}$ to x. For each point $x \in H$, we know that there exists a unique nearest point in C, denoted by $P_C x$, s.t. $\|x - P_C x\| \leq \|x - y\|\ \forall y \in C$. The operator P_C is called the metric projection of H onto C.

Lemma 2.1. *The following hold:*

(i) $\langle y - z, P_C y - P_C z\rangle \geq \|P_C y - P_C z\|^2\ \forall y, z \in H$;
(ii) $\langle y - P_C y, z - P_C y\rangle \leq 0\ \forall y \in H, z \in C$;
(iii) $\|y - z\|^2 \geq \|y - P_C y\|^2 + \|z - P_C y\|^2\ \forall y \in H, z \in C$;
(iv) $\|y - z\|^2 = \|y\|^2 - \|z\|^2 - 2\langle y - z, z\rangle\ \forall y, z \in H$;
(v) $\|\lambda y + (1-\lambda)z\|^2 = \lambda\|y\|^2 + (1-\lambda)\|z\|^2 - \lambda(1-\lambda)\|y - z\|^2\ \forall y, z \in H, \lambda \in [0, 1]$.

The following lemma is an immediate consequence of the inner product in H.

Lemma 2.2. *The inequality holds:* $\|y + z\|^2 \leq \|y\|^2 + 2\langle z, y + z\rangle\ \forall y, z \in H$.

Lemma 2.3 ([14]). *Let* $\{x_n\}$ *and* $\{z_n\}$ *be bounded sequences in a Banach space* X *and let* $\{\beta_n\}$ *be a sequence in* $[0, 1]$*, satisfying the condition* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. *Suppose that* $x_{n+1} = \beta_n x_n + (1 - \beta_n)z_n\ \forall n \geq 0$ *and* $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then* $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Lemma 2.4 ([17]). *Let $\{a_n\}$ be a sequence in $[0,+\infty)$ satisfying $a_{n+1} \le (1-s_n)a_n + \delta_n$ $\forall n \ge 0$, where $\{s_n\}$ and $\{\delta_n\}$ lie in $\mathbf{R} := (-\infty,\infty)$ s.t. (a) $\{s_n\} \subset (0,1)$ and $\sum_{n=0}^{\infty} s_n = \infty$, and (b) $\limsup_{n\to\infty} \frac{\delta_n}{s_n} \le 0$ or $\sum_{n=0}^{\infty} |\delta_n| < \infty$. Then $a_n \to 0$ as $n \to \infty$.*

Let $\{S_n\}_{n=0}^{\infty}$ be a countable family of nonexpansive self-mappings on C, and $\{\lambda_n\}_{n=0}^{\infty}$ be a sequence in $[0,1]$. For any $n \ge 0$, we define a mapping $W_n : C \to C$ as follows:

$$
(2.1)\qquad \begin{cases}
U_{n,n+1} = I, \\
U_{n,n} = \lambda_n S_n U_{n,n+1} + (1-\lambda_n)I, \\
U_{n,n-1} = \lambda_{n-1} S_{n-1} U_{n,n} + (1-\lambda_{n-1})I, \\
\cdots \\
U_{n,k} = \lambda_k S_k U_{n,k+1} + (1-\lambda_k)I, \\
\cdots \\
U_{n,1} = \lambda_1 S_1 U_{n,2} + (1-\lambda_1)I, \\
W_n = U_{n,0} = \lambda_0 S_0 U_{n,1} + (1-\lambda_0)I.
\end{cases}
$$

Such a mapping W_n is nonexpansive and it is called a W-mapping generated by $S_n, ..., S_1, S_0$ and $\lambda_n, ..., \lambda_1, \lambda_0$.

Lemma 2.5 ([13]). *Let $\{S_n\}_{n=0}^{\infty}$ be a countable family of nonexpansive self-mappings on C with $\cap_{n=0}^{\infty}\mathrm{Fix}(S_n) \ne \emptyset$, and $\{\lambda_n\}_{n=0}^{\infty}$ be a real sequence such that $0 < \lambda_n \le b < 1$ $\forall n \ge 0$. Then the following statements hold:*

(i) *W_n is nonexpansive and $\mathrm{Fix}(W_n) = \cap_{i=0}^{n}\mathrm{Fix}(S_i)$ $\forall n \ge 0$;*
(ii) *the limit $\lim_{n\to\infty} U_{n,k}x$ exists for all $x \in C$ and $k \ge 0$;*
(iii) *the mapping $W : C \to C$ defined by $Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,0}x$ $\forall x \in C$, is a nonexpansive mapping satisfying $\mathrm{Fix}(W) = \cap_{n=0}^{\infty}\mathrm{Fix}(S_n)$ and it is called the W-mapping generated by $S_0, S_1, ...$ and $\lambda_0, \lambda_1, ...$*

Lemma 2.6 ([11]). *Let $\{S_n\}_{n=0}^{\infty}$ and $\{\zeta_n\}_{n=0}^{\infty}$ be as in Lemma 2.5. If D is any bounded subset of C, then the following statements hold:*

(i) *$\lim_{n\to\infty} \sup_{x\in D} \|W_n x - Wx\| = 0$;*
(ii) *$\sum_{n=0}^{\infty} \sup_{x\in D} \|W_{n+1}x - W_n x\| < \infty$.*

Lemma 2.7 ([8]). *Let the mapping $A : C \to H$ be α-inverse-strongly monotone. Then, for a given $\lambda \ge 0$, $\|(I-\lambda A)x-(I-\lambda A)y\|^2 \le \|x-y\|^2+\lambda(\lambda-2\alpha)\|Ax-Ay\|^2$. In particular, if $0 \le \lambda \le 2\alpha$, then $I - \lambda A$ is nonexpansive.*

The following lemma is an immediate consequence of Lemma 2.7.

Lemma 2.8 ([8]). *Let the mappings $A_1, A_2 : C \to H$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let the mapping $G : C \to C$ be defined as $G := P_C(I - \mu_1 A_1)P_C(I - \mu_2 A_2)$. If $0 \le \mu_1 \le 2\alpha$ and $0 \le \mu_2 \le 2\beta$, then $G : C \to C$ is nonexpansive.*

Lemma 2.9 ([9]). *Let X be a Banach space which admits a weakly continuous duality mapping, C be a nonempty closed convex subset of X, and $T : C \to C$ be an asymptotically nonexpansive mapping with a fixed point. Then $I - T$ is demiclosed at zero, i.e., if the sequence $\{x_n\} \subset C$ satisfies $x_n \rightharpoonup x \in C$ and $(I - T)x_n \to 0$, then $(I - T)x = 0$, where I is the identity mapping of X.*

3. Main results

In this section, we always assume that the following conditions hold:

$\{S_n\}_{n=0}^{\infty}$ is a countable family of nonexpansive self-mapping on C, and $\{\lambda_n\}_{n=0}^{\infty} \subset (0, b]$ for some $b \in (0, 1)$.

$T : C \to C$ is asymptotically nonexpansive with $\{\theta_n\}$ and the mappings $A_1, A_2 : C \to H$ are α-inverse-strongly monotone and β-inverse-strongly monotone, respectively.

$f : C \to C$ is a δ-contraction with $\delta \in [0, 1)$, and $W_n : C \to C$ is a W-mapping in (2.1) generated by $S_n, ..., S_1, S_0$ and $\lambda_n, ..., \lambda_1, \lambda_0$.

$\Omega := \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, A_1, A_2) \cap \mathrm{Fix}(T) \neq \emptyset$, where $\mathrm{GSVI}(C, A_1, A_2)$ is the fixed point set of $G := P_C(I - \mu_1 A_1)P_C(I - \mu_2 A_2)$ for $0 < \mu_1 < 2\alpha$ and $0 < \mu_2 < 2\beta$.

$\{t_n\} \subset (0, 1]$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$ are such that:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0$ and $\alpha_n + \beta_n + \gamma_n = 1\ \forall n \geq 0$;

(iii) $0 < \liminf_{n\to\infty} \gamma_n$ and $0 < \liminf_{n\to\infty} t_n \leq \limsup_{n\to\infty} t_n < 1$.

Algorithm 3.1. Suppose that the above hypotheses are satisfied. Given an arbitrary $x_0 \in C$. Let $\{x_n\}$ be the sequence generated by

$$\begin{cases} v_n = P_C(x_{n+1} - \mu_2 A_2 x_{n+1}), \\ u_n = P_C(v_n - \mu_1 A_1 v_n), \\ y_n = t_n x_n + (1 - t_n) W_n u_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T^n y_n \quad \forall n \geq 0. \end{cases} \tag{3.1}$$

We are now in a position to state and prove the first main result of this paper.

Theorem 3.2. *Let $\{x_n\}$ be constructed by Algorithm 3.1. Assume $T^{n+1}x_n - T^n x_n \to 0$. Then $x_n \to x^* \in \Omega \Leftrightarrow x_n - x_{n+1} \to 0$, where $x^* \in \Omega$ is the unique solution to the hierarchical variational inequality (HVI): $\langle (I - f)x^*, x - x^* \rangle \geq 0\ \forall x \in \Omega$.*

Proof. First of all, we note that the mapping $G : C \to C$ is defined as $G := P_C(I - \mu_1 A_1)P_C(I - \mu_2 A_2)$, where $0 < \mu_1 < 2\alpha$ and $0 < \mu_2 < 2\beta$. So, by Lemma 2.8, we know that G is nonexpansive. Meantime, by Lemma 2.5 (i), we know that W_n is nonexpansive. Since $\theta_n = \circ(\alpha_n)$, without loss of generality, we may assume that $\theta_n \leq \frac{(1-\delta)\alpha_n}{2}\ \forall n \geq 0$. For each $n \geq 0$ we define the mapping $F_n : C \to C$ as follows:

$$F_n(x) = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T^n(t_n x_n + (1 - t_n) W_n G x) \quad \forall x \in C.$$

It is easy to see that for all $x, y \in C$,

$$\begin{aligned} \|F_n(x) - F_n(y)\| &= \gamma_n \|T^n(t_n x_n + (1 - t_n) W_n G x) - T^n(t_n x_n + (1 - t_n) W_n G y)\| \\ &\leq \gamma_n (1 + \theta_n) \|(t_n x_n + (1 - t_n) W_n G x) - (t_n x_n + (1 - t_n) W_n G y)\| \\ &= \gamma_n (1 + \theta_n)(1 - t_n) \|W_n G x - W_n G y\| \\ &\leq \gamma_n (1 + \theta_n)(1 - t_n) \|x - y\|. \end{aligned}$$

Since $\gamma_n(1+\theta_n)(1-t_n) = \gamma_n(1-t_n) + \theta_n\gamma_n(1-t_n) \leq \gamma_n + \theta_n \leq \gamma_n + \frac{(1-\delta)\alpha_n}{2} < 1 - \beta_n$, by the Banach Contraction Principle, we deduce the existence and uniqueness of a fixed point $x_{n+1} \in C$ for the operator F_n, i.e.,

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T^n(t_n x_n + (1-t_n)W_n G x_{n+1}). \tag{3.2}$$

This ensures that the sequence $\{x_n\}$ generated by (3.1) is well defined.

It is now clear that the necessity of the theorem is true. In fact, if $x_n \to x^* \in \Omega$, then we have

$$\|x_{n+1} - x_n\| \le \|x^* - x_{n+1}\| + \|x^* - x_n\| \to 0 \quad (n \to \infty).$$

In order to prove the sufficiency of the theorem, we suppose $x_n - x_{n+1} \to 0$ and divide the proof of the sufficiency into several steps.

Step 1. We claim the boundedness of $\{x_n\}$. In fact, take an arbitrary $p \in \Omega$. Then $Tp = p,\ Gp = p$ and $W_n p = p\ \forall n \ge 0$. Choose a constant $M > 0$ sufficiently large such that $\max\{\|x_0 - p\|, \frac{2}{1-\delta}\|f(p) - p\|\} \le M$. We proceed by induction to show that $\|x_n - p\| \le M\ \forall n \ge 0$. Assume $\|x_n - p\| \le M$ for some $n \ge 0$. We show that $\|x_{n+1} - p\| \le M$. From (3.1) it follows that

$$\begin{aligned}
\|x_{n+1} - p\| &\le \alpha_n(\|f(x_n) - f(p)\| + \|f(p) - p\|) + \beta_n\|x_n - p\| \\
&\quad + \gamma_n(1+\theta_n)\times \\
&\quad \times \|t_n x_n + (1-t_n)W_n G x_{n+1} - p\| \\
&\le [\alpha_n\delta + \beta_n + \gamma_n(1+\theta_n)t_n]\|x_n - p\| \\
&\quad + \gamma_n(1+\theta_n)(1-t_n)\|x_{n+1} - p\| + \alpha_n\|f(p) - p\|,
\end{aligned}$$

which immediately yields

$$\begin{aligned}
\|x_{n+1} - p\| &\le \frac{\alpha_n\delta + \beta_n + \gamma_n(1+\theta_n)t_n}{1 - \gamma_n(1+\theta_n)(1-t_n)}\|x_n - p\| \\
&\quad + \frac{\alpha_n}{1 - \gamma_n(1+\theta_n)(1-t_n)}\|f(p) - p\| \\
&\le [1 - \frac{\alpha_n(1-\delta)}{2(1 - \gamma_n(1+\theta_n)(1-t_n))}]\|x_n - p\| \\
&\quad + \frac{\alpha_n(1-\delta)}{2(1 - \gamma_n(1+\theta_n)(1-t_n))} \cdot \frac{2}{1-\delta}\|f(p) - p\| \\
&\le \max\{\|x_n - p\|, \frac{2}{1-\delta}\|f(p) - p\|\} \\
&\le M
\end{aligned}$$

(it is remarkable that $\theta_n \le \frac{(1-\delta)\alpha_n}{2} \Rightarrow \alpha_n\delta + \beta_n < \alpha_n + \beta_n - \gamma_n\theta_n \Rightarrow \frac{\alpha_n\delta+\beta_n+\gamma_n(1+\theta_n)t_n}{1-\gamma_n(1+\theta_n)(1-t_n)} < 1$). Thus, $\{x_n\}$ is bounded, and so are the sequences $\{u_n\}, \{v_n\}, \{y_n\}, \{Gx_n\}, \{W_n u_n\}, \{T^n y_n\}$.

Step 2. We claim $x_n - Gx_n \to 0$. In fact, we write $q := P_C(p - \mu_2 A_2 p)$. Then $p = P_C(q - \mu_1 A_1 q) = Gp$. Note that $v_n = P_C(x_{n+1} - \mu_2 A_2 x_{n+1})$ and $u_n = P_C(v_n - \mu_1 A_1 v_n)$. Hence $u_n = Gx_{n+1}$. By Lemma 2.7 we have

$$\|v_n - q\|^2 \le \|x_{n+1} - p\|^2 - \mu_2(2\beta - \mu_2)\|A_2 x_{n+1} - A_2 p\|^2,$$

and $\|u_n - p\|^2 \leq \|v_n - q\|^2 - \mu_1(2\alpha - \mu_1)\|A_1 v_n - A_1 q\|^2$. Combining the last two inequalities, we obtain

$$\|u_n - p\|^2 \leq \|x_{n+1} - p\|^2 - \mu_2(2\beta - \mu_2)\|A_2 x_{n+1} - A_2 p\|^2 - \mu_1(2\alpha - \mu_1)\|A_1 v_n - A_1 q\|^2. \tag{3.3}$$

Also, using (3.1) and Lemma 2.1 (v), we get

$$\begin{aligned}\|y_n - p\|^2 &= t_n\|x_n - p\|^2 + (1 - t_n)\|W_n u_n - p\|^2 - t_n(1 - t_n)\|x_n - W_n u_n\|^2 \\ &\leq t_n\|x_n - p\|^2 + (1 - t_n)\|u_n - p\|^2 - t_n(1 - t_n)\|x_n - W_n u_n\|^2.\end{aligned}$$

Hence, using Lemma 2.2 we deduce from (3.1) and the convexity of the function $h(t) = t^2$ $\forall t \in \mathbf{R}$ that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq \|\alpha_n(f(x_n) - f(p)) + \beta_n(x_n - p) + \gamma_n(T^n y_n - p)\|^2 \\ &\quad + 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle \\ &\leq [\alpha_n\delta\|x_n - p\| + \beta_n\|x_n - p\| + \gamma_n(1 + \theta_n)\|y_n - p\|]^2 \\ &\quad + 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle \\ &\leq \alpha_n\delta\|x_n - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n(1 + \theta_n)\|y_n - p\|^2 \\ &\quad + 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle \\ &\leq (\alpha_n\delta + \beta_n + \gamma_n(1 + \theta_n)t_n)\|x_n - p\|^2 \\ &\quad + \gamma_n(1 + \theta_n)[(1 - t_n)\|u_n - p\|^2 \\ &\quad - t_n(1 - t_n)\|x_n - W_n u_n\|^2] + 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle.\end{aligned} \tag{3.4}$$

Substituting (3.3) for (3.4), we obtain

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq (\alpha_n\delta + \beta_n + \gamma_n(1 + \theta_n)t_n)\|x_n - p\|^2 \\ &\quad + \gamma_n(1 + \theta_n)\{(1 - t_n)[\|x_{n+1} - p\|^2 \\ &\quad - \mu_2(2\beta - \mu_2)\|A_2 x_{n+1} - A_2 p\|^2 - \mu_1(2\alpha - \mu_1)\|A_1 v_n - A_1 q\|^2] \\ &\quad - t_n(1 - t_n)\|x_n - W_n u_n\|^2\} + 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle,\end{aligned}$$

which immediately leads to

$$\begin{aligned}&(\alpha_n\delta + \beta_n + \gamma_n(1 + \theta_n)t_n)\|x_{n+1} - p\|^2 \\ &\leq (1 - \gamma_n(1 + \theta_n)(1 - t_n))\|x_{n+1} - p\|^2 \\ &\leq (\alpha_n\delta + \beta_n + \gamma_n(1 + \theta_n)t_n)\|x_n - p\|^2 - \gamma_n(1 + \theta_n)(1 - t_n)\times \\ &\quad \times\{\mu_2(2\beta - \mu_2)\|A_2 x_{n+1} - A_2 p\|^2 + \mu_1(2\alpha - \mu_1)\|A_1 v_n - A_1 q\|^2 \\ &\quad + t_n\|x_n - W_n u_n\|^2\} + 2\alpha_n\|f(p) - p\|\|x_{n+1} - p\|.\end{aligned} \tag{3.5}$$

(it is remarkable that $\theta_n \leq \frac{(1-\delta)\alpha_n}{2} \Rightarrow \alpha_n\delta + \beta_n < \alpha_n + \beta_n - \gamma_n\theta_n \Rightarrow \frac{\alpha_n\delta+\beta_n+\gamma_n(1+\theta_n)t_n}{1-\gamma_n(1+\theta_n)(1-t_n)} < 1$). Since $0 < \liminf_{n\to\infty}\gamma_n$ and $0 < \liminf_{n\to\infty} t_n \leq \limsup_{n\to\infty} t_n < 1$, we may assume, without loss of generality, that $\{\gamma_n\} \subset [c, 1)$ and $\{t_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. So it follows from (3.5) that

$$\begin{aligned}&c(1 + \theta_n)(1 - d)[\mu_2(2\beta - \mu_2)\|A_2 x_{n+1} - A_2 p\|^2 \\ &\quad + \mu_1(2\alpha - \mu_1)\|A_1 v_n - A_1 q\|^2 + c\|x_n - W_n u_n\|^2]\end{aligned}$$

$$\begin{aligned}&\le \gamma_n(1+\theta_n)(1-t_n)[\mu_2(2\beta-\mu_2)\|A_2x_{n+1}-A_2p\|^2\\&\quad+\mu_1(2\alpha-\mu_1)\|A_1v_n-A_1q\|^2+t_n\|x_n-W_nu_n\|^2]\\&\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\|.\end{aligned}$$

Since $\alpha_n \to 0$, $\theta_n \to 0$, $x_n - x_{n+1} \to 0$, $0 < \mu_1 < 2\alpha$, $0 < \mu_2 < 2\beta$, from the boundedness of $\{x_n\}$ we infer that

$$\lim_{n\to\infty}\|A_2x_{n+1}-A_2p\|=0,\quad \lim_{n\to\infty}\|A_1v_n-A_1q\|=0 \\ \text{and}\ \lim_{n\to\infty}\|x_n-W_nu_n\|=0. \tag{3.6}$$

On the other hand, from Lemma 2.1 (i) and (iv), we have

$$\|u_n-p\|^2\le\|v_n-q\|^2-\|v_n-u_n+p-q\|^2+2\mu_1\|A_1v_n-A_1q\|\|u_n-p\|.$$

Similarly, we obtain

$$\|v_n-q\|^2\le\|x_{n+1}-p\|^2-\|x_{n+1}-v_n+q-p\|^2+2\mu_2\|A_2x_{n+1}-A_2p\|\|v_n-q\|.$$

Combining the last two inequalities, we obtain

$$\begin{aligned}\|u_n-p\|^2&\le\|x_{n+1}-p\|^2-\|x_{n+1}-v_n+q-p\|^2-\|v_n-u_n+p-q\|^2\\&\quad+2\mu_1\|A_1v_n-A_1q\|\|u_n-p\|+2\mu_2\|A_2x_{n+1}-A_2p\|\|v_n-q\|.\end{aligned}\tag{3.7}$$

Substituting (3.7) for (3.4), we get

$$\begin{aligned}\|x_{n+1}-p\|^2&\le(\alpha_n\delta+\beta_n+\gamma_n(1+\theta_n)t_n)\|x_n-p\|^2\\&\quad+\gamma_n(1+\theta_n)(1-t_n)\|u_n-p\|^2\\&\quad+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\|\\&\le(\alpha_n\delta+\beta_n+\gamma_n(1+\theta_n)t_n)\|x_n-p\|^2\\&\quad+\gamma_n(1+\theta_n)(1-t_n)[\|x_{n+1}-p\|^2\\&\quad-\|x_{n+1}-v_n+q-p\|^2-\|v_n-u_n+p-q\|^2\\&\quad+2\mu_1\|A_1v_n-A_1q\|\|u_n-p\|\\&\quad+2\mu_2\|A_2x_{n+1}-A_2p\|\|v_n-q\|]\\&\quad+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\|,\end{aligned}$$

which immediately leads to

$$\begin{aligned}&(\alpha_n\delta+\beta_n+\gamma_n(1+\theta_n)t_n)\|x_{n+1}-p\|^2\\&\le(1-\gamma_n(1+\theta_n)(1-t_n))\|x_{n+1}-p\|^2\\&\le(\alpha_n\delta+\beta_n+\gamma_n(1+\theta_n)t_n)\|x_n-p\|^2-\gamma_n(1+\theta_n)(1-t_n)\times\\&\qquad\times\{\|x_{n+1}-v_n+q-p\|^2+\|v_n-u_n+p-q\|^2\\&\qquad-2\mu_1\|A_1v_n-A_1q\|\|u_n-p\|\\&\qquad-2\mu_2\|A_2x_{n+1}-A_2p\|\|v_n-q\|\}+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\|\\&\le(\alpha_n\delta+\beta_n+\gamma_n(1+\theta_n)t_n)\|x_n-p\|^2-\gamma_n(1+\theta_n)(1-t_n)\times\\&\qquad\times[\|x_{n+1}-v_n+q-p\|^2+\|v_n-u_n+p-q\|^2]\\&\qquad+2(1+\theta_n)[\mu_1\|A_1v_n-A_1q\|\|u_n-p\|\end{aligned}$$

$$+ \mu_2\|A_2x_{n+1} - A_2p\|\|v_n - q\|] + 2\alpha_n\|f(p) - p\|\|x_{n+1} - p\|.$$

So it follows that

$$\begin{aligned}
&c(1+\theta_n)(1-d)[\|x_{n+1} - v_n + q - p\|^2 + \|v_n - u_n + p - q\|^2] \\
&\le \gamma_n(1+\theta_n)(1-t_n)[\|x_{n+1} - v_n + q - p\|^2 + \|v_n - u_n + p - q\|^2] \\
&\le (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| + 2(1+\theta_n)[\mu_1\|A_1v_n - A_1q\|\|u_n - p\| \\
&\quad + \mu_2\|A_2x_{n+1} - A_2p\|\|v_n - q\|] + 2\alpha_n\|f(p) - p\|\|x_{n+1} - p\|.
\end{aligned}$$

Since $\alpha_n \to 0$, $\theta_n \to 0$ and $x_n - x_{n+1} \to 0$, from (3.6) and the boundedness of $\{x_n\}, \{u_n\}, \{v_n\}$ we obtain that $\lim_{n\to\infty} \|x_{n+1} - v_n + q - p\| = 0$, $\lim_{n\to\infty} \|v_n - u_n + p - q\| = 0$. Consequently,

$$\begin{aligned}
\|x_{n+1} - Gx_{n+1}\| =&\|x_{n+1} - u_n\| \le \|x_{n+1} - v_n + q - p\| \\
&+ \|v_n - u_n + p - q\| \to 0 \quad (n \to \infty).
\end{aligned} \tag{3.8}$$

Step 3. We claim $x_n - Tx_n \to 0$ and $x_n - Wx_n \to 0$. In fact, we observe from (3.2) that

$$\begin{aligned}
&\|x_{n+1} - T^n(t_nx_n + (1-t_n)W_nGx_{n+1})\| \\
&\quad \le \alpha_n\|f(x_n) - T^n(t_nx_n + (1-t_n)W_nGx_{n+1})\| + \beta_n\|x_n - x_{n+1}\| \\
&\qquad + \beta_n\|x_{n+1} - T^n(t_nx_n + (1-t_n)W_nGx_{n+1})\|,
\end{aligned}$$

This implies that

$$\begin{aligned}
&(1-\beta_n)\|x_{n+1} - T^n(t_nx_n + (1-t_n)W_nGx_{n+1})\| \\
&\quad \le \alpha_n\|f(x_n) - T^n(t_nx_n + (1-t_n)W_nGx_{n+1})\| + \beta_n\|x_n - x_{n+1}\|.
\end{aligned}$$

Since $x_n - x_{n+1} \to 0$, $\alpha_n \to 0$ and $\liminf_{n\to\infty}(1-\beta_n) = \liminf_{n\to\infty}(\alpha_n + \gamma_n) > 0$, we get $\|x_{n+1} - T^n(t_nx_n + (1-t_n)W_nu_n)\| \to 0 \quad (n \to \infty)$, which together with (3.6), implies that as $n \to \infty$,

$$\begin{aligned}
\|x_n - T^nx_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T^n(t_nx_n + (1-t_n)W_nu_n)\| \\
&\quad + \|T^n(t_nx_n + (1-t_n)W_nu_n - T^nx_n\| \\
&\le \|x_n - x_{n+1}\| + \|x_{n+1} - T^n(t_nx_n + (1-t_n)W_nu_n)\| \\
&\quad + (1+\theta_n)(1-t_n)\|W_nu_n - x_n\| \to 0.
\end{aligned} \tag{3.9}$$

Note that

$$\|x_n - Tx_n\| \le \|T^nx_n - T^{n+1}x_n\| + (2+\theta_1)\|T^nx_n - x_n\|.$$

So, using (3.9) and the assumption $T^{n+1}x_n - T^nx_n \to 0$, we have

$$\|x_n - Tx_n\| \to 0 \quad (n \to \infty). \tag{3.10}$$

In addition, using Lemma 2.6 (i), we have

$$\|WGx_n - W_nGx_n\| \le \sup_{x\in D} \|Wx - W_nx\| \to 0 \quad (n \to \infty). \tag{3.11}$$

for the bounded subset $D := \{Gx_n : n \ge 0\} \subset C$. Thus, using the assumption $x_n - x_{n+1} \to 0$, from (3.6), (3.8) and (3.11) we deduce that as $n \to \infty$,

$$\begin{aligned}
\|Wx_n - x_n\| \le& \|x_n - Gx_n\| + \|WGx_n - W_nGx_n\| + \|x_n - x_{n+1}\| \\
&+\|W_nu_n - x_n\| \to 0.
\end{aligned} \tag{3.12}$$

Step 4. We claim $\limsup_{n\to\infty}\langle x^* - f(x^*), x^* - x_n\rangle \le 0$, where $x^* = P_\Omega f(x^*)$. In fact, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x^* - f(x^*), x^* - x_n\rangle = \lim_{k\to\infty}\langle x^* - f(x^*), x^* - x_{n_k}\rangle.$$

By the boundedness of $\{x_n\}$ we know that there exists a subsequence of $\{x_n\}$ converging weakly to $\hat{x} \in C$. We may assume, without loss of generality, that $x_{n_k} \rightharpoonup \hat{x}$. Using Lemma 2.9, we conclude from (3.8), (3.10) and (3.12) that $\hat{x} \in \mathrm{Fix}(G) = \mathrm{GSVI}(C, A_1, A_2)$, $\hat{x} \in \mathrm{Fix}(T)$ and $\hat{x} \in \mathrm{Fix}(W) = \cap_{n=0}^\infty \mathrm{Fix}(S_n)$ (due to Lemma 2.5 (iii)). Therefore, $\hat{x} \in \Omega := \bigcap_{n=0}^\infty \mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, A_1, A_2) \cap \mathrm{Fix}(T)$. This together with the property of the metric projection implies that

(3.13)

$$\limsup_{n\to\infty}\langle x^* - f(x^*), x^* - x_n\rangle = \lim_{k\to\infty}\langle x^* - f(x^*), x^* - x_{n_k}\rangle = \langle x^* - f(x^*), x^* - \hat{x}\rangle \le 0.$$

Step 5. We claim $x_n \to x^*$, where $x^* = P_\Omega f(x^*)$. In fact, putting $p = x^*$, we obtain from (3.4) that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\le [1 - \frac{\alpha_n(1-\delta) - \gamma_n\theta_n}{1 - \gamma_n(1+\theta_n)(1-t_n)}]\|x_n - x^*\|^2 \\ &\quad + \frac{2\alpha_n}{1 - \gamma_n(1+\theta_n)(1-t_n)}\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ &\le [1 - \frac{\alpha_n(1-\delta)}{2(1 - \gamma_n(1+\theta_n)(1-t_n))}]\|x_n - x^*\|^2 \\ &\quad + \frac{\alpha_n(1-\delta)}{2(1 - \gamma_n(1+\theta_n)(1-t_n))} \cdot \frac{4}{1-\delta}\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \end{aligned} \tag{3.14}$$

(it is remarkable that $\theta_n \le \frac{(1-\delta)\alpha_n}{2} \Rightarrow \alpha_n\delta + \beta_n < \alpha_n + \beta_n - \gamma_n\theta_n \Rightarrow \frac{\alpha_n\delta+\beta_n+\gamma_n(1+\theta_n)t_n}{1-\gamma_n(1+\theta_n)(1-t_n)} < 1$). Since $\alpha_n\delta + \beta_n + \gamma_n(1+\theta_n)t_n \ge (\alpha_n\delta + \beta_n + \gamma_n(1+\theta_n))t_n = (1 - \alpha_n(1-\delta) + \gamma_n\theta_n)t_n$, we get $\lim_{n\to\infty}\frac{\alpha_n(1-\delta)}{2(1-\gamma_n(1+\theta_n)(1-t_n))} \le \lim_{n\to\infty}\frac{\alpha_n(1-\delta)}{2(1-\alpha_n(1-\delta)+\gamma_n\theta_n)t_n} = 0$, which implies that $\exists n_0 \ge 1$ s.t. $\{\frac{\alpha_n(1-\delta)}{2(1-\gamma_n(1+\theta_n)(1-t_n))}\}_{n=n_0}^\infty \subset (0,1)$. Note that $\frac{\alpha_n(1-\delta)}{2(1-\gamma_n(1+\theta_n)(1-t_n))} \ge \frac{\alpha_n(1-\delta)}{2(1-c(1-d))}$ and $\sum_{n=0}^\infty \alpha_n = \infty$. So it follows that $\sum_{n=0}^\infty \frac{\alpha_n(1-\delta)}{2(1-\gamma_n(1+\theta_n)(1-t_n))} = \infty$. Therefore, using (3.13) and Lemma 2.4, we conclude from (3.14) that $\|x_n - x^*\| \to 0$ as $n \to \infty$. This completes the proof. □

Theorem 3.3. *Let $\{x_n\}$ be constructed by Algorithm 3.1. Assume additionally that*

(i) $\sum_{n=0}^\infty |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=0}^\infty |\beta_{n+1} - \beta_n| < \infty$ *and* $\sum_{n=0}^\infty |t_{n+1} - t_n| < \infty$*;*

(ii) $\sum_{n=0}^\infty \sup_{x\in D}\|T^{n+1}x - T^n x\| < \infty$ *for any bounded subset D of C.*

Then $x_n \to x^* \in \Omega$, which is the unique solution to the HVI: $\langle (I-f)x^*, x - x^*\rangle \ge 0\ \forall x \in \Omega$.

Proof. In terms of Theorem 3.2, we only need to show $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. From (3.1) we get

$$\begin{aligned} \|y_n - y_{n-1}\| &\le t_n\|x_n - x_{n-1}\| + (1-t_n)\|W_nGx_{n+1} - W_{n-1}Gx_n\| \\ &\quad + |t_n - t_{n-1}|\|x_{n-1} - W_{n-1}Gx_n\| \\ &\le t_n\|x_n - x_{n-1}\| + (1-t_n)(\|x_{n+1} - x_n\| \\ &\quad + \|W_nGx_n - W_{n-1}Gx_n\|) \\ &\quad + |t_n - t_{n-1}|\|x_{n-1} - W_{n-1}Gx_n\|. \end{aligned} \tag{3.15}$$

Also, it follows from (3.2) that

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|\beta_n(x_n - x_{n-1}) + (\beta_n - \beta_{n-1})(x_{n-1} - T^n y_{n-1}) \\ &\quad + (\alpha_n - \alpha_{n-1})(f(x_{n-1}) - T^n y_{n-1}) \\ &\quad + \alpha_n(f(x_n) - f(x_{n-1})) + \gamma_n(T^n y_n - T^n y_{n-1}) \\ &\quad + \gamma_{n-1}(T^n y_{n-1} - T^{n-1} y_{n-1})\| \\ &\le (\beta_n + \alpha_n\delta + \gamma_n(1+\theta_n)t_n)\|x_n - x_{n-1}\| \\ &\quad + \gamma_n(1+\theta_n)(1-t_n)\|x_{n+1} - x_n\| + \{|\alpha_n - \alpha_{n-1}| \\ &\quad + |\beta_n - \beta_{n-1}| + |t_n - t_{n-1}| + \|W_nGx_n - W_{n-1}Gx_n\| \\ &\quad + \|T^n y_{n-1} - T^{n-1} y_{n-1}\|\}M_1, \end{aligned}$$

where

$$\sup_{n\ge 1}\{\|x_{n-1} - T^n y_{n-1}\|, \|f(x_{n-1}) - T^n y_{n-1}\|, (1+\theta_n)(1+\|x_{n-1} - W_{n-1}Gx_n\|)\} \le M_1$$

for some $M_1 > 0$. This implies that

$$\begin{aligned} \|x_{n+1} - x_n\| &\le [1 - \frac{\alpha_n(1-\delta) - \gamma_n\theta_n}{1 - \gamma_n(1+\theta_n)(1-t_n)}]\|x_n - x_{n-1}\| \\ &\quad + \{|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| \\ &\quad + |t_n - t_{n-1}| + \|W_nGx_n - W_{n-1}Gx_n\| \\ &\quad + \|T^n y_{n-1} - T^{n-1} y_{n-1}\|\}\frac{M_1}{1 - \gamma_n(1+\theta_n)(1-t_n)} \\ &\le [1 - \frac{\alpha_n(1-\delta)}{2(1 - \gamma_n(1+\theta_n)(1-t_n))}]\|x_n - x_{n-1}\| \\ &\quad + \{|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| \\ &\quad + |t_n - t_{n-1}| + \|W_nGx_n - W_{n-1}Gx_n\| \\ &\quad + \|T^n y_{n-1} - T^{n-1} y_{n-1}\|\}\frac{M_1}{1 - \gamma_n(1+\theta_n)(1-t_n)} \end{aligned} \tag{3.16}$$

(it is remarkable that $\theta_n \le \frac{(1-\delta)\alpha_n}{2} \Rightarrow \alpha_n\delta + \beta_n < \alpha_n + \beta_n - \gamma_n\theta_n \Rightarrow \frac{\alpha_n\delta+\beta_n+\gamma_n(1+\theta_n)t_n}{1-\gamma_n(1+\theta_n)(1-t_n)} < 1$). Since $\alpha_n\delta + \beta_n + \gamma_n(1+\theta_n)t_n \ge (\alpha_n\delta + \beta_n + \gamma_n(1+\theta_n))c = (1 - \alpha_n(1-\delta) + \gamma_n\theta_n)c$, we get $\limsup_{n\to\infty} \frac{M_1}{1-\gamma_n(1+\theta_n)(1-t_n)} \le \limsup_{n\to\infty} \frac{M_1}{(1-\alpha_n(1-\delta)+\gamma_n\theta_n)c} = \frac{M_1}{c}$. Thus, we may assume, without loss of generality, that $\frac{M_1}{1-\gamma_n(1+\theta_n)(1-t_n)} \le M_2\ \forall n \ge 0$. So it follows from (3.16) that for all

$n \geq 0$,

$$\|x_{n+1} - x_n\| \leq [1 - \frac{\alpha_n(1-\delta)}{2(1-\gamma_n(1+\theta_n)(1-t_n))}]\|x_n - x_{n-1}\| + \{|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + |t_n - t_{n-1}| + \|W_nGx_n - W_{n-1}Gx_n\| + \|T^n y_{n-1} - T^{n-1}y_{n-1}\|\}M_2. \tag{3.17}$$

Putting $D = \{Gx_n\}_{n=0}^{\infty} \cup \{y_n\}_{n=0}^{\infty}$, we know that D is a bounded subset of C. Hence, by Lemma 2.6 (ii) we have that $\sum_{n=1}^{\infty} \|W_nGx_n - W_{n-1}Gx_n\| \leq \sum_{n=1}^{\infty} \sup_{x\in D} \|W_nx - W_{n-1}x\| < \infty$. Note that the condition (ii) ensures $\sum_{n=1}^{\infty} \|T^n y_{n-1} - T^{n-1}y_{n-1}\| \leq \sum_{n=1}^{\infty} \sup_{x\in D} \|T^n x - T^{n-1}x\| < \infty$. Also, by the condition (i) we get

$$\sum_{n=1}^{\infty}\{|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + |t_n - t_{n-1}| + \|W_nGx_n - W_{n-1}Gx_n\| + \|T^n y_{n-1} - T^{n-1}y_{n-1}\|\}M_2 < \infty. \tag{3.18}$$

Since $1-\gamma_n(1+\theta_n)(1-t_n) > \alpha_n\delta + \beta_n + \gamma_n(1+\theta_n)t_n \geq (1-\alpha_n(1-\delta)+\gamma_n\theta_n)c$, we get $\lim_{n\to\infty} \frac{\alpha_n(1-\delta)}{2(1-\gamma_n(1+\theta_n)(1-t_n))} \leq \lim_{n\to\infty} \frac{\alpha_n(1-\delta)}{2(1-\alpha_n(1-\delta)+\gamma_n\theta_n)c} = 0$, which implies that $\exists n_0 \geq 1$ s.t. $\{\frac{\alpha_n(1-\delta)}{2(1-\gamma_n(1+\theta_n)(1-t_n))}\}_{n=n_0}^{\infty} \subset (0,1)$. Note that $\frac{\alpha_n(1-\delta)}{2(1-\gamma_n(1+\theta_n)(1-t_n))} \geq \frac{\alpha_n(1-\delta)}{2(1-c(1-d))}$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. So it follows that $\sum_{n=0}^{\infty} \frac{\alpha_n(1-\delta)}{2(1-\gamma_n(1+\theta_n)(1-t_n))} = \infty$. Therefore, using (3.18) and Lemma 2.4, we conclude from (3.17) that $\|x_{n+1}-x_n\| \to 0$ as $n \to \infty$. This completes the proof. $\square$

Theorem 3.4. *Let $\{x_n\}$ be constructed by Algorithm 3.1. Assume additionally that*

(i) $\limsup_{n\to\infty} \gamma_n < 1$, $\lim_{n\to\infty} |\beta_{n+1} - \beta_n| = 0$ *and* $\lim_{n\to\infty} |t_{n+1} - t_n| = 0$;

(ii) $\lim_{n\to\infty} \sup_{x\in D} \|T^{n+1}x - T^n x\| = 0$ *for any bounded subset D of C.*

Then $x_n \to x^* \in \Omega$, which is the unique solution to the HVI: $\langle (I-f)x^*, x - x^*\rangle \geq 0 \ \forall x \in \Omega$.

Proof. By Theorem 3.2, we only need to show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. In fact, set $z_n := \frac{x_{n+1}-\beta_n x_n}{1-\beta_n}$ $\forall n \geq 0$. Then we have

$$\begin{aligned} z_{n+1} - z_n &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(f(x_{n+1} - f(x_n)) + (\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n})f(x_n) \\ &\quad - \frac{\alpha_{n+1}}{1-\beta_{n+1}}(T^{n+1}y_{n+1} - T^n y_n) - (\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n})T^n y_n \\ &\quad + (T^{n+1}y_{n+1} - T^n y_n) \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(f(x_{n+1} - f(x_n)) \\ &\quad + (\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n})(f(x_n) - T^n y_n) \\ &\quad + (1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}})(T^{n+1}y_{n+1} - T^n y_n) \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(f(x_{n+1} - f(x_n)) \end{aligned}$$

$$+ (\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n})(f(x_n) - T^n y_n)$$
$$+ (1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}})(T^{n+1} y_{n+1} - T^n y_{n+1})$$
$$+ (1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}})(T^n y_{n+1} - T^n y_n).$$

It follows from (3.15) and (3.16) that

$$\begin{aligned}
\|z_{n+1} - z_n\| &\le \frac{\delta\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1} - x_n\| + |\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|M_1 \\
&\quad + \sup_{x\in D}\|T^{n+1}x - T^n x\| \\
&\quad + (1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}})(1+\theta_n)\{t_{n+1}\|x_{n+1} - x_n\| \\
&\quad + (1 - t_{n+1})\|x_{n+2} - x_{n+1}\| \\
&\quad + \|W_{n+1}Gx_{n+1} - W_n G x_{n+1}\| + |t_{n+1} - t_n|M_1\} \\
&\le (1 - \frac{(1-\delta)\alpha_{n+1}}{1-\beta_{n+1}} + \theta_n)\|x_{n+1} - x_n\| \\
&\quad + \frac{|\alpha_{n+1}(1-\beta_n) - \alpha_n(1-\beta_{n+1})|}{(\alpha_{n+1}+\gamma_{n+1})(\alpha_n+\gamma_n)}M_1 \\
&\quad + \sup_{x\in D}\|T^{n+1}x - T^n x\| \\
&\quad + (1+\theta_n)\{(|\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n| + |t_{n+1} - t_n| \\
&\quad + \sup_{x\in D}\|W_{n+1}x - W_n x\| + \sup_{x\in D}\|T^{n+1}x - T^n x\|)M_2 \\
&\quad + \sup_{x\in D}\|W_{n+1}x - W_n x\| + |t_{n+1} - t_n|M_1\} \\
&\le (1+\theta_n)\|x_{n+1} - x_n\| + \frac{\alpha_{n+1}+\alpha_n}{c^2}M_1 \\
&\quad + ((1+\theta_n)M_2 + 1)\sup_{x\in D}\|T^{n+1}x - T^n x\| \\
&\quad + (1+\theta_n)(\alpha_{n+1} + \alpha_n + |\beta_{n+1} - \beta_n| + |t_{n+1} - t_n| \\
&\quad + \sup_{x\in D}\|W_{n+1}x - W_n x\|)M_3,
\end{aligned}$$

where $1 + M_1 + M_2 \le M_3$ for some $M_3 > 0$. This ensures that

$$\begin{aligned}
\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| &\le \theta_n\|x_{n+1} - x_n\| + \frac{\alpha_{n+1}+\alpha_n}{c^2}M_1 \\
&\quad + ((1+\theta_n)M_2 + 1)\sup_{x\in D}\|T^{n+1}x - T^n x\| \\
&\quad + (1+\theta_n)(\alpha_{n+1} + \alpha_n + |\beta_{n+1} - \beta_n| + |t_{n+1} - t_n| \\
&\quad + \sup_{x\in D}\|W_{n+1}x - W_n x\|)M_3.
\end{aligned}$$

Using Lemma 2.6 (ii) and conditions (i), (ii), we deduce from $\alpha_n \to 0$, $\theta_n \to 0$ and the boundedness of $\{x_n\}$ that $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0$. By

Lemma 2.3, we have $\lim_{n\to\infty}\|z_n - x_n\| = 0$. So we obtain $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. This completes the proof. □

References

[1] G. Aviv and Y. Shehu, *An efficient iterative method for finding common fixed point and variational inequalities in Hilbert spaces*, Optimization **68** (2019), 13–32.

[2] G. Cai, Y. Shehu and O. S. Iyiola, *Strong convergence results for variational inequalities and fixed point problems using modified viscosity implicit rules*, Numer. Algo. **77** (2018), 535–558.

[3] Y. Cao and K. Guo, On the convergence of inertial two-subgradient extragradient method for variational inequality problems, (2019), 10.1080/02331934.2019.1686632.

[4] L. C. Ceng, A. Petrusel, J. C. Yao and Y. Yao, *Hybrid viscosity extragradient method for systems of variational inequalities, fixed points of nonexpansive mappings, zero points of accretive operators in Banach spaces*, Fixed Point Theory **19** (2018), 487–503.

[5] L. C. Ceng, A. Petrusel, J. C. Yao and Y. Yao, *Systems of variational inequalities with hierarchical variational inequality constraints for Lipschitzian pseudocontractions*, Fixed Point Theory **20** (2019), 113–133.

[6] L. C. Ceng and M. Shang, *Generalized Mann viscosity implicit rules for solving systems of variational inequalities with constraints of variational inclusions and fixed point problems*, Mathematics **7** (2019), 933.

[7] L. C. Ceng, C. Y. Wang and J. C. Yao, *Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities*, Math. Methods Oper. Res. **67** (2008), 375–390.

[8] L. C. Ceng and C. F. Wen, *Systems of variational inequalities with hierarchical variational inequality constraints for asymptotically nonexpansive and pseudocontractive mappings,* Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **113** (2019), 2431–2447.

[9] L. C. Ceng, H. K. Xu and J. C. Yao, *The viscosity approximation method for asymptotically nonexpansive mappings in Banach spaces*, Nonlinear Anal. **69** (2008), 1402–1412.

[10] C. E. Chidume, O. M. Romanus and U. V. Nnyaba, *An iterative algorithm for solving split equilibrium problems and split equality variational inclusions for a class of nonexpansive-type maps*, Optimization **67** (2018), 1949–1962.

[11] S. S. Chang, H. W. J. Lee and C. K. Chan, *A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization,* Nonlinear Anal. **70** (2009), 3307–3319.

[12] X. Qin and J. C. Yao, *A viscosity iterative method for a split feasibility problem*, J. Nonlinear Convex Anal. **20** (2019), 1497–1506.

[13] K. Shimoji and W. Takahashi, *Strong convergence to common fixed points of infinite nonexpansive mappings and applications,* Taiwanese J. Math. **5** (2001), 387–404.

[14] T. Suzuki, *Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals,* J. Math. Anal. Appl. **305** (2005), 227–239.

[15] H. K. Xu, M. A. Alghamdi and N. Shahzad, *The viscosity technique for the implicit midpoint rule of nonexpansive mappings in Hilbert spaces,* Fixed Point Theory Appl. **2015** (2015), 41.

[16] Q. Yan and G. Cai, *Convergence analysis of modified viscosity implicit rules of asymptotically nonexpansive mappings in Hilbert spaces,* Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **112** (2018), 1125–1140.

[17] Z. Xue, H. Zhou and Y. J. Cho, *Iterative solutions of nonlinear equations for m-accretive operators in Banach spaces,* J. Nonlineaer Convex Anal. **1** (2000), 313–320.

Manuscript received July 7 2019
revised November 30 2019

L.C. Ceng
Department of Mathematics, Shanghai Normal University, Shanghai 200234, China
E-mail address: zenglc@hotmail.com